The Conjugate Gradient Method for Least Squares Solutions of Constrained Singular Linear Operater Equations

Kim Do-hyun

규정된 선형연산자 방정식의 최소자승해에 관한 Conjugate gradient 방법

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Ⅰ. Introduction and Preliminaries

A class of ill-posed operator equations subject to a weighted minimization constraint is investigated by a regularization-iteration method. The problem is as follows: Among all least squares solutions of an equation involving a bounded linear operator with non-closed range, find the one of minimum weighted norm.

The results of this work can be applied to integral equations of the first kind and ill-posed constrained minimization problems.

The existence and uniqueness of the weighted minimum norm solution were shown. A family of regularization operators was obtained and the convergence of the regularized solution to the exist solution was proved.

In this paper, we establish the convergence of the conjugate gradient method to a solution of any equation. We also determine the error bound.

Let X, Y and Z be (real or complex) Hilbert spaces. Let $A: X \rightarrow Y$ and $L: X \rightarrow Z$ be bounded linear operators. We assume that the range R(L) of L is closed in Z, but the range R(A) of A is not necessarily closed in Y. Let A^+ denote generalized inverse of A, which will be defined later. For Y in the domain $D(A^+)$ of A^+, let (1, 1) $S_Y = \{u \in X : \|Au-Y\|_Y = \inf \|Ax-Y\|_Y,\ x \in X\}$.

Then the problems to find $w \in S_Y$ such that $\|Lw\|_Z = \inf \{\|Lu\|_Z : u \in S_Y\}$.

Definition 1,1. For a given $y \in Y$, an element $u \in X$ is called a least squares solution of the operator equation $Ax=y$ if and only if

$$\|Au-y\| \leq \|Ax-y\| \text{ for all } x \in X$$

Definition 1,2. An element $\bar{u}$ is called a least squares solution of minimal norm of $Ax=y$ if and only if $\bar{u}$ is a least squares solution of $Ax=y$ and $\|\bar{u}\| \leq \|u\|$ for all least squares solutions u of $Ax=y$

Definitin 1,3. Let A be a bounded linear operator from X into Y. The generalized inverse, denoted by A^+, is a linear operator from the subspace $R(A) \oplus R(A)^{\perp}$ into X, defined by $A^+y=\bar{u}$ where $\bar{u}$ is the least squares solution of minimal norm of the equation $Ax=y$.

Throughout this paper, we assume that $N(A) \cap N(L) = \{0\}$ and $N(A)+N(L)$ is closed. We define a new inner product in X:

$$[u, v] = \langle Au, Av \rangle_Y + \langle Lu, Lv \rangle_Z \text{ for } u, v \in X.$$

We denote the space X with the inner product $[\cdot, \cdot]$ by X_L.

Theorem 1,4. An element $w \in X$ is a solution to the problem (1,1) if and only if $A^*Aw = A^*y$ and $L^*Lw \in N(A)$

Proof) Refer to Nashed.

Our interest is in the case that the range of A is not closed. Instead of solving this ill-posed problem directly, we will regularize it by a family of stable minimization problems.

Let W be the product space of Y and Z with the usual inner product: $W = YxZ$, $\langle (y_1, z_1), (y_2, z_2)\rangle_W = \langle y_1, y_2\rangle_Y + \langle z_1, z_2\rangle_Z$ for $y_1, y_2 \in Y$ and $z_1, z_2, \in Z$. For $\alpha > 0$, let C_α be a linear operator from X into W defined by $C_\alpha x = (Ax, \sqrt{\alpha} Lx)$ for $x \in X$. We denote by U_α the unique least squares solution of minimal norm of the equation $C_\alpha x = \bar{b}$ for each $\alpha > 0$. That is, $U_\alpha = C^+{}_\alpha \bar{b}$. Let us write $J_\alpha(x) = \|Ax - y\|^2 + \alpha \|Lx\|^2$.

Theorem 1,5. Let $\alpha > 0$. An element X_α in X^* minimizes the quadratic functional $J_\alpha(x)$ if and only if (1, 2) $C^*_\alpha C_\alpha x = C^*_\alpha \bar{b}$

proof) Refer to Song.

Theorem 1,6. For $\alpha > 0$, let U_α be the unique solution of the operator equation (1,2). Then $\lim_{\alpha \to 0} U_\alpha$ exists and $\lim_{\alpha \to 0} U_\alpha = A_L^+ y$.

Proof) Refer to Song.

II. Convergence of the conjugate gradient method.

In this section, using the conjugate gradient method, we find an approximate solution U_α of the regularized operator equation $C^*_\alpha C_\alpha x = C^*_\alpha \bar{b}$ We prove the convergence of the conjugate gradient method to a solution of $C^*_\alpha C_\alpha x = C^*_\alpha \bar{b}$.

Let $J_\alpha(x) = \|Ax - y\|^2 + \alpha \|Lx\|^2$ for > 0. The conjugate gradient method for minimizing $J_\alpha(x)$ is generated by the following prescription:

$x_0 \in X$ is arbitrary,

$p_0 = r_0 = C^*_\alpha C_\alpha x_0 - C^*_\alpha b$,

$\alpha_0 = \|r_0\|^2 / \|C_\alpha r\|^2$,

$x_1 = x_0 - \alpha_0 p_0$, and for $n = 1, 2 \cdots\cdots$, we compute $r_n = C^*_\alpha C_\alpha x_n - C^*_\alpha \bar{b} = r_{n-1} - \alpha_{n-1} C^*_\alpha C_\alpha p_{n-1}$ where $\alpha_{n-1} = \langle r_{n-1}, p_{n-1}\rangle / \|C^*_\alpha P_{n-1}\|^2$ If $r_n \neq 0$, we compute $p_n = r_n + \beta_{n-1} p_{n-1}$ where $\beta_{n-1} = -\langle r_n, C^*_\alpha C_\alpha p_{n-1}\rangle / \|C_\alpha p_{n-1}\|^2$. Finally, we set $x_{n+1} = x_n - \alpha_n p_n$.

Theorem 2,1. Suppose H_1 is a Hilbert space satisfying $H_1 = \bigcup_{n=0}^{\infty} \text{span}\{p_0, \cdots, p_{n-1}\}$ and $T \in L(H_1, H_2)$ is invertible. Then the conjugate gradient method converges to the unique solution u of $Tx = b$ for any $x_0 \in H_1$

Proof) Refer to Groetsch.

Theorem 2,2. Let C_n be the closed convex hull of $\{x_0, x_1, \cdots\cdots, x_n\}$. Then x_n is the unique vector in C_n which is closest to the solution u of $Tx = b$.

Proof) Refer to Groetsch.

The functional g appearing below is defined by $g(x) = \|C_\alpha x = P\bar{b}\|^2$ where P is the projection of W onto $R(C_\alpha)$.

Theorem 2,3. In the assumptions of section 1 the sequence generated by conjugate gradient method converges monotonically to the least squares solution $u = C^+{}_\alpha \bar{b} + (I_1 - P_H)x_0$ of the equation $C_\alpha x = \bar{b}$, where P_H is the projection of X onto the closed subspace $H = R(C^*_\alpha)$ Moreover, if m and M are positive numbers such that $mI \leq C^*{}_\alpha C_\alpha \mid H \leq MI$ where I is the identity on H, then $\|x_n - u\|^2 \leq \frac{g(x_o)}{m} \left(\frac{M-m}{M+m}\right)^{2n}$

Proof) Note that for any $x_0 \in X$, $\{r_i\} \subset H$ and $\{p_i\} \subset H$

Therefore $\{x_i\} \subset x_0 + H$. Also the mapping $R: x_0 + H \to H$ obtained by restricting P_H to $x_0 + H$ is an isometry onto H and the conjugate gradient method applied to the operator $s \in L(H, R(C_\alpha))$ defined by $S = C_\alpha \mid H$ generates a sequence $\{x'_n\}$ which is related to $\{x_n\}$ by $x'_n = Rx_n$. Also S has a bounded inverse and hence (2,1) and (2,2) apply to the sequence $\{x'_n\}$, showing that $\{x'_n\}$ converges monotonically to $C^+{}_\alpha \bar{b}$, the unique solution of $Sx = P\bar{b}$. Daniel's error bound gives $\|x'_n - C^+_\alpha \bar{b}\|^2 \leq \frac{g(x_o)}{m} \left(\frac{M-m}{M+m}\right)^{2n}$ By applying the isometry R^{-1} we see that $R^{-1} C^+{}_\alpha \bar{b} = C^+{}_\alpha \bar{b} + (I_1 - P_H)x_0 = u$ and $R^{-1} x'_n = x_n$. Therefore $\{x_n\}$ converges monotonically to u and the error bound holds.

Corollary 2,4. If C_α has rank r, then for any $x_0 \in X$ the conjugate gradient method for $C_\alpha x = \bar{b}$ converges in at most r steps to the least squares solution $C^+_\alpha b + (I_1 - P_H)x_0$.

Proof) Refer to Groetsch.

Literature cited

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國 文 抄 錄

이 논문에서는 어떤 초기 근사치에 대하여 Conjugate gradient 방법에 의해서 형성되는 수열은 규정된 방정식의 근에 수렴한다는 것을 보이고, 오차범위를 결정하였다.