Two Dimensional Quantum Mechanics on the Noncommutative Space

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1 Introduction

There has been considerable progress in understanding the field theory [1] and the string theory on the noncommutative space[2]. Especially. the presence of magnetic field in the string theories yields to a noncommutative structure for the space time[3]. Basing on these results, there have been many attempts to reformulate Ouantum Mechanics (QM) on the noncommutative space[4,5]. In QM on the noncommutative space, Schrödinger equation is replaced by the $\star$-genvalue, while the wave functions become Wigner functions. The ordinary product is replaced by the associative noncommutative $\star$-product[6]. Using these replacements, the variations of energy eigenvalue and Aharonov-Bhom effect, etc, are newly discussed[7].

In the present work it is shown that the $\star$-genvalue prolem is equivalent to the Schrödinger problem in an appropriate transformation of variables. Instead of the above replacement in the two dimensional theory, selecting the commutating variables and then finding the representation for these variables, we can derive a various physical quantities as the usual QM. Especially the energy eigenvalues are easily calculated.

This paper is organized as follows. In section 2, the commutation relations are discussed and the transformations of variables are treated. Also the transformation of the wave function on the configuration space into that on the momentum space is descibed. In section 3, the eigenvalues and the wave functions of the free particle on the noncommutative space are calculated.

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2 Representation

Let us consider position opreators and momenetum operators in the two dimensional noncommutative space. They are satisfied the following commuation relations

$$
\begin{aligned}
&[\hat{q}_1, \hat{q}_2] = i\theta, \\
&[\hat{q}_j, \hat{p}_k] = i\hbar\delta_{jk}, \ j,k = 1,2 \\
&[\hat{p}_1, \hat{p}_2] = iB
\end{aligned} \tag{1}
$$

where θ, B are dimensionful parameters. $\hat{q}_1$ and $\hat{p}_2$, $\hat{q}_2$ and $\hat{p}_1$ commute each other respectively. Thus we can redefine these operators as following

$$
\begin{aligned}
&\hat{q}_1 \rightarrow \hat{z}_1 \, , \ \hat{p}_2 \rightarrow \hat{z}_2 \, , \\
&\hat{p}_1 \rightarrow \hat{w}_1 \, , \ -\hat{q}_2 \rightarrow \hat{w}_2 \, .
\end{aligned} \tag{2}
$$

Using these redefined operators, the commutation relations (1) can be rewritten as follows.

$$
\begin{aligned}
&[\hat{z}_j \, , \hat{z}_k] = 0 \, , \\
&[\hat{w}_j \, , \hat{w}_k] = 0 \, , \\
&[\hat{z}_j \, , \hat{w}_k] = i\tilde{M}_{jk} \, , \ j,k = 1,2
\end{aligned} \tag{3}
$$

where

$$
(M_{jk}) = \begin{pmatrix} \hbar & -B \\ -\theta & \hbar \end{pmatrix} , \tag{4}
$$

and $\tilde{M}$ is transpose of matrix M. Then QM on the noncommutative space described by relations (1) can be replaced with the usual QM basing on the relations (3) and we can treat $\hat{z}_j$ as position operators and $\hat{w}_k$ as conjugate momentum operators.

Before writing down an equation, one has to search for a suitable representations for the operators satisfying the commutation relations (3) (z-representation)

$$
\hat{z}_j \rightarrow z_j \, , \ \hat{w}_j \rightarrow -iM_{jk}\partial_k \, , \ j,k = 1,2 \tag{5}
$$

where $\partial_k = \partial/\partial z_k$. The Hamiltonian and the Schrödinger equation with respect to the new variables become

$$\tilde{H}(\hat{p},\hat{q}) \to H(\hat{w},\hat{z}) = H\left(-iM_{jk}\partial_k, z_j\right), \tag{6}$$

$$H(\hat{w},\hat{z})\psi(z_1,z_2) = E\psi(z_1,z_2) \tag{7}$$

where $\psi(z_1,z_2)$ and E are wave function and energy eigenvalue respectively. So, instead of solving the $\star$-genvalue problem, one can try to solve the Schrödinger equation to find the wave functions and the energy eigenvalues.

In order to find the wave functions given by functions of w_1, w_2, we must transfom functions of z_1, z_2 into functions of w_1, w_2. Because $\hat{z}_1, \hat{z}_2$ do not commute with $\hat{w}_2, \hat{w}_1$ respectively, the ordinary Fourier transfomation could not be used. Let's use the following Ansatz

$$\tilde{\psi}(w_1,w_2) = \frac{1}{2\pi\sqrt{D}}\int dz_1 dz_2\, \psi(z_1,z_2) e^{-iz_j M^{jk} w_k} \tag{8}$$

where D is the determinant of M_{jk}, M^{jk} inverse matrix of M_{jk}. The inverse transformation is

$$\psi(z_1,z_2) = \frac{1}{2\pi\sqrt{D}}\int dw_1 dw_2\, \tilde{\psi}(w_1,w_2) e^{iz_j M^{jk} w_k}, \tag{9}$$

and the Dirac delta function becomes

$$\delta(\vec{w}-\vec{w}') = \frac{1}{\left(2\pi\sqrt{D}\right)^2}\int dz_1 dz_2\, e^{iz_j M^{jk}(w_k - w'_k)}. \tag{10}$$

Using these transformations, we can get the representations for $\hat{z}_j$ and $\hat{w}_j$ on the w-space (w-representation):

$$\hat{z}_j \to i\tilde{M}_{jk}\frac{\partial}{\partial w_k}, \; \hat{w}_j \to w_j, \; j,k = 1,2. \tag{11}$$

From these representations and Eq. (9), the Hamiltonian on the w-space is represented as follows

$$H(\hat{w},\hat{z}) = H\left(w_j, i\tilde{M}_{jk}\partial/\partial w_k\right), \tag{12}$$

and the Schrödinger equation becomes

$$H\left(w_j, i\tilde{M}_{jk}\partial/\partial w_k\right)\tilde{\psi}(w_1,w_2) = E\tilde{\psi}(w_1,w_2)\,. \tag{13}$$

Thus we can get the wave functions on the w-space from the wave function on the z-space using the transformations (8) or from the Schrödinger equation (13) directly.

3 Free particle on noncommutative space

Hamiltonian of the free particle on the noncommutative space is given by

$$\tilde{H} = \frac{1}{2}\hat{p}^2 = \frac{1}{2}\left(\hat{p}_1^2 + \hat{p}_2^2\right) \tag{14}$$

where parameters are absorbed in the variables. Using the redefined vairables (2) and their representations (5), this Hamiltonian is rewrtten as follows

$$\begin{aligned} H &= \frac{1}{2}\left(\hat{w}_1^2 + \hat{z}_2^2\right) \\ &= \frac{1}{2}\left[-\left(\hbar\partial_1 - B\partial_2\right)^2 + z_2^2\right]. \end{aligned} \tag{15}$$

From the Schrödinger equation the following equation is got

$$\left[\left(\hbar\partial_1 - B\partial_2\right)^2 + \left(2E - z_2^2\right)\right]\psi(z_1,z_2) = 0\,. \tag{16}$$

Because z_1 is cyclic variable, the solution to the above equation has a form

$$\psi(z_1,z_2) = e^{ikz_1}\psi_2(z_2)\,, \tag{17}$$

where k is real. If k is complex, the imaginary part must vanish in order to satisfy the continuity property of the wave function. Then the equation (16) becomes

$$\left[\left(i\hbar k - B\partial_2\right)^2 + \left(2E - z_2^2\right)\right]\psi_2(z_2) = 0\,. \tag{18}$$

We may define

$$\psi_2(z_2) = e^{i\hbar k z_2/B} u(z_2), \tag{19}$$

$$z_2 = \sqrt{B}\xi, \tag{20}$$

then the equation (18) is changed to

$$u''(\xi) + (2E/B - \xi^2)u(\xi) = 0. \tag{21}$$

This is the differential equation for a quantum mechanical, simple harmonic oscillator. Thus its solution and the eigenvalues are

$$u(\xi) = Ae^{-\xi^2/2} H_n(\xi), \tag{22}$$

$$E = (n + \frac{1}{2})B, \ n = 0, 1, 2, \cdots \tag{23}$$

where A is an arbitrary constant and $H_n(\xi)$ the Hermite polynomials. The general solution is

$$\begin{aligned} \psi(z_1, z_2) &= \frac{A}{2\pi} \int dk\, e^{ik(z_1 + \hbar z_2/B)} e^{-\frac{z_2^2}{2B}} H_n(z_2/\sqrt{B}) \\ &= A\delta\,(z_1 + \hbar z_2/B)\, e^{-\frac{z_2^2}{2B}} H_n(z_2/\sqrt{B}). \end{aligned} \tag{24}$$

But, from the first line of the Hamiltonian (15) we know that the degree of freedom of the configuration space is one and the coordinate of that is z_2. Thus integrating equation (24) over z_1, the true wave function of the system is obtained as follows

$$\psi_n(z_2) = Ae^{-\frac{z_2^2}{2B}} H_n(z_2/\sqrt{B}). \tag{25}$$

These wave functions are also derived from the Schrödinger equation on the momentum space and the transformation relations (8) and have the eigenvalues (23) for the operator (15).

Let's consider a wave function on the momentum space (w-space). Using the transformation (8) and the solution (24), the wave function on the w-space becomes

$$\begin{aligned} \tilde{\psi}(w_1, w_2) &= \frac{A}{2\pi\sqrt{D}} \int dz_1 dz_2\, \delta\,(z_1 + \hbar z_2/B) \\ &\quad \times e^{-\frac{z_2^2}{2B}} H_n(z_2/\sqrt{B}) e^{-iz_j M^{jk} w_k} \\ &= A' e^{-\frac{w_1^2}{2B}} H_n(w_1/\sqrt{B}), \end{aligned} \tag{26}$$

is independent of w_2. where A' is an arbitrary constant. Because this wave function is independent of w_2, we can denote as $\tilde{\psi}_n(w_1)$. Thus the wave function on the w-space is

$$\tilde{\psi}_n(w_1) = A' e^{-\frac{w_1^2}{2B}} H_n(w_1/\sqrt{B}) . \tag{27}$$

These wave functions and the eigenvalues are also obtained from the algebraic method. The eigenvalues and the ground state wave function are shown at the Appendix. The wave functions (25), (27) are those of the simple harmonic oscillators in the usual QM. From these and the eigenvalues (23) the free particle systems in the noncommutative QM are considered as the simple harmonic oscillators on the usual QM.

4 Simple harmonic oscillator

The Hamiltonian of the simple harmonic oscillator on the two dimensional noncommutative space is

$$\tilde{H} = \frac{1}{2}\left(\hat{p}^2 + \hat{q}^2\right) . \tag{28}$$

In terms of the transformed variables Eq. (2), the Hamiltonian is represented as

$$\begin{aligned} H &= \frac{1}{2}\left(\hat{w}^2 + \hat{z}^2\right) \\ &= \frac{1}{2}\left[-\left(\hbar^2 + \theta^2\right)\partial_1^2 - \left(\hbar^2 + B^2\right)\partial_2^2 \right. \\ &\left. +2\hbar\,(\theta + B)\,\partial_1\partial_2 + z_1^2 + z_2^2\right] . \end{aligned} \tag{29}$$

Consider special cases, i.e., $(B = -\theta)$-case and $(B = \theta)$-case. First, in the $(B = -\theta)$-case the cross term in Hamiltonian (29) vanishes. Setting $\hbar^2 + \theta^2 = \kappa^2$, the Hamiltonian becomes

$$H = \frac{1}{2}\left[-\kappa^2(\partial_1^2 + \partial_2^2) + z_1^2 + z_2^2\right] . \tag{30}$$

This Hamiltonian is same as that of the two dimensional simple harmonic oscillator with angular frequency κ in the usual QM. Thus the energy eigenvalues are

$$E_{nm} = \kappa\,(n+m+1)\,,\; n,m = 0,1,2,\cdots\,. \tag{31}$$

This is equal to result of Ref[6], where these energy eigenvalues are obtained by solving the $\star$-genvalue problem. But the method proposed here is easier than that of Ref[6].

The wave functions for this system is

$$\psi(z_1,z_2) = C\exp\left[-\frac{1}{2\kappa}\left(z_1^2+z_2^2\right)\right] H_n(z_1/\sqrt{\kappa})H_m(z_2/\sqrt{\kappa})\,. \tag{32}$$

The wave function on the momentum space is obtained from the transformation relation (8) as follows

$$\begin{aligned}\tilde{\psi}(w_1,w_2) \;\sim\; & \exp\left[-\frac{1}{2\kappa}(\hbar w_1+\theta w_2)^2 - \frac{1}{2\kappa}(\theta w_1-\hbar w_2)^2\right] \\ & \times H_n((\hbar w_1+\theta w_2)/\sqrt{\kappa})H_m((\theta w_1-\hbar w_2)/\sqrt{\kappa})\,. \end{aligned} \tag{33}$$

Using the transformation propreties of Hermite polynomials, these functions are represented

$$\tilde{\psi}(w_1,w_2) \sim \exp\left[-\frac{1}{2\kappa}\left(w_1^2+w_2^2\right)\right] H_n(w_1/\sqrt{\kappa})H_m(w_2/\sqrt{\kappa})\,. \tag{34}$$

These are same to the wave functions obtained from the Hamiltonian on the momentum space.

Secondly, consider the $(B=\theta)$-case. Hamiltonian (29) is

$$H = \frac{1}{2}\left[-\kappa^2(\partial_1^2+\partial_2^2) + 4\hbar\theta\partial_1\partial_2 + z_1^2 + z_2^2\right]. \tag{35}$$

In order to solve the eigenvalue problem, change the variables z_1, z_2 into

$$\begin{aligned} z_1 &= \frac{1}{\sqrt{2}}\,(x_1+x_2)\,, \\ z_2 &= \frac{1}{\sqrt{2}}\,(x_1-x_2)\,. \end{aligned} \tag{36}$$

Then Hamiltonian (35) becomes

$$H = \frac{1}{2}\left[-(\hbar-\theta)^2\frac{\partial^2}{\partial x_1^2} - (\hbar+\theta)^2\frac{\partial^2}{\partial x_2^2} + x_1^2 + x_2^2\right]. \tag{37}$$

This Hamiltonian describes the simple harmonic oscillator oscillating with angular frequency $\hbar - \theta$ in one direction and oscillating with angular frequency $\hbar + \theta$ in other direction. Thus the energy eigenvalues are

$$\begin{aligned} E_{nm} &= (\hbar-\theta)\left(n+\frac{1}{2}\right) + (\hbar+\theta)\left(m+\frac{1}{2}\right) \\ &= \hbar\,(n+m+1) + \theta(m-n),\ n,m = 0,1,2,\cdots. \end{aligned} \tag{38}$$

Here one can consider that the first term is corresponding to the energy eigenvalue of the two dimensional simple harmonic oscillator in usual QM ($\theta = 0$ case) and the second term is due to the noncommutativity of space.

5 Conclusion

In this paper, we considered Quantum Mechanics on the noncommutative space from the point of view that the $\star$-genvalue problem is equal to the usual Schrödinger problem through the appropriate transformation of variables.

Instead of the usual replacement in the two dimensional theory, selecting the commutating variables and then finding the representation for these variables, we can derive a various physical quantities as the usual QM. Especially the energy eigenvalues are easily calculated.

The free particle systems in the noncommutative QM are equivalent to the simple harmonic oscillators on the usual QM with the usual energy eigenvalues.

The simple harmonic oscillator on the two dimensional noncommutative space is transformed into the simple harmonic oscillator in usual QM. But the energy eigenvalues are constituted of those of the two dimensional simple harmonic oscillator in usual QM ($\theta = 0$ case) and of those the noncommutativity of space.

6 Appendix

The Hamiltonian (15) is similar to the usual Hamiltonian on the commutative space having the coordinate z_2, the momentum w_1. Thus following the usual procedure, we introduce the operators

$$a_{\mp} = \frac{1}{\sqrt{2B}} \left(\hat{z}_2 \pm i\hat{w}_1 \right) . \tag{39}$$

Using the commutation relation (3) these operators can be shown to satisfy

$$[a_- \, , a_+] = 1. \tag{40}$$

As usual the vacuum state $|0\rangle$ is defined to satisfy $a_- \, |0\rangle = 0$.Then the Hamitonian (15) becomes

$$H = \left(a_+ a_- + \frac{1}{2} \right) B \, . \tag{41}$$

This is Hamiltonian of the simple harmonic oscillator replacing the angular frequency with B. Its energy eigenvalues

$$E_n = \left(n + \frac{1}{2} \right) B \, , \; n = 0, 1, 2, \cdots \tag{42}$$

This result is same as the case of the simple harmonic oscillator in usual QM.

The ground state wave function is found by using the equations $a_- \, |0\rangle = 0$, and representations (5) and (39). Let

$$\psi_0(z_1, z_2) = \langle z_1, z_2 \, |0\rangle \, . \tag{43}$$

Then from $a_- \psi_0(z_1, z_2) = 0$,

$$(z_2 + \hbar\partial_1 - B\partial_2)\psi_0 = 0. \tag{44}$$

Its solution is

$$\psi_0(z_1, z_2) = A e^{ik\left(\frac{z_1}{\hbar} + \frac{z_2}{B}\right)} e^{-\frac{z_2^2}{2B}} \, , \tag{45}$$

and the true wave function is

$$\psi_0(z_2) = \frac{1}{2\pi}\int dk dz_1\, \psi_0(z_1, z_2) = Ae^{-\frac{z_2^2}{2B}}. \tag{46}$$

Thus we can treat the free particle on the two dimensional noncommutative space as the simple harmonic oscillator in usual QM on the commutative space.

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