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RIEMANNIAN MANIFOLD $\hat{M}$ IN SPACE $(\mathbb{X}, I)$ UNDER INVERSION

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0. Introduction.

It is our purpose in this paper to define and study some properties of manifold $\hat{M}$ under inversion, especially structure of inversion manifold $\hat{M}$.

As in the case of C^∞-manifold, which are Riemannian manifold, we begin with Euclidean space E^n under inversion.

Transformation $I : E^n - (0, \cdots, 0) \to E^n$ defined by $I(\mathbb{X}) = R^2\mathbb{X}^{-1}$ made up Inversion space $(\mathbb{X}, I)$.

In theorem.1.3, function $\hat{f}$ is continuousely differentiable on $(\mathbb{X}, I)$, and so theorem.2.3 show the inversion manifold $\hat{M}$ is Riemannian manifold.

1. Inversion space.

Let E^n be on Euclidean space of dimension n, and let $S^n_R(0) = \{(x^1, x^2, \cdots, x^n) \in R^{n+1} | \sum_{i=1}^{n+1}(x^i)^2 = R^2\}$ is a n-sphere in E^n.

Definition.1.1. Two points $\mathbb{P}, \hat{\mathbb{P}} \in E^n$ are said to be inverse with respect to a given sphere $S^n_R(0)$, if

$$O\mathbb{P} \cdot O\hat{\mathbb{P}} = R^2$$

where $\mathbb{P}, \hat{\mathbb{P}}$ are on the same side of O and $O, \mathbb{P}, \hat{\mathbb{P}}$ are collinear.

A sphere $S^n_R(0)$ is called the sphere of inversion and the transformation which sends a point $\mathbb{P}$ into $\hat{\mathbb{P}}$ is called an inversion.

From now on, we take the center O as an origin of the coordinate system in E^n and denote the distance from O to inside point $\mathbb{X}$ of $S^n_R(0)$ by $|\mathbb{X}|$.

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Then the inversion mapping defined by

$$I : E^n - (0, \cdots, 0) \to E^n$$

for all $\mathbb{X} \in E^n - (0, \cdots, 0)$, such that

$$I(\mathbb{X}) = \frac{R^2}{< \mathbb{X}, \mathbb{X} >}\mathbb{X}$$

That is, the inversion $I(\mathbb{X})$ is the vector of length $R^2|\mathbb{X}|^{-1}$ on the ray of $\mathbb{X}$.

Definition.1.2. Inversion space $(\mathbb{X}, I)$ is a collection of all space which for every point $\mathbb{X}$ inside of $S^n_R(0)$ there is a space whose image consists of a neighborhood of $\mathbb{X}$ under inversion and outside of $S^n_R(0)$.

E^n-space inside of $S^n_R(0)$ is a subspace of E^n denote by $(\mathbb{X}, \mathbb{S})$.

Let M be a C^∞-Riemannian manifold in $(\mathbb{X}, S)$ not through O and let

$$\hat{M} = I(M) = \{I(\mathbb{X}) \in (\mathbb{X}, I) | I(\mathbb{X}) = \frac{R^2}{|\mathbb{X}|^2}\mathbb{X}\}.$$

Then $\hat{M}$ be a n-dimensional metric space.

Let $\hat{U} = I(U)$ be an open set of $\hat{M}$ about open set U of M.

For every $\mathbb{X} \in U$, inversion $I|_U : U \to \hat{U}$ is one to one and continuous, there exists I^{-1} such that $I^{-1} : I(U) \to U$ is continuous in $\hat{M}$.

Hence for every open $U \subset M$, $I : M \to \hat{M}$ is homeomorphism.

Let an open set $U \subset R^n$ for function f in $(\mathbb{X}, S)$ if $f : U \to R$ define $f(\mathbb{X}) = (x^1, x^2, \cdots, x^n)$ denotes its value $\mathbb{X} = (x^1, x^2, \cdots, x^n) \in U$.

Then there is a function $\hat{f} : \hat{U} \to R$ in $(\mathbb{X}, I)$ and we have the followings.

Theorem 1.3. *Let f is continuously differentiable on U. Then $\hat{f}$ is continuously differentiable on $\hat{U}$.*

Proof. Let a point $\hat{\mathbb{X}} \in (\mathbb{X}, I)$ corresponding to $\mathbb{X} \in (\mathbb{X}, S)$. For every open set U in $(\mathbb{X}, S)$, the mapping $\hat{f} \circ I(U) \to R$ given by, for each $\mathbb{X} \in U$,

$$\hat{f}(I(\mathbb{X})) = \hat{f}(\hat{\mathbb{X}}) = \frac{R^2}{\sum_{i=1}^n (x^i)^2} f(x^1, x^2, \cdots, x^n).$$

The limit value of j-partial derivative of $\hat{f}$ as follows :

$$\begin{aligned}
D_j\hat{f}(\mathbb{X}) &= \lim_{h^j\to 0}\frac{\hat{f}(\hat{x^1},\cdots,\hat{x^j}+h^j,\cdots,\hat{x^n})-\hat{f}(\hat{x^1},\cdots,\hat{x^j},\cdots,\hat{x^n})}{h^j}\\
&= \lim_{h^j\to o}\frac{R^2}{(x^1)^2+\cdots+(x^j+h^j)^2+\cdots+(x^n)^2}\\
&\quad\cdot\frac{f(x^1,\cdots,x^j+h^j,\cdots,x^n)}{h^j}\\
&\quad-\lim_{h^j\to 0}\frac{R^2}{\sum_{i=1}^n(x^i)^2}\cdot\frac{f(x^1,\cdots,x^j,\cdots,x^n)}{h^j}\\
&= \frac{R^2}{\sum_{i=1}^n(x^i)^2}D_jf(\mathbb{X})
\end{aligned}$$

where D_jf is j-partial derivative of f. Thus

$$\begin{aligned}
D\hat{f}(\hat{\mathbb{X}}) &= D_1\hat{f}(\hat{\mathbb{X}})+\cdots+D_j\hat{f}(\hat{\mathbb{X}})+\cdots+D_n\hat{f}(\hat{\mathbb{X}})\\
&= \frac{R^2}{\sum_{i=1}^n(x^i)^2}D_1f(\mathbb{X})+\cdots+\frac{R^2}{\sum_{i=1}^n(x^i)^2}D_jf(\mathbb{X})\\
&\quad+\cdots+\frac{R^2}{\sum_{i=1}^n(x^i)^2}D_nf(\mathbb{X})\\
&= \frac{R^2}{\sum_{i=1}^n(x^i)^2}Df(\mathbb{X}).
\end{aligned}$$

Thus $\hat{f}$ is differentiable.

2. Inversion Manifold.

Let M be a metric space and let the coordinate chart (U,ϕ) of one-to-one continuous function $\phi : U \to R^n$ is proper. For $\hat{\mathbb{X}} \in \hat{M}$, take a coordinate chart $(\hat{U},\hat{\phi})$ about $I(\mathbb{X}) = \hat{\mathbb{X}}$ and atlas $\hat{\mathfrak{a}}$ of $\hat{M}$. Since I is homeomorphism, there exists a inverse $\hat{\phi}^{-1} : \hat{\phi}(I(U)) \to \hat{U}$ is continuous. Thus $\hat{U},\hat{\phi})$ is proper. Let $(\hat{U},\hat{\phi}),(\hat{V},\hat{\psi}) \in \hat{\mathfrak{a}}$ with respect to coordinate chart $(U,\phi),(V,\psi)$ of M with $\hat{U}\cap\hat{V} = \emptyset$, respectively.

If M be a C^∞ manifold, then $\psi\circ\phi^{-1} : \phi(U\cap V) \to \psi(U\cap V)$ is a C^∞ diffeomorphism. By means of theorem 1.3,

$$\hat{\psi}\circ\hat{\phi}^{-1} : \hat{\phi}(\hat{U}\cap\hat{V}) \to \hat{\psi}(\hat{U}\cap\hat{V})$$

is also C^∞ deffeomorphism.

Thus we have the followings.

Lemma.2.1. *Manifold $\hat{M}$ in $(\mathbb{X}, I)$ is an n-dimensional c^∞-manifold.*

Let M be a manifold in $(\mathbb{X}, S)$ and $\mathbb{X}_p = \sum_{i=1}^{n} \alpha^i \mathbb{E}_{ip}$ be a vector field at $p \in M$. If $\mathbb{E}_{ip}$ is natural basis, then the vector field $\mathbb{X}_{I(p)}$ of $(\mathbb{X}, I)$ at $I(p)$ is defined by

$$\mathbb{X}_{I(p)} = \frac{R^2}{\sum_{i=1}^{n}(\alpha^i)^2}\mathbb{X}_p$$

Note : The direction of component vectors of $\mathbb{X}_{I(p)}$ is at least one at least one opposite direction of the component vectors of $\mathbb{X}_p$.

Definition.2.2. The tangent space to $\hat{M}$ at $I(p)$ is the set of all tangent vector to $\hat{M}$ at $I(p)$ denote by $T_{I(p)}(\hat{M})$

Let $\hat{\Phi} : T(\hat{M}) \times T(\hat{M}) \to R$ be an inner product on a manifold $\hat{M}$, that is, for each $I(p) \in \hat{M}$, the map $\hat{\Phi}_{I(p)} : T_{I(p)}(\hat{M}) \times T_{I(p)}(\hat{M}) \to R$ is a C^∞ bilinear form satisfying

i) $\hat{\Phi}_{I(p)}(\mathbb{X}_{I(p)}, \mathbb{Y}_{I(p)}) = \hat{\Phi}_{I(p)}(\mathbb{Y}_{I(p)}, \mathbb{X}_{I(p)})$ (symmetric)

ii) $\hat{\Phi}_{I(p)}(\mathbb{X}_{I(p)}, \mathbb{Y}_{I(p)}) \geq o$ and $\hat{\Phi}_{I(p)}(\mathbb{X}_{I(p)}, \mathbb{Y}_{I(p)}) = 0 \Leftrightarrow \mathbb{X}_{I(p)} = 0$ (positive definite).

Let $(\hat{U}, \hat{\phi})$ be a local coordinate system of $\hat{M}$, then the basis of $T_{I(p)}(\hat{M})$ is $(\mathbb{E}_{1I(p)}, \cdots, (\mathbb{E}_{nI(p)})$. That is

$$(\mathbb{E}_{iI(p)} = I(\mathbb{E}_{ip}) = \frac{R^2}{|\mathbb{E}_{ip}|^2}\mathbb{E}_{ip} = \frac{R^2}{n}\mathbb{E}_{ip} = I \circ \phi^{-1}(\frac{\partial}{\partial x^i})$$

where $p \in U, \phi(p) = (x^1, \cdots, x^n) \in R^n$

Theorem 2.3. *The manifold $\hat{M}$ in $(\mathbb{X}, I)$ is Riemannian manifold.*

Proof. Let M be a manifold with Riemannian metric tensor g and let (U, ϕ) be a local coordinate system of M. For a basis of $T_p(M)$ of $(\mathbb{E}_{1p}, \cdots, \mathbb{E}_{np})$,

$$\mathbb{E}_{ip} = \phi^{-1}(\frac{\partial}{\partial x^i})_{i=1,\cdots,n}, \text{ for } p \in U, \phi(U) = (x^1, \cdots, x^n) \in R^n$$

If the map $\Phi : T(M) \times T(M) \to R$ be an inner product on M defined by

$$\Phi_p(\mathbb{E}_{ip}, \mathbb{E}_{jp}) = g_{ij}(p)$$

Then the map $\hat{\Phi} : T(\hat{M}) \times T(\hat{M}) \to R$ is given by

$$\begin{aligned}\hat{\Phi}_{I(p)}(\mathbb{E}_{iI(p)}, \mathbb{E}_{jI(p)}) &= \hat{\Phi}_{I(p)}(I(\mathbb{E}_{ip}), I(\mathbb{E}_{jp})) \\ &= (\frac{R^2}{n})^2 \Phi_p(\mathbb{E}_{ip}, \mathbb{E}_{jp}) \\ &= (\frac{R^2}{n})^2 g_{ij}(p).\end{aligned}$$

We put $\hat{\Phi}_{I(p)}(\mathbb{E}_{iI(p)}, \mathbb{E}_{jI(p)}) = \hat{g}_{ij}(I(p))$ is metric on $\hat{M}$.

$$\hat{g}_{ij}(I(p)) = (\frac{R^2}{n})^2 g_{ij}(p).$$

Let $\hat{g}$ be a determinant of $\hat{\Phi}_{I(p)}(\mathbb{E}_{iI(p)}, \mathbb{E}_{jI(p)})$, then

$$\begin{aligned}\hat{g} &= det(\hat{g}_{ij}(I(p))) \\ &= det\begin{pmatrix} \hat{g}_{11}(I(p)) & \cdots & \hat{g}_{1n}(I(p)) \\ \vdots & & \vdots \\ \hat{g}_{n1}(I(p)) & \cdots & \hat{g}_{nn}(I(p)) \end{pmatrix} \\ &= det\begin{pmatrix} (\frac{R^2}{n})^2 g_{11}(p) & \cdots & (\frac{R^2}{n})^2 g_{1n}(p) \\ \vdots & & \vdots \\ (\frac{R^2}{n})^2 g_{n1}(p) & \cdots & (\frac{R^2}{n})^2 g_{nn}(p) \end{pmatrix} \\ &= (\frac{R^2}{n})^{2n} g,\end{aligned}$$

where g is determinent of metric $g_{ij}(p)$ on M.

For $\mathbb{X}_{I(p)} = \sum \alpha^i \mathbb{E}_{iI(p)}, \mathbb{Y}_{I(p)} = \sum \beta^j \mathbb{E}_{jI(p)}$ vectors in $T_{I(p)}(\hat{M})$, we have

$$\begin{aligned}\hat{\Phi}_{I(p)}(\mathbb{X}_{I(p)}, \mathbb{Y}_{I(p)}) &= \hat{\Phi}_{I(p)}(\sum \alpha^i \mathbb{E}_{iI(p)}, \sum \beta^j \mathbb{E}_{jI(p)}) \\ &= (\frac{R^2}{n})^2 \sum_{i,j=1}^{n} \alpha^i \beta^j g_{ij}(p)\end{aligned}$$

manifold $\hat{M}$ have the Riemannian metric g.

References

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<국문초록>

반전에 의한 공간 $(\mathbb{X}, I)$에 있는 리만 다양체 $\hat{M}$

현 진 오

반전구 $S^n_R(0)$의 내부공간 $(\mathbb{X}, S)$에 있는 다양체가 반전 $I : (\mathbb{X}, S) \to (\mathbb{X}, I)$에 의하여 재구성 되는 반전 다양체 $\hat{M}$의 성질을 구명하기 위하여 기본적으로 가장 필요한 $\hat{M}$을 정의하고 정리 1.3에서는 $(\mathbb{X}, I)$에서의 함수 $\hat{f}$가 연속이 되고 미분가능함을 보이고, 정리2.3에서는 $\hat{M}$도 리만 다양체가 됨을 보인다.