On the Symmetric Riemannian Manifold**

Hyun Jin-Oh*

對稱 Riemann 多樣體에 관하여**

玄 進 五*

I. INTRODUCTION

Let M be a C^∞manifold, and let $\theta : R\times M \rightarrow M$ be a C^∞mapping satisfying the condition

1) $\theta(0,p)=p$ for every $p \in M$
2) $\theta_t \cdot \theta_s(p)=\theta_{t+s}(p)=\theta_s \cdot \theta_t(p)$ for every $s,t \in R$ and $p \in M$ where $\theta_t(p)=\theta(t,p)$

Then θ is called a *C^∞ action or a one parameter group* of M. For each one parameter group $\theta : R\times M \rightarrow M$ there exists a unique C^∞ vector field X, which is called the *infinitesimal generator* of θ, such that

$$X_p f = \lim_{\Delta t \to 0} \frac{1}{\Delta t} [f(\theta_{\Delta(t)}(p)) - f(p)]$$

for each $f \in C^\infty(p)$

In this case, for all $t \in R$ and $\theta_t : M \rightarrow M$ we have

$$\theta_{t_*}(x_p) = X_{\theta_t}(p)$$

where $\theta_{t_*} : T(M) \rightarrow T(M)$ is a map which commutes the following diagram;

$$\begin{array}{ccc} T(M) & \xrightarrow{\theta_{t_*}} & T(M) \\ \downarrow & & \downarrow \pi \\ M & \xrightarrow{\theta_t} & M \end{array}$$

Note that $\pi : T(M) \rightarrow M$ is the tangent vector bundle of M. Hence we have the following results

1) The infinitesimal generator X of θ is invariant under the action θ
2) Each orbit of the action θ is an integral curve of X : that is

$$\frac{d}{dt}\theta_t(p) = X_{\theta_t(p)}$$

II. BI-INVARIANT RIEMANNIAN MATRIX

Let G be a Lie group. For each $a \in G$, let L_a $[R_a]$ be a left[right] transformation, that is

$L_a : G \rightarrow G$, $L_a(g)=ag$ for every $g \in G$

and

* 師範大學 副敎授

** 本 論文은 文敎部, 學術硏究 助成費에 의한 硏究임.

$R_a : G \to G,\ R_a(g) = ga$ for every $g \in G$

If a C^∞ vector field X of G has the property that $L_{a_*}(Xg) = X_{ag}$ for every $a, g \in G$, then X is said to be ***left invariant.*** We put

$\mathcal{L} = \{X \in \mathfrak{X}(M) \mid X$ is a left invariant C^∞ vector field$\}$

where $\mathfrak{X}(M)$ is the set of all C^∞ vector field defined on the C^∞ manifold M.

Then $\mathcal{L}$ is a Lie algebra. In this case $\mathcal{L} \cong T_e(G)$ where e is the identity of G as Lie algebra (S.Helgason. 1962, W.Klingenberg. 1982). The Lie algebra $\mathcal{L}$ is called the ***Lie algebra*** of G

Let $F : R \to G$ be a group homomorphism, where R is a Lie group with addition and G is a Lie group. Then $F(R) = H \subset G$ is called a ***one parameter subgroup*** of G.

PROPERTY 2.1 *Let G be a Lie group. Then there is an one-to-one correspondence between Lie algebra $\mathcal{L}$ and the set of all one-parameter group of G, equally, every left invariant vector field of G is complete* (H. Karcher. 1968)

Let $F : R \to G$ be a one parameter subgroup of a Lie group G and X the left invariant vector field on G defined by

$$X_e = \frac{dF}{dt}\Big|_{t=0} \qquad (= \dot{F}(0))$$

Then we have a unique one parameter group

$$\theta : R \times G \to G \quad (\theta(t,g) = gF(t) = R_{F(t)}(g))$$

of G (see property 2.1).

conversely, let X be a left-invariant vector field of G and $\theta : R \times G \to G$ the corresponding one parameter group of G to X. Then $F : R \to G$, defined by $F(t) = \theta(t,e)$ is an one parameter subgroup of G such that $\theta(t,g) = gF(t)$, where e is the itentity of G.

Therefore, there exists the one-to-one correspandence between $T_e(G)$ and the set of all one parameter subgroups of G. In consequence, we have the following one-to-one correspondence ([4]):

$\mathcal{L} \leftrightarrow$ the set of all parameter group of $G \leftrightarrow T_e(G)$.

Therefore, we can define

$$F : R \times T_e(G) \to G$$

such that F is a function of class C^∞ with respect to $t \in R$ $z \in T_e(G)$ and $\dot{F}(0,z) = Z$

PROPOSITION 2.2 *For $s, t \in R$ and $z \in Te(G)$, $F(st, z) = F(t, sz)$*

proof. Put $st = T$. Then $\frac{dF}{dT}\Big|_{T=0} = Z$ and also

$$\frac{dF}{dt}\Big|_{t=0} = \frac{dT}{dt}\,\frac{dF}{dT}\Big|_{T=0} = s\dot{F}(0,z) = sz$$

Hence the map $t \to F(st, z)$ is a group homomorphism and we have

$$F(st,z) = F(t,sz) \quad ///$$

Let $\Phi : T(M) \times T(M) \to R$ be an inner product on a manifold M, that is for each $p \in M$ the map

$\Phi_p : T_p(M) \times T_p(M) \to R$ is a C^∞-bilinear form satisfying

1) $\Phi_p(X_p, Y_p) = \Phi_p(Y_p, X_p)$ (symmetric)

2) $\Phi_p(X_p, X_p) \geq 0$ and $\Phi_p(X_p, X_p) = 0 \leftrightarrow X_p = 0$ (positive definite)

Let (U, φ) be a local coordinate system of M. then $E_{ip} = \varphi_*^{-1}(\frac{\partial}{\partial x^i})$ $i = 1, 2, \cdots, n$ is called the ***coordinate frame,*** where $p \in U$, $\varphi(p) = (x^1, x^2, \cdots, x^n) \in R^n$.

It is clear that $(E_{1p}, E_{2p}, \cdots, E_{np})$ forms a basis

of $T_p(M)$.
We put

$$\Phi_p(E_{ip},\ E_{jp}) = g_{ij}(p)$$

Then, in $T_p(M)$ we have

$$\Phi_p(X_p,\ Y_p) = \sum_{i,j=1}^{n} g_{ij}(p)a^i b^j$$

for $X_p = \sum_{i=1}^{n} a^i E_{ip}$, $Y_p = \sum_{j=1}^{n} b^j E_{jp}$. We put

$$g(p) = \begin{pmatrix} g_{11}(p) \cdots\cdots g_{1n}(p) \\ \vdots \\ g_{n1}(p) \cdots\cdots g_{nn}(p) \end{pmatrix}$$

which is called a *Riemannian matrix* of M.

A manifold M with Riemannian metric is called *a Riemannian manifold*. It is well-known that every manifold M has a Riemannian metric and a manifold M is orientable (A. Besse 1978, E. Marsden 1973)

PROPERTY 2.3. *Every Lie group is orientable* (W. M. Boothby. 1975).

Let G be a Lie group. For each $a \in G$, we define $I_a : G \to G$ by $I_a(g) = aga^{-1}$. We can easily prove the following : For $a, b \in G$.

$$L_a^{-1} = L_{a^{-1}},\ R_a^{-1} = R_{a^{-1}},\ L_a \cdot R_a \cdot L_a$$
$$I_a = L_a \cdot R_{a^{-1}},\ I_{ab} = I_a \cdot I_b$$

Therefore we can get the following : For $X, Y \in \mathscr{L}$

$$\left.\begin{array}{l} (1)\ L_{b*}(R_{a*}X) = R_{a*}(L_{b*}X) = R_{a*}X \in \mathscr{L} \\ (2)\ I_{a*}(X) = L_{a*}(R_{a*}X) = R_{a*}X \in \mathscr{L} \\ (3)\ I_{a*}([X,Y]) = [I_{a*}X,\ I_{a*}Y] \in \mathscr{L} \\ (4)\ R_{a*} \text{ and } I_{a*} \text{ are automorphism of } \mathscr{L} \end{array}\right] (*)_1$$

We put $I_{a*} = Ad(a)$. Then

$$Ad : G \to Aut(\mathscr{L})$$

defined by $Ad(g) = I_{g*}$ is a function of C^∞, where $Aut(\mathscr{L})$ is the set of all automorphisms of $\mathscr{L}$.

Let Φ be an m-form of a Lie group G. If for all $a, g \in G$

$$L_a^* \Phi_{ag} = \Phi_g \quad (R_a^* \Phi_{ga} = \Phi_g)$$

then Φ is said to be *left invariant (right inuariant)*, where

$$L_a^* : \wedge^m(T(G)) \to \wedge^m(T(G))$$

is defined from $L_a : G \to G$.

It is bi-invariant if it is both left-and right-invariant. If a Lie group G is compact and connected, then there exists a unique bi-invariant volume element Ω such that the volume of G is 1 (W.M.Boothby. 1975) .... $(*)_2$

PROPOSITION 2.4. *It is possible to defined a bi-invariant Riemannian matrix $\tilde{\Phi}$ on a compact connected Lie group G.*

Proof. We have note that $\tilde{\Phi}_e$ determines a bi-invariant tensor field of order 2 on G if and only if $Ad(g)\tilde{\Phi}_e = \tilde{\Phi}_e$ for all $g \in G$ (W.M. boothby. 1975). By $(*)_2$, there exists a unique bi-invariant volume element Ω of G with the Riemannian matrix Φ.

Given $X_e,\ Y_e \in T_e(G)$, define a function $f : G \to R$ by

$$f(g) = (Ad(g)^* \Phi_e)(X_e, Y_e) = \Phi_e(Ad(g)X_e,\ Ad(g)Y_e)$$

for each $g \in G$ and $\tilde{\Phi}_e(X_e, Y_e) = \int_G f(g)\,\Omega$.
Thus, for $a \in G$, we have

$$\begin{aligned}
Ad(a)^* \tilde{\Phi}_e(X_e, Y_e) &= \tilde{\Phi}_e(Ad(a)X_e,\ Ad(a)Y_e) \\
&= \int_G (Ad(g)^*\Phi_e)(Ad(a)X_e,\ Ad(a)Y_e)\,\Omega \\
&= \int_G (Ad(a)^* Ad(g)^*\ \Phi_e(X_e, Y_e)\,\Omega \\
&= \int_G Ad(ga)^*\ \Phi_e(x_e, Y_e)\,\Omega \\
&= \int_G f(R_a(g))\,\Omega
\end{aligned}$$

Since $I_a : g \to G$ is a diffeomorphism by $(*)_1$, we have

$$\int_{I_a(G)} f(g)\,\Omega = \int_G f(R_a(g))R_a^*\Omega.$$

Note that $I_{a^{-1}}^*\Omega = R_a^*\Omega$, $I_a(G) = G$. Moreover, Since $R_a^*\Omega = \Omega$, we have

$$\tilde{\Phi}_e(X_e, Y_e) = \int_G f(g)\,\Omega = \int_G f(R_a(g))\,\Omega.$$

Thus we have

$$Ad(a)^*\tilde{\Phi}_e(X_e, Y_e) = \tilde{\Phi}_e(X_e, Y_e)$$

Since $\tilde{\Phi}_e$ is symmetric, positive definit and bilinear, so is $\tilde{\Phi}$. Hence $\tilde{\Phi}$ is a bi-invariant matrix on G. ///

Ⅲ. SYMMETRIC RIEMANNIAN MANIFOLD

Let M be a connected Riemannian manifold. If to each $p \in M$ there exists an isometry $\sigma_p : M \to M$ which is

1) σ_p is involute (i.e $\sigma_p^2 = \sigma_p$), and

2) there exists an open neighborhood U of P such that $\sigma_p|U$ has the only fixed point P,

then M is said to be *Symmetric.*

Sometimes P is called the ***isolated point*** of a symmety at P

Let M be a symmetric manifold and let $\sigma_p : M \to M$ be a symmetry at P. Then for $X_p \in T_p(M)$, we define

$$\sigma_{p*} : T_p(M) \to T_p(M)$$

by $\sigma_{p*}(X_p) = -X_p$ (W. Klingenberg. 1982) and a symmetric Riemannian manifold M is complete (W.M.Boothby. 1975).

PROPOSITION 3.1. ***Every compact and connected Lie group G is the symmetric space with respect to the bi-invariant metric. Thus with the bi-invariant metric G is complete.***

Proof. By proposition 2.4, G has the bi-invariant metric. Define $\psi : G \to G$ by $\psi(x) = x^{-1}$ for each $x \in G$. If follows that ψ is involute because that ψ has only one fixed point e (identity of G). Recall that for each $X_e \in T_e(G)$ there exists a unique one parameter subgroup $F : R \to G$ such that $X_e = \dot{F}(0)$. If $x = F(t)$ then $x^{-1} = F(-t)$ and thus $\psi(F(t)) = F(-t)$. Hence

$$\begin{aligned}
\psi_*(X_e) = \psi_*(\dot{F}(0)) &= \frac{d}{dt}(\psi(F(t)))\Big|_{t=0} \\
&= \frac{d}{dt}F(-t)\Big|_{t=0} = -\dot{F}(0) = -X_e
\end{aligned}$$

If follows that for X_e, $Y_e \in T_e(G)$

$$\begin{aligned}
(\psi_{*e}X_e,\ \psi_{*e}Y_e) &= (-X_e, -Y_e) \\
&= (X_e,\ Y_e)
\end{aligned}$$

where (,) is the bi-invariant inner product on $T_e(G)$. That is, ψ_{*e} is an isometry on $T_e(G)$. Note that L_a and $R_a (a \in G)$ are isometries with respect to the bi-invariant metric of G. Since

$$\psi(x) = x^{-1} = (a^{-1}x)^{-1}a^{-1} = R_{a^{-1}} \cdot \psi \cdot L_{a^{-1}}(x)$$

for each $x \in G$ $\psi_{*a} : T_a(G) \to T_{a^{-1}}(G)$ may written as

$$\psi_{*a} = (R_{a^{-1}*})_e \cdot \psi_{*e} \cdot (L_{a^{-1}*})a.$$

Thus ψ_{*a} is an isometry. In consequence, $\psi : G \to G$ is an isometry. For each $g \in G$ define σ_g by

$$\sigma g = L_g \cdot R_g \cdot \text{짜}, \text{ that is } \sigma_g(x) = gx^{-1}g$$

Then if follows that σ_g is the symmetry at g. It is well known (W.M.Boothby. 1975) that for a complet Riemannian maniftold M if two isometries $F_1, F_2 : M \rightarrow M$ have property such that for some $p \in M$ $F_1(p) = F_2(p)$ and $F_{1*}|T_p(M) = F_{2*}|T_p(M)$, then $F_1 = F_2$.

Using this fact we can easily prove that for a complete Riemannian manifold M and $p \in M$ there can be at most one involutive isometry σ_p with P as isolated fixed point. ///

COROLLARY 3. 2. *Every point of a connected compact Lie group G is a one parameter subgroup*

Proof. By proposition 3.1. G is complete with respect to bi-invariant matrix, and thus there is only one minimal geodesic $p(t)(-\infty < t < \infty)$ joining e (identity of G) and $g \in G$.

Let G be a compact connected Lie group. Then each geodesic through the identity e of G is a one parameter subgroup of G (W.M.boothby. 1975)

Thus this geodesic P(t) is an one parameter subgroup.

Hence g is a one paramerter subgroup. ///

DEFINITION 3. 3. *A C^∞ connection ∇ on C^∞* manifold M is a mapping

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$$

defined by $\nabla(X,Y) = \nabla_X Y$, which is satisfying conditions : For all $f, g \in C^\infty(M)$, and X, X', Y, $Y' \in \mathfrak{X}(M)$

1) $\nabla_{fx+gx'} Y = f\nabla_x Y + g\nabla_{x'} Y$

2) $\nabla_x(fY + gY') = f\nabla_x Y + g\nabla_x Y' + (xf)Y + (XgY')$

on a Riemannian manifold.

A C^∞ connection ∇ is called a ***Riemannian connection*** if it satisfies the following two further properties;

3) $[X,Y] = \nabla_x Y - \nabla_Y X$

4) $X(Y,Y') = (\nabla_x Y, Y') + (Y, \nabla_x Y')$

where (,) is the inner product on M.

Let M be a Riemannian manifold. For C^∞ vector fields X.Y over M, the ***curvature operator*** R(X.Y) is defined by

$$R(X.Y) \cdot Z = \nabla_x(\nabla_Y Z) - \nabla_Y(\nabla_x Z) - \nabla_{[X,Y]} Z$$

for each C^∞ vector field Z over M, where ∇ is Riemannian connection of M.

TEOREM 3. 4. *Let G be a compact connected Lie group and let $\mathcal{L}$ be the Lie algebra of G. For $X, Y, Z \in \mathcal{L}$*

$$R(X,Y)Z = \frac{1}{4}[Z,[X,Y]]$$

with bi-invariant Riemannian metric where R(X, Y) is the curvature operator.

Proof. Let ∇ be the Riemannian connection with bi-invariant metric of G. Take $X \in \mathcal{L}$ then $\nabla_x X = 0$. In fact, X_e define a unique one parameter subgroup $F : R \rightarrow G$ such that $F(0) = e$ and $\dot{F}(0) = X_e$. For a C^∞ vector field Y over M.

$$\nabla_{X_e} Y = \frac{D}{dt} Y_{F(t)}\Big|_{t=0}$$

Hence

$$\nabla_{X_e} X = \frac{D}{dt} X_{F(t)}\Big|_{t=0}$$

F(t) is geodesic and thus

$$\frac{D}{dt} X_{F(t)} = \frac{D}{dt}\left(\frac{dF}{dt}\right) = 0$$

This means that $\nabla_{X_e}X=0$. Since our metric is left-invariant and X is also left-invariant, $\nabla_X X=0$ everywhere on G.

Since if X and Y are left invariant vector fields then so are X+Y and [X,Y], we have

$$0=\nabla_{X+Y}(X+Y)=\nabla_X Y+\nabla_Y X\ (\nabla_X X=0=\nabla_Y Y)$$

If X and Y are left invariant, then

$$\nabla_X Y+\nabla_Y X=0,\ [X,Y]=\nabla_X Y-\nabla_Y X$$

and so we get

$$\nabla_X Y=\tfrac{1}{2}[X,Y]$$

For X, Y and Z in $\mathscr{L}$, since

$$\nabla_X(\nabla_Y Z)=\tfrac{1}{2}[X,\nabla_Y Z]=\tfrac{1}{2}[X,\tfrac{1}{2}[Y,Z]]$$

$$=\tfrac{1}{4}[X,[Y,Z]]$$

$$\nabla_Y(\nabla_X Z)=\tfrac{1}{4}[Y,[X,Z]]$$

$$\nabla_{[X,Y]}Z=\tfrac{1}{2}[[X,Y],Z]$$

we have following;

$$R(X,Y)Z=\nabla_X(\nabla_Y Z)-\nabla_Y(\nabla_X Z)-\nabla_{[X,Y]}Z$$

$$=\tfrac{1}{4}[X,[Y,Z]]-\tfrac{1}{4}[Y,[X,Z]]$$

$$-\tfrac{1}{2}[[X,Y],Z]$$

$$=\tfrac{1}{4}\{[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]\}$$

$$+\tfrac{1}{4}[Z,[X,Y]]$$

$$=\tfrac{1}{4}[Z,[X,Y]]$$

REFERENCES

Bangert. V. 1980. Closed geodesics on complete surfaces. *Math. Ann.* Vol.251, p.83~96.

Besse. A. 1978. Manifolds all of whose geodesics are closed. Springer.

Bishop. R.L. 1964. Geometry of manifold. Academic Press. INC

Boothby. w.M. 1975. An introduction to differential manifolds and Riemannian Geometry. Academic Press.

Farkers. H.M. 1980. Riemann Surfaces. Springer verlag

Grove, K.H. Karcher and E. Rub. 1977. A generalized Sphere theorem. *Ann. of Math.* Vol.106. 201~211.

Helgason. S. 1962. Differential geometry and symmetric spaces, Academic Press.

Karcher. H. 1968. Schnittort und konvex megen in vollstas digen Riemannchen mannifaltigkeiten. *Math. Ann.* Vol.177, 105~121.

Kleingenberg. W. 1982. Riemannian Geometry. Walter de Gruyter.

Kleingenberg. W. 1971. Geodatischer fluss auf mannifatigkeiten typ. *Inv. Math.* Vol.14. 63~82.

Marsden. E. 1973. On completeness of homogeneous Psedo-Riemannian manifolds. *Indian Univ. Math. Jounal* Vol.22, 1065~1066.

Ozels. V. 1976. Largest normal neighborhoods. *Proc. Amer. Math. Soc.* 99~101.

Sakai. T. 1977. On the cut loci of symmetric spaces. *Hokkaido Math.* Vol.6. 136~161.

Warner. F. W. 1971. Foundations of differentiable manifolds and Lie group, Scott. Foresmann and Company.

Wolf. J.A. 1964. On the classification of hermitian symmetric space. *Journal of Math. and Mechanics* Vol.13. 489~496.

國文抄錄

對稱 Riemann 多樣體에 관하여

C^∞ 多樣體上에서 1-媒介變數群과 Lie群을 정의하고, 모든 Compact이고 連結인 Lie群 G는 雙不變 距離에 대하여 對稱空間이 됨을 보이고 나아가서는 對稱 Riemann 多樣體上의 모든 점들은 1-媒介變數部分群이 되며 Riemann 接續 ∇와 曲率作用素 $R(X,Y)$의 성질을 이용하여 Compact이고 連結인 Lie群 G와 Lie 代數 $\mathscr{L}$에서, $\mathscr{L}$의 원소 X,Y,Z에 대하여 $R(X,Y)Z=\frac{1}{4}[Z,[X,Y]]$가 됨을 보였다.