濟大論文集(自然), 39, 19-22, 1994
Cheju Univ. Jour. (Natural Sci.),
39, 19-22, 1994

On Extensions of Commutative Banach Algebras

Hyun Jin-Oh, Kim Seong-Won**

可換 Banach 代數의 擴大에 關하여

玄進五*, 金成原*

Introduction

In this note, all algebra will mean a commutative complex algebra with identity.

Lemma1.If $T: A^n \to A^m$ is an A-module homomorphism then T is bounded as a linear mapping of Banach spaces.

proof) Let(t_{ij}) be the matrix of T in the standand bases on A^n and A^m

and let $M = \max(\|t_{ij}\|)$.

since $\|T_x\| = \sum_{j=1}^{m} \|t_{i1}x_1 + \cdots + t_{im}x_m\|$ for each

$x = (x_1, \cdots, x_n) \in A^n$

We get that $\|T_x\| = \sum_{ij} \|t_{ij}\|\|x_j\| \leq \sum_{j=1}^{n} mM\|x_j\|$

and hence $\|T_x\| \leq mM\|x\|$

Therefore T is bounded and T is continuous

Theorem2. Let A be a Banach algebra and B a faithful A-algebra finitely generated and projective as an A-module.

Then B is a Banach algebra.

proof) Since B is a faithful Aalgebra and finitely generated. B is integral over A.

There exist elements $b_1, \cdots, b_n$ in B that generate B as an A-algebra. and each b_i satisfies a monic polynom ial $f_i(x)$ in $A[X]$ of degree d_i.

Put

$B_0 = A$ and $B_i = B_{i-1}(x)/(f_i(x))$ for $i = 1, \cdots, n$

If B_{i-1} is normed so that it is a Banch algebra. then we can extend this norm to B_i so that it is a Banach algebra isometric to B_{i-1}.

We see that every Banach norm on A extends to a Banach norm on B_n. where B_n is isometric (as a Banach A-module) to

* 사범대학 수학교육과(Dept. of Mathematics Education, Cheju Univ., Cheju-do, 690-756, Korea)

$A^d(d=d_1 \cdot d_2 \cdots d_n)$

Thus we have an A-algebra homomorphism $f: B_n \to B$.

Since B is projective, the kernel of f is a direct summand of B_n

Thus if we define norm B by using the quotient seminorm,this seminorm is actualy a norm making B a Banach algebra.

Main Theorems

If B is an integral extension of A, then it is well known that an ideal M in A is a maximal ideal in A if and only if there is a maximal ideal N in B such that M=N ∩ A.

From this it easily follow that R(A)=R(B)∩A, where R(A) denotes the radical of A.

For an extension B of a Banach algebra A, define the mapping by

$$\Pi_A^B \mid M(B) \Rightarrow M(A)\ \Pi_A^B(\phi) = \phi / A \text{ for each } \phi \in M(B).$$

clearly. Π_A^B is continuous mapping with respect to the Gelfand toplogy.

Lemma3. If B is an integral extension of A.

then Π_A^B is onto and $\hat{a} \to \Pi_A^B(\hat{a})$is an isomorphism of $\hat{A}$ into $\hat{B}$. Thus B is an integral extension of A.

proof)

For $\Phi \in M(A)$, Put $M=\phi^{-1}(0)$.

Since M is a maximal ideal in A, there exists a maximal ideal N in B such that N ∩ A= M.

Let Ψ denote the canonical projection of B onto B/A.

If $a \in A$,then $a-\phi(a)\psi(e)$ and thus there is an element $m \in M$ such that $a=\phi(a)+m$. hence we have

$$\psi(a)=\phi(a)\psi(e) \text{ and } \psi(A)=C\psi(e).$$

Since B is integral over A. B/A in integral over $\Psi(A)$ so that $B/A=\Psi(A)=C\Psi(e)$.

If put $\Phi(b)=\lambda_b\Psi(e)$, we have $\Phi \in M(B)$. $\Phi|A=\Pi_A^B(\Phi)=\Phi$ and $\phi^{-1}(0)=N$.

Hence Π_A^B is onto.

Since A seperates points of M(A), it follow immediately from Π_A^B is onto.

For a polynomial

$$\beta(x)=\sum \beta_i x^i \in A(x), \quad put\ \beta_\phi(x)=\sum \phi(B_i)x^i.$$

Theorom4.If B is an integral extension of A, then M(A) is compact if and only if M(B) is compact.

proof) Suppose M(A) is compact.

Let $\| \|_\infty$ denotes the sup norm over M(A). Then $\|a\|_\infty < +\infty$ for each $a \in A$

we show that every element of B has a bounded transform.

Let $b \in B$, and $\beta(x)=x^n+\sum_{i=0}^{n-1}\beta_i x^i$

be any monic polynomial over A such that

$\beta(b)=0$. if $t>0$ is any positive number satisfying $t^n \geq \sum_{i=0}^{n-1}\|\beta_i\|_\infty t^i$, then for

$$\tilde{\phi} \in M(B), \ |\tilde{\phi}(b)| \leq t \ \text{ for all } \tilde{\phi} \in M(B),$$

and B is a normed algebra with respect to sup norm so that M(B) is compact in the Gelfand topology.

From now on B will denote a finitely generated projective extension of a fixed Banach algebra A.

If $\phi \in M(B)$, $\Pi_A^B(\Psi) = \phi$ and $m_\phi = Ker\phi$.

then we define the multiplicity m(ψ) ofψ as the complex dimension of $e(B/m_\phi B)$, where e is the idempotent element of B/m$_\phi$B such that the support of e is {ψ}.

If $\alpha(x) = \sum_{i=0}^{n} a_i x^i \in A(x)$ and $\phi \in M(A)$.

we set $Z(\alpha_\phi) = \{ \lambda \in C \mid \alpha_\phi(\lambda) = 0 \}$.

If α(x) is monic we write

$A_\alpha = A(x)/(\alpha(x))$and Π_α for the projection of $M(A_\alpha)$ onto M(A).

we recall the a finitely generated projective module M is said to be have a well-defined rank n if for any prime ideal p of A the localized module M_p. Conversely, since Π_A^B is continuous and onto,hold.

References

C. E.Rickart, Banach Algebra, D.Van Nostrand Co.Inc.(1960)

J. B.Conway, A course in Funtional Analysis, springer-Vevlag New York Inc.(1985)

S. W.Kim, On extensions Banach lgebra, (1990)

〈國文抄錄〉

可換 Banach 代數의 擴大에 關하여

本 論文에서는 (α(x))∈A(x)에 의하여 生成된 ideal이라 할때 또한 A가 複素數體C上의] 單位元을 갖는 可換 Banach代數이고 α(x)가 A上의 monic Polynomial이라 할 때 A(x)/(α(x))는 A의 하나의 擴大인데 이러한 性質을 活用하여 A의 擴大代數 B가 몇가지 條件하에서 하나의 Banach 代數가 됨을 證明하였다.