The Property of Group as a Semigroup

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반군으로서의 군의성질

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I. Introduction and Definition

In [1] J.M. Howie has induced the definition of group by the property of semigorup. In [2] T.K. Dutta has studied the relative ideals in group.

Now the object of this paper is to study properties of group as a semigroup.

Definition (1-1). We shall say that (S,·) is a semigroup if $(xy)z=y(yz)$ for any $s,y,z \in S$.

Definition (1-2). If a semigroup (S,·) has the additional property that $xy=yx$ for any $x,y,x \in S$, it is called a commutative semigroup.

Definition (1-3). If a semigroup (S,·) has an element 1 such that $x1=1x=x$ for any $x \in S$, 1 is called an identity (element) of S and S is called a semigroup with identity, or monoid.

Definition (1-4). If A and B are subsets of a semigroup S, we write $AB=\{ab: a \in A, b \in B\}$ and $\{a\}B=aB=\{ab : b \in B\}$ for $a \in S$.

Definition (1-5). If (S,·) is a semigroup, then a nonempty subset T of S is called a subsemigroup of S if $xy \in T$ for any $x,y \in T$.

Definition (1-6).A nonempty subset I of a semigroup S is called a left ideal is $SI \subseteq I$, a right ideal if $IS \subseteq I$, and a (two-sided) ideal if it is both a left and a right ideal.

Remark (1-7). Every ideal (whether one- or two-sided) is a subsemigroup, but not every subsemigroup is an ideal.

Counter-example). Let S be a semigroup with identity. Then $\{1\}$ is a subsemigroup of S. But $s\{1\}=S \not\subset \{1\}$. Hence $\{1\}$ is not an ideal.

Definition (1-8). An ideal I of S such that $\{0\} \subset I \subset S$ (strictly) is called a proper ideal.

Definition (1-9). A ring such that $a^2=a$ for all $a \in R$ is called a Boolean ring.

Example (1-10). Let S be the set of all subsets of some fixed set U. For A,B S, define $A+B=(A-B) \cup (B-A)$ and $AB=A \cap B$. Then S is a Boolean ring.

II. The Properties of a Group is a Semigroup

Proposition (2-1). Let S be a semigroup. S is a group iff complement of every ideal (both left and right) is also an ideal.

Proof: Suppose that I is an ideal of S and x belongs to S-I. Now we must show that tx and xt belong to S-I for any $t \in S$. Here if $tx \in S-I$, then $t^{-1}(tx)=x \in I$, which is a contradiction. So $tx \in S-I$ and $xt \in S-I$. Conversely, suppose that I is an ideal. Then S-I is an ideal of S. Let $t \in S$ and $i \in I$. Then $ti \in I$ and $ti \in S-I$ since S-I is an ideal of S. Thus S has no any proper ideal. That is, $S=Sa=aS$ for any for any $a \in S$ since Sa is a left ideal and aS is a right ideal.

Here $\exists$ $e \in S \ni ae = a$ and $\exists$ $e' \in S \cdot \ni e'a = a$ for any a S. Thus $e = e'e = e'$ and $ae = ea = a$. That is, e is a unique identity in S. Since $e \in S$ and $aS = Sa = S$ for any $a \in S$, so $\exists$ $a_1, a_2 \in S \cdot \ni e = aa_1$, and $e = a_2a$ for any $a \in S$. Thus $a_2e = a_2aa_1 = ea_1$. Hence $a_1 = a_2 = a^{-1}$ is a unique inverse of a.

Proposition (2-2). Let S be a semigroup. S is a group if the difference A-B of two ideals is an ideal (assuming that ϕ is an ideal).

Proof: Let $s \in S$ and $a \in A\text{-}B$ where A, B are ideals. Then $sa \in A$ since A is an ideal, but $sa \notin B$. (if $sa \in B$, then $s^{-1} sa = a \in B$). By similary method $as \in A\text{-}B$. Hence A-B is an ideal in S. Conversely, consider S and A which is any ideal of S. Then S-A is an ideal. Let $s \in S\text{-}A$ and $a \in A$. Then $sa \in A$ and $sa \in S\text{-}A$. Thus S has no proper ideal. Hence we can hold the proof (by proposition 2.1).

Definition (2-3). $I_B(S)$ is the set of all ideals of a semigroup S, $I_L(S)$ is the set of all left ideals of S and $I_R(S)$ is the set of all right ideals of S.

$P_L(S)$ is the set of all left ideals such that $sa \in A$ imply $a \in A$ for any $s \in S$, $P_R(S)$ is the set of all right ideals such that $as \in A$ imply $a \in A$ for any $s \in S$ and $P_B(S)$ is the set of all ideals such that $sa \in A$ imply $a \in A$ and $as \in A$ imply $a \in A$ for any $s \in S$.

Proposition (2-4). If a semigroup S is a group, then $I_L(S) = P_L(S)$ and $I_R(S) = P_R(S)$. Furthermore S is a group iff $I_B(S) = P_B(S)$.

Proof: (1) Evidently $I_L(S) \supseteq P_L(S)$. Let L be a left ideal and $ta \in L$ for any $t \in S$. Then $(t^{-1})ta = a \in L$. Thus $L \in P_L(S)$. Hence $I_L(S) = P_L(S)$. And $I_R(S) = P_R(S)$ (by similary method).

(2) Evidently, $P_B(S) \subseteq I_B(S)$. Let $A \in I_B(S)$ and let $at \in A$ and $ta \in A$ for any $t \in S$. Then $(at)t^{-1} = a \in A$ and $(t^{-1})ta = a \in A$. Hence $I_B(S) = P_B(S)$. Conversely, let A be an ideal. Then we must show that S-A is an ideal (by Proposition 2.1). Let $a \in S\text{-}A$ and $t \in S$. Then $ta \in S\text{-}A$ and $at \in S\text{-}A$ (if $ta \in A$ and $at \in A$, $a \in A$). Thus S-A is an ideal. Hence we can hold the proof.

Proposition (2-5). Let S be a monoid and let $M_1(S)$ be the set of all ideals of S which contain an identity. Then $M_1(S)$ is a monoid with zero and $M_1(S) = \{S\}$.

Proof: Define an operation by the definition 1.4. Then $(AB)C = A(BC)$ for A,B,C $M_1(S)$. Here $S(AB) = (SA)B \subseteq AB$ and $(AB)S = A(BS) \subseteq AB$ and $1 \in AB$ since $1 \in A$, $1 \in B$ and $AB = \{ab : a \in A, b \in B\}$. Thus $AB \in M_1(S)$. And $SA \subseteq A$ and $AS \subseteq A$. Since S has an identity, so $SA \supseteq A$ and $AS \supseteq A$. Thus $SA = AS = A$ that is, S is an identity in $M_1(S)$. Hence $M_1(S)$ is a monoid. Furthermore $AS = SA = S$ since A has an identity. Thus S is a zero in $M_1(S)$ and $M_1(S) = \{S\}$.

Proposition (2-6). Let T be a subgroup of a monoid M. Then $P = \{Tm : m \in M\}$ is a partition of M.

Proof: Let $m \in M$. Then $m \in Tm$ and $M = \bigcup_{m \in M} Tm$. If $x \in Tm \cap Tn$, then $\exists$ $t_1, t_2 \in T \cdot \ni \cdot x = t_1m$ and $x = t_2n$. Here $m = t_1^{-1}(t_2n) = (t_1^{-1}t_2)n$ and $Tm = T(t_1^{-1}t_2)n \subseteq Tn$ and $Tn \subseteq Tm$. Hence P is a partition of M.

Definition (2-7). A semigroup S will be called left (right) simple iff S is the only left (right) ideal of S. A semigroup S which is both left and right simple is called simple.

Definition (2-8). $(a, b) \in \mathcal{L}$ iff $Sa = Sb$, $(a, b) \in R$ iff $aS = bS$ for a monoid S.

Proposition (2-9). A monoid S is left simple iff $\mathcal{L} = S \times S$.

Proof: Let S be left simple and let $a \in S$. Then $sa \in Sa$ and $S(Sa) = (SS)a = Sa$ is a left ideal. Thus $S = Sa$ and $Sa = Sb$ for any $a, b \in S$. Hence $S \times S = \mathcal{L}$. Conversely, since $Sa = Sb$ for any $a, b \in S$ and S has an identity, so $S = Sa$ for any $a \in S$. Let A be a left ideal of S and $a \in A \subseteq S$. Then $S = Sa \subseteq A \subseteq S$ and $S = A$. Thus S contains no proper left ideal. Hence we can hold the proof.

Corollary (2-10). A monoid S is right simple iff $R = S \times S$.

Proposition (2-11). A semigroup S will be a group iff it is simple.

Proof: If $x \in S$, then $x = aa^{-1}x$ aS and $x = xa^{-1}a \in Sa$. And $aS \subseteq S$ and $Sa \subseteq S$ for any $a \in S$. Thus $S = Sa$ and $aS = S$ for any $a \in S$. Let L be a left ideal of S and let $m \in L \subseteq S$. Then $S = Sm \subset L \subset S$. Thus S contains no proper left ideal. And S also contains no proper right ideal (by the similary method). Hence S is simple. Conversely, let S be simple.

Then S is left and right simple. If S is left simple and $a \in S$, then $sa \in Sa$ for any $s \in S$. Since $S(Sa) \subseteq Sa$, so Sa is a left ideal. Thus $Sa = S$ and $aS = S$ for any $a \in S$. Hence we can hold the proof by the Proposition 1.1.

Proposition (2-12). $P_g(S)$ is a Boolean ring on assuming that $\phi \in P_g(S)$.

Proof: Let $A, B \in P_g(S)$. Then $A\text{-}B \in P_g(S)$. For if $a \in A\text{-}B$ and $t \in S$ then $ta \in A\text{-}B$ and $at \in A\text{-}B$, since $ta \in B$ imply $a \in B$ which contradicts $a \in A\text{-}B$. If $ta \in A\text{-}B$ and $at \in A\text{-}B$ for any $t \in S$ then $a \in A\text{-}B$, since $a \in B$ imply $ta \in B$ and $at \in B$. Thus $A\text{-}B \in P_g(S)$. And we can easily check that $A \cup B$, $A \cap B \in P_g(S)$. Hence we can complete the proof by the Example 1. 10.

Literature cited

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국 문 초 록

이 논문은 반군으로서의 군의 여러가지 성질을 새로운 정의를 통해서 다루었다.