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A NOTE ON BIRGET-RHODES EXPANSIONS OF TOPOLOGICAL GROUPS

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ABSTRACT. Let G be a compact group with identity 1 and let $C_1(G)$ be the semilattice of all compact subsets of G containing 1. In this paper, we investigate some structures of the compact F-inverse monoid

$$\tilde{G}_c^{\mathcal{R}} = \{(A, g) \in C_1(G) \times G \ : \ g \in A\}.$$

1. INTRODUCTION

For any finite sequence $(s_1, s_2, \ldots, s_n)$ of elements $s_1, s_2, \ldots, s_n$ in a semigroup S, put

$$P(s_1, s_2, \ldots, s_n) := \{1, s_1, s_1 s_2, \ldots, s_1 s_2 \cdots s_n\},$$

where 1 is the identity of S^1. Define

$$\tilde{S}^{\mathcal{R}} := \{(P(s_1, s_2, \ldots, s_n), s_1 s_2 \cdots s_n) : s_1, s_2, \ldots, s_n \in S, n \geq 1\}$$

with the multiplication

$$\begin{aligned}(P(s_1, s_2, \ldots, s_n), s_1 s_2 \cdots s_n)&(P(t_1, t_2, \ldots, t_m), t_1 t_2 \cdots t_m) \\ &= (P(s_1, s_2, \ldots, s_n) \cup (s_1 s_2 \cdots s_n) \cdot P(t_1, t_2, \ldots, t_m), s_1 s_2 \cdots s_n t_1 t_2 \cdots t_m)\end{aligned}$$

where $s \cdot U = \{su : u \in U\}$ for $s \in S$ and $U \subset S$. Then $\tilde{S}^{\mathcal{R}}$ is a semigroup, which is called the *Birget-Rhodes expansion* of the semigroup S (see [1]). It turns out [12] that when $S = G$ a group,

$$\tilde{G}^{\mathcal{R}} = \{(A, g) \in P_1(G) \times G : g \in G\},$$

where $P_1(G)$ denotes the set of all finite subsets of G containing the identity 1 of G. In particular, the Birget-Rhodes expansion $\tilde{G}^{\mathcal{R}}$ of a group G is an F-inverse monoid whose maximum group image is isomorphic to the group G. This also leads to a new approach to the Burnside problem [2]. Recently, Lawson [11] proves that the Birget-Rhodes expansion $\tilde{G}^{\mathcal{R}}$ of a group G is isomorphic to the Exel's semigoup

* [illegible]

$S(G)$ [7] constructed by generators and relations. In [6], the author introduces an inverse monoid $\tilde{G}_c^{\mathcal{F}}$ containing the Birget-Rhodes expansion $\tilde{G}^{\mathcal{F}}$ of a topological group G obtained by replacing "finite" with "compact" of $P_1(G)$ and describe some algebraic structures of the monoid $\tilde{G}_c^{\mathcal{F}}$ and give a topology on it so that the Birget-Rhodes Expansion $\tilde{G}^{\mathcal{F}}$ of a compact group G is dense in $\tilde{G}_c^{\mathcal{F}}$ and its maximum group image is topologically isomorphic to the group G. Furthermore, it was shown that Green's relations on $\tilde{G}^{\mathcal{F}}$ are dense in those of $\tilde{G}_c^{\mathcal{F}}$.

In this paper, we investigate some structures of the compact F-inverse monoid $\tilde{G}_c^{\mathcal{F}}$.

Throughout this paper, we shall use basic results from (inverse) semigroup theory and topological semigroup theory; see [4], [5], [8], and [10].

2. Birget-Rhodes Expansions of Topological Groups

Let G be a topological group and let $C(G)$ be the set of all non-empty compact subsets of G. Then $C(G)$ with the set product multiplication,

$$(A, B) \to AB := \{ab : a \in A, b \in B\},$$

is a topological semigroup under the Vietoris topology [3]. And also $C(G)$ with the set union multiplication,

$$(A, B) \to m(A, B) := A \cup B,$$

is a semilattice.

For each $g \in G$, the map $\alpha_g : C(G) \to C(G)$ defined by

$$\alpha_g(A) = gA := \{g\}A$$

is an endomorphism of the semilattice $(C(G), m)$ since

$$\alpha_g(m(A, B)) = m(\alpha_g(A), \alpha_g(B)).$$

It is easy to check that the semidirect product $(C(G), m) \rtimes_\lambda G$ of the semilattice $C(G)$ and G is an inverse semigroup, where

$$\lambda : G \to \mathrm{End}(C(G)),\ \lambda(g) = \alpha_g.$$

Let

$$\tilde{G}_c^{\mathcal{F}} := \{(A, g) \in C_1(G) \times G : g \in A\}$$

where $C_1(G)$ denotes the set of all compact subsets of G which contains the identity element 1 of G. Then we can easily show that $\tilde{G}_c^{\mathscr{P}}$ is an inverse submonoid of the inverse monoid $(C(G), m) \times_\phi G$.

Let G be a locally compact group. Define a map

$$\alpha : G \times C(G) \to C(G), \ \alpha(g, A) = gA$$

Then α is continuous action on $C(G)$ from the fact that $\alpha = p \circ (i \times 1_{C(G)})$, where p is the set product multiplication on $C(G)$ and $i : G \to C(G)$, $g \mapsto \{g\}$, is a homeomorphic embedding of G into $C(G)$.

For each $g \in G$, the map $\alpha_g : C(G) \to C(G)$ defined by $\alpha_g(A) = gA$ is a continuous endomorphism of the topological semilattice $(C(G), \cup)$.

Throughout, if G is a locally compact group, we denote by $\tilde{G}_c^{\mathscr{P}}$ the topological inverse submonoid of the topological inverse semigroup $C(G) \times_\lambda G$. In particular, if G is a compact group, then $\tilde{G}_c^{\mathscr{P}}$ is a compact F-inverse monoid (see, [6]).

An inverse semigroup is E^*-*unitary* (also termed '0-E-*unitary*') if every element above a non-zero idempotent is also an idempotent. Let S and T be inverse semigroups with zero, say 0. A function $\theta : S \to T$ is said to be a 0-*morphism* if $\theta(ab) = \theta(a)\theta(b)$ for all $ab \neq 0$; It is called 0-*restricted if* $\theta(0) = 0$; and it is said to be *idempotent pure* if a is idempotent whenever $\theta(a)$ is idempotent.

An inverse semigroup S with zero is said to be *strongly* E^*-*unitary* [9] if there is an idempotent pure, 0-restricted, 0-morphism θ from S to a group with zero adjoined.

The importance of (strongly) E^*-unitary semigroups within inverse semigroup theory is described in detail in [9] and [10].

Lemma 2.1. *Let G be a compact group and let $M = \{(G, g) : g \in G\}$. Then M is a minimal ideal of $\tilde{G}_c^{\mathscr{P}}$.*

Proof. We can easily show that M is an ideal of $\tilde{G}_c^{\mathscr{P}}$ and it is also a group. Thus we have M is the minimal ideal of $\tilde{G}_c^{\mathscr{P}}$. □

Theorem 2.2. *If G is a compact group, then the Rees quotient $\tilde{G}_c^{\mathscr{P}}/M$ of $\tilde{G}_c^{\mathscr{P}}$ mod the minimal ideal M of $\tilde{G}_c^{\mathscr{P}}$ is a strongly E^*-unitary inverse semigroup.*

Proof. We note that $\tilde{G}_c^{\mathscr{P}}$ is an E-unitary inverse monoid whose maximal group homomorphic image is G. By Theorem 4 in [9], the inverse semigroup $\tilde{G}_c^{\mathscr{P}}/M$ is strongly E^*-unitary associated with G. □

If X and Y are disjoint spaces, the we give $X \sqcup Y$ the topology which is coherent with that of X and Y, i.e., a subset U of $X \sqcup Y$ is open if and only if $U \cap X$ is open in X and $U \cap Y$ is open in Y. Notice that if X and Y are both (locally) compact, then $X \sqcup Y$ is (locally) compact.

Let S and T be disjoint topological semigroups and let $\phi : S \to T$ be a continuous homomorphism, then define continuous multiplication on $S \sqcup T$ by

$$(x, y) \mapsto \begin{cases} m_S(x, y) & \text{if } x, y \in S; \\ m_T(x, y) & \text{if } x, y \in T; \\ m_T(\phi(x), y) & \text{if } x \in S \text{ and } y \in T; \\ m_T(x, \phi(y)) & \text{if } x \in T \text{ and } y \in S, \end{cases}$$

where m_S and m_T are the multiplication on S and T, respectively.

We denote $S \sqcup T$ with this multiplication by $S \sqcup_\phi T$. Observe that $S \sqcup_\phi T$ is a topological semigroup with this multiplication under the topology which is coherent with that of S and T. Let I be a closed ideal of S and let R be the congruence on $S \sqcup_\phi T$ generated by $\{(x, \phi(x)) : x \in I\}$. If S and T are locally compact σ-compact semigroups, then $(S \sqcup_\phi T)/R$ is a topological semigroup [4] which is called the *adjunction semigroup of S and T relative to ϕ and I*, and denoted by $S \bigcup_{\phi, I} T$.

Observe that the restriction on T of the natural map

$$\pi : S \sqcup_\phi T \to (S \sqcup_\phi T)/R = S \bigcup_{\phi, I} T$$

is a topological embedding of T into $S \bigcup_{\phi, I} T$.

Lemma 2.3. *Let S and T be disjoint compact inverse monoids, $\phi : S \to T$ be an identity preserving continuous homomorphism. Then $S \bigcup_{\phi, I} T$ is a compact inverse monoid.*

Proof. Since $S \sqcup_\phi T$ is a compact inverse semigroup, the continuous homomorphic image $S \bigcup_{\phi, I} T$ of π is also a compact inverse semigroup. And since the map ϕ preserves identity, the identity of S is exactly the identity of $S \bigcup_{\phi, I} T$.

Theorem 2.4. *Let G be a compact group, H be a compact group, and let ϕ be a topological embedding from the minimal ideal M of $\tilde{G}^*$ to H. Define $\psi : \tilde{G}^* \to H$*

by $\phi: x \mapsto \lambda_{(G,1)}$. Then the adjunction semigroup $\tilde{G}_c^{\mathscr{R}} \bigcup_{\phi, M} H$ is a compact F-inverse monoid whose maximal group image is isomorphic to H.

Proof. Observe that $M = \{(G, g) : g \in G\}$ which is a group with identity $(G, 1)$. By the definition of ϕ, we have that ϕ is continuous homomorphism which preserves identity. Hence the adjunction semigroup $\tilde{G}_c^{\mathscr{R}} \bigcup_{\phi, M} H$ is a compact inverse semigroup with the identity $(\{1\}, 1)$ by Lemma 2.3. Furthermore, it has a minimal ideal which is topologically isomorphic to H. In particular, if ϕ is a topological embedding, then the minimal ideal of $\tilde{G}_c^{\mathscr{R}} \bigcup_{\phi, M} H$ is the set of the form

$$\{[x] : x \in M\} \cup (H \setminus \phi(M)),$$

where $[x]$ is the R-congruence class of $x \in M$, in fact, $[x] = \{x, \phi(x) : x \in M\}$.

Let σ be a minimum group congruence of $\tilde{G}_c^{\mathscr{R}} \bigcup_{\phi, M} H$. Then the σ-class $\sigma_{[x]}$ containing $[x]$ is of the form

$$\sigma_{[x]} = \begin{cases} \{(A, g) : g \in G\} & \text{if } x = (G, g) \in M \\ \{x\} & \text{if } x \in H \setminus \phi(M). \end{cases} \tag{2.1}$$

Thus the σ-class $\sigma_{[x]}$ containing $[x]$ has its maximal element of the form: $(\{1, g\}, g)$ in the first case of (2.1) and x in the second case of (2.1).

It follows that $\tilde{G}_c^{\mathscr{R}} \bigcup_{\phi, M} H$ is F-inverse monoid whose maximal group image is isomorphic to H. □

A *partially ordered space (pospace)* is a pair $(X, \leq)$ such that X is a Hausdorff space and $\leq$ is a closed partial order on X, i.e., $\leq$ is a closed subset of $X \times X$. Observe that if X is a compact pospace, then $\downarrow x := \{b \in X : b \leq x\}$ is closed for each $x \in X$.

Lemma 2.5. *Let S be a compact F-inverse semigroup with a minimum group congruence σ. Then we have*

(i) *Let $\leq$ be the natural partial order on S. Then if $s \leq t$, then $s \sigma t$.*

(ii) *$(S, \leq)$ is a partially ordered space (pospace).*

(iii) *Every σ-class of S has a unique minimal element.*

(iv) *Two elements are σ-related if and only if they are bounded above by the same maximal element.*

Proof. (i) Straightforward.

(ii) Since S is Hausdorff, it suffices to show that the natural partial order $\leq$ is closed. Let $\{(x_\alpha, y_\alpha)\}$ be a net in $\leq$ which converges to (x, y). Then $\{x_\alpha\} \to x$ and $\{y_\alpha\} \to y$. Since $(x_\alpha, y_\alpha) \in \leq$ for each α, there exists $e_\alpha \in E(S)$ such that $x_\alpha = e_\alpha y_\alpha$ for each α. Notice that $\{e_\alpha\}$ cluster to a point $e \in E(S)$ from the compactness of $E(S)$. By considering subnet, we can assume that $\{e_\alpha\} \to e$. By the continuity of multiplication yields that $x = ey$. We conclude that $(x, y) \in \leq$ and $\leq$ is closed.

(iii) Let H be a σ-class of S and let $x, y \in H$. Then there exists $e \in E(S)$ such that $ex = ey$. Let $s = ex = ey$. Then $s \in H$ and $s \leq x$, $s \leq y$. Hence H is down-directed and hence a net. Since S is compact pospace, by B.4 Theorem in [5], $\inf H$ exists and $H \to \inf H$. To show that $\inf H \in H$, let m be the greatest element of H. By (i), $H = \downarrow m$ and hence H is closed since S is a compact pospace. Thus we have $\inf H \in H$ and hence $\inf H$ is a unique minimal element of H.

(iv) Suppose that $s \sigma t$ for $s, t \in S$. Then s and t are contained in some σ-class H of S. Since S is F-inverse, s and t are bounded above by the greatest element of H. Conversely, if s, t are bounded above a maximal element m, then $s = em, t = fm$ for some $e, f \in E(S)$. Let $w = ef$. Then $w \in E(S)$ and $ws = wt$. It follows that s, t are σ-related. □

Lemma 2.6. *Let G be a group, S be a inverse semigroup with a minimum group congruence σ, and let $\varphi : S \to G$ be a surmorphism with* $\ker \varphi = \sigma$. *Then every σ-class of S is of the form $\varphi^{-1}(g)$ for some $g \in G$.*

Proof. Let H be a σ-class of S containing s. Then $t \in H$ if and only if $(t, s) \in \ker \varphi$ if and only if $\varphi(t) = \varphi(s)$ if and only if $t \in \varphi^{-1}(\varphi(s))$. It follows that any σ-class of S is of the form $\varphi^{-1}(g)$ for $g \in G$. □

Define a map η by

$$\eta : \tilde{G}_c^{\#} \to G, \quad (A, g) \mapsto g.$$

Then η is semigroup homomorphism and the kernel of it is equal to the minimum group congruence of $\tilde{G}_c^{\#}$.

Theorem 2.7. *For any compact group G, the pair $(\tilde{G}_c^{\#}, \eta)$ has the property that, whenever S is a compact F-inverse semigroup with a minimum group congruence σ, φ is a surmorphism of S onto G with* $\ker \varphi = \sigma$, *and the set of all minimal elements* *of S forms an ideal of S, then there exists a homomorphism ξ of $\tilde{G}_c^{\mathscr{A}}$ into S*

$$\begin{array}{ccc} \tilde{G}_c^{\mathscr{A}} & \xrightarrow{\eta} & G \\ {\scriptstyle \xi}\downarrow & \nearrow {\scriptstyle \varphi} & \\ S & & \end{array}$$

mapping the greatest element of each σ-class to the greatest element of a σ-class such that $\varphi \circ \xi = \eta$.

Proof. By Lemma 2.6, every σ-class of S is of the form $\varphi^{-1}(g)$ for some $g \in G$. Let m_g and l_g be the unique maximal and minimal elements of σ-class $\varphi^{-1}(g)$ for each $g \in G$, respectively. Define a map $\xi : \tilde{G}_c^{\mathscr{A}} \to S$ by

$$\xi(A, g) = \begin{cases} m_{g_1} m_{g_1^{-1} g_2} \cdots m_{g_k^{-1} g} & \text{if } A = \{1, g_1, g_2, \ldots, g_k, g\} \in P_1(G) \\ l_g & \text{otherwise} \end{cases}$$

Then ξ is well-defined by Lemma 2.5 and Lemma 2.6. Now we shall show that ξ is a homomorphism. If $(A, g), (B, h) \in \tilde{G}^{\mathscr{A}}$ with $A = \{1, g_1, g_2, \ldots, g_k, g\}$ and $B = \{1, h_1, h_2, \ldots, h_m, h\}$, then

$$\begin{aligned} \xi(A, g)\xi(B, h) &= m_{g_1} m_{g_1^{-1} g_2} \cdots m_{g_k^{-1} g} m_{h_1} m_{h_1^{-1} h_2} \cdots m_{h_m^{-1} h} \\ &= m_{g_1} m_{g_1^{-1} g_2} \cdots m_{g_k^{-1} g} m_{g^{-1}(gh_1)} m_{(gh_1)^{-1}(gh_2)} \cdots m_{(gh_m)^{-1}(gh)} \\ &= \xi((A, g)(B, h)). \end{aligned}$$

In the other cases, we can easily show that ξ is a homomorphism using the fact that the set of all minimal elements of S forms an ideal of S. Clearly, ξ maps the greatest element of each σ-class to the greatest element of a σ-class such that $\varphi \circ \xi = \eta$. □

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