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Maximal Column Rank Preservers of Nonnegative Integer Matrices

Dedicated to Professor U-Hang Ki on his sixtieth birthday

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The maximal column rank of an m by n matrix over a semiring is the maximal number of the columns of A which are linearly independent. We characterize the linear operator which preserve the maximal column ranks of nonnegative integer matrices.

1. Introduction

Suppose F is a field and $\mathbf{M}$ is the set of all $m \times n$ matrices over $\mathbf{F}$. If T is a linear operator on $\mathbf{M}$ and f is a function defined on $\mathbf{M}$, then T preserves f if $f(T(A)) = f(A)$ for all A in $\mathbf{M}$.

Frobenius (1897), Marcus and Moyls (1959), Marcus and May (1976), Marcus and Purves (1959), Beasley (1970), Minc (1976) and Kovacs (1977) characterized those linear operators on $\mathbf{M}$ that preserve : determinant and characteristic polynomial, rank, permanent, the rth symmetric function ($r \geq 4$), and so on (respectively).

In 1983, McDonald found that the characterizations of the first three functions were valid over more general rings. Typically, the first operations that come to mind for preserving f turn out to be the only ones. For example, T

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preserves the characteristic polynomial if and only if T is a similarity transformation, transposition, or composition of such operations.

In 1984 and 1985 analogues of Marcus and Moyls's work on rank were obtained by Beasley, Gregory, and Pullman [3,4] for certain type of semirings. These semirings included such combinatorially significant systems as the nonnegative integers and the Boolean algebra of two elements.

There are many papers on the study of linear operators that preserve the rank functions of matrices over several algebraic structures. We can find them in [5].

Recently Song characterized linear operators that preserve the column rank [7], and those that preserve the maximal column rank [6] of matrices over binary Boolean algebra with Hwang and Kim.

In this paper that work is continued. We obtain characterizations of those linear operators on m by n matrices over nonnegative integer semiring that preserve the maximal column rank.

2. Preliminaries

A semiring is essentially a ring in which only the zero is required to have an additive inverse. Thus all rings are semirings. The nonnegative integers, Z^+, and nonnegative reals, R^+, (with the usual arithmetic) are combinatorially interesting examples of semirings. Algebraic operations on matrices over a semiring and such notions as *linearity* and *invertibility* are also defined as if the underlying scalars were in a field.

The set of $m \times n$ matrices with entries in Z^+ is denoted by $\mathbf{M}_{m,n}(Z^+)$.

A set of vectors ($m \times 1$ matriecs) is a *semimodule* [1] if it is closed under addition and scalar multiplication. A subset $\mathcal{W}$ of a semimodule $\mathcal{V}$ is a *spanning set* if each vector in $\mathcal{V}$ can be written as a sum of scalar multiples (i.e. a linear combination) of elements of $\mathcal{W}$.

The $m \times n$ matrix all of whose entries are zero except its (i,j)th, which 1, is denoted E_{ij}. We call E_{ij} a *cell*. The set of cells spans $M_{m,n}(Z^+)$. Let $\mathbf{e}_i$ be the $n \times 1$ matrix with a "1" in the ith position and zero elsewhere. We say that A is the *column matrix* if $A = \mathbf{a}\mathbf{e}_i^t$ for some $1 \leq i \leq n$ and some $\mathbf{a} \in M_{m,1}(Z^+)$.

The *column space* of a matrix $A \in M_{m,n}(Z^+)$ is the semimodule spanned by the columns of A over Z^+. Since the column space is spanned by a finite set of vectors, it contains a spanning set of minimum cardinality; that cardinality is the *column rank* [2] of $A, \chi(A)$.

A set G of vectors over Z^+ is *linearly dependent* [2] if for some $g \in G$, g is a linear combination of elements in $G - \{g\}$. Otherwise G is *linearly independent*.

The *maximal column rank* [5], $\psi(A)$, of an $m \times n$ matrix $A \in M_{m,n}(Z^+)$ is the maximal number of the columns of A which are linearly independent over Z^+.

It follows that

$$0 \le \chi(A) \le \psi(A) \le n \tag{1.1}$$

for all $m \times n$ matrix A over Z^+.

The inequality in (1.1) may be strict over Z^+. For example, we consider the matrix

$$A = [1, 2, 3]$$

over Z^+. Then the column rank of A is one, while the maximal column rank of it is two since the last two columns of A are linearly independent over Z^+.

Hwang, Kim and Song [6] compared the column rank and the maximal column rank for matrices over certain semirings, and found that except for small values of m and n, the two ranks did not agree in general. In particular, they obtained the following relations between column rank and maximal column rank over $M_{m,n}(Z^+)$.

Theorem 2.1. ([6]) *Let $\alpha(Z^+, m, n)$ be the largest k such that for all $m \times n$ matrices A over Z^+, $\chi(A) = \psi(A)$ if $\chi(A) \le k$ and there is at least one $m \times n$ matrix A over Z^+ with $\chi(A) = k$. Then for $m \ge 1$,*

$$\alpha(Z^+, m, n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ 0 & \text{if } n \ge 3 \end{cases}$$

If A is a matrix over Z^+ and $A = \mathbf{u}\mathbf{a}^t$, then $\mathbf{a}, \mathbf{u}$ are called right and left factors of A respectively.

Lemma 2.2. *For $A \in M_{m,n}(Z^+), \psi(A) = 1$ if and only if A can be factored as $\mathbf{u}\mathbf{a}^t$ for some nonzero $\mathbf{u} \in M_{m,1}(Z^+)$ and $\mathbf{a} \in M_{m,1}(Z^+)$ with $\psi(\mathbf{a}^t) = 1$.*

Proof. If $\psi(A) = 1$, then there exists one column $\mathbf{a}_k$ of A such that all the other columns $\mathbf{a}_i$ are linearly dependent in each other, and hence all $\mathbf{a}_i$ are expressed as a scalar multiple of $\mathbf{a}_k$, that is $\mathbf{a}_i = \alpha_i \mathbf{a}_k$ for some $\alpha_i \in Z^+$. Therefore $A = \mathbf{a}_k[\alpha_1, \cdots, \alpha_n]$.

Let $\mathbf{u} = \mathbf{a}_k, \mathbf{a}^t = [\alpha_1, \cdots, \alpha_n]$. Then the fact that $\psi(\mathbf{a}^t) = 1$ follows from $\psi(A) = 1$.

The converse is clear.

3. Maximal column rank preserver over $M_{m,n}(Z^+)$

A function T mapping $M_{m,n}(Z^+)$ into itself is called an *operator* on $M_{m,n}(Z^+)$ The operator T

(1) is *linear* if $T(\alpha A+\beta B)=\alpha T(A)+\beta T(B)$ for all $\alpha,\beta\in Z^+$ and all $A,B\in M_{m,n}(Z^+)$

(2) *preserves maximal column rank* if $\psi(A)=\psi(T(A))$ for all $A\in M_{m,n}(Z^+)$.

(3) *strongly preserves maximal column rank 1* provided that $\psi(T(A))=1$ if and only if $\psi(A)=1$ for all $A\in M_{m,n}(Z^+)$.

In this section we obtain characterizations of the linear operator which preserves maximal column rank over nonnegative integer matrices.

Lemma 3.1. *Let* $A=\mathbf{h}(\mathbf{e}_1)^t, B=\mathbf{h}(\mathbf{e}_2)^t$ *be the matrices in* $M_{m,n}\ (Z^+)$ *with* $\mathbf{h}\in M_{m,1}(Z^+)$. *Suppose* T *is a linear operator from* $M_{m,n}(Z^+)$ *into itself which strongly preserves maximal column rank 1. If* $T(A)=\mathbf{u}\mathbf{a}^t, T(B)=\mathbf{v}\mathbf{b}^t$, *where* $\mathbf{a}^t=[a_1,\cdots,a_n], \mathbf{b}^t=[b_1,\cdots,b_n]$ *and* $c\mathbf{u}=d\mathbf{v}$ *for some nonzero* $c,d\in Z^+$, *then* $a_ib_i=0$ *for some* i.

Proof. Suppose to the contrary that $a_ib_i\neq 0$ for all i. Since $c\mathbf{u}=d\mathbf{v}$, we have

$$\begin{aligned} dT(\alpha A+\beta B) &= \alpha d\mathbf{u}\mathbf{a}^t+\beta d\mathbf{v}\mathbf{b}^t=\alpha d\mathbf{u}\mathbf{a}^t+\beta c\mathbf{u}\mathbf{b}^t \\ &= \mathbf{u}(\alpha d\mathbf{a}+\beta c\mathbf{b})^t. \end{aligned} \tag{3.1}$$

By permuting columns, if necessary, we may assume that $b_1\leq b_i$ for all i and $a_1\leq a_j$ for all j such that $b_j=b_1$. Since c and d are nonzero, we have for all sufficiently large β, that $da_1+\beta b_1c\leq da_i+\beta b_ic$ for all i. Since $dT(A+\beta B)$ has maximal column rank 1 for all β, any two columns are linearly dependent over Z^+. Thus one column is a scalar multiple of the other column for any two columns. Since $\psi(A+\beta B)=1$, from (3.1) we have

$$(da_1+\beta b_1c)|(da_i+\beta b_ic) \tag{3.2}$$

for sufficiently large β, and all i. Choose k large and let $\beta=dka_1$. Then $(da_1+dka_1b_1c)|(da_i+dka_1b_ic)$ for all i by (3.2). So we have $a_1|a_i$ for all i. It follows that we may assume that $a_1=1$. Since $a_1=1$ and $b_1\leq b_i, \alpha d+b_1c\leq \alpha da_i+b_ic$ for all α and all i. Thus $(\alpha d+b_1c)|(\alpha da_i+b_ic)$ for all α and all i since $dT(\alpha A+B)$ has maximal column rank 1. Letting $\alpha=b_1c$ we have $(b_1cd+b_1c)|(b_1cda_i+b_ic)$ for all i. It now follows that $b_1|b_i$ for all i, so we also may assume that $b_1=1$. Therefore we have

$$(d+\beta c)|(da_i+\beta b_ic) \tag{3.3}$$

for all β and all i from (3.2).

Suppose that $a_i\neq b_i$ for some i. Say $a_i<b_i$. Letting $\beta=b_i$ and $\beta=b_i+1$ from (3.3), we have that for some $r,s\in Z^+$

$$da_i+b_i^2c=r(d+b_ic) \tag{3.4}$$

$$da_i + (b_i^2 + b_i)c = s(d + (b_i + 1)c), \tag{3.5}$$

respectively. Subtracting (3.4) from (3.5) we have

$$b_i c = (s - r)(d + b_i c) + sc \tag{3.6}$$

If $s = r$, then from (3.6) we have $b_i = s$. So (3.4) gives

$$da_i + s^2 c = s(d + sc) = sd + s^2 c,$$

that is, $a_i = s = b_i$, a contradiction since $a_i < b_i$. If $s > r$, then (3.6) gives

$$b_i c = (s - r)(d + b_i c) + sc > b_i c,$$

a contradiction. If $s < r$, then (3.6) gives

$$b_i c = (s - r)(d + b_i c) + sc < sc.$$

So $b_i < s < r$. From (3.4) we have

$$da_i + b_i^2 c = r(d + b_i c) > s(d + b_i c) > sd + b_i^2 c,$$

that is, $da_i > sd$. Thus we have $a_i > s > b_i$, which contradicts $a_i < b_i$.

For the case $b_i < a_i$, we also get contradictions by symmetric arguments.

Thus $a_i = b_i$ for all i, that is, $\mathbf{a} = \mathbf{b}$. Then $\alpha A + \beta B$ has maximal column rank 2 for relatively prime positive integers α and β since the first two columns of $\alpha A + \beta B$ are linearly independent. But

$$\begin{aligned} T(\alpha A + \beta B) &= \alpha T(A) + \beta T(B) \\ &= \alpha \mathbf{u}\mathbf{a}^t + \beta \mathbf{v}\mathbf{b}^t \\ &= (\alpha \mathbf{u} + \beta \mathbf{v})\mathbf{a}^t \end{aligned}$$

has maximal column rank 1 since $\mathbf{a}^t$ has maximal column rank 1 from the construction. Hence we have a contradiction that T strongly preserves maximal column rank 1.

Lemma 3.2. *Let T be a linear operator from $M_{m,n}(Z^+)$ into itself. If T strongly preserves maximal column rank 1, then T maps column matrices to column matrices.*

Proof. Suppose to the contrary that T maps a column matrix to a matrix which is not a column matrix. Say $X_1 = \mathbf{x}(\mathbf{e_1})^t$ and $T(X_1)$ has more than one nonzero column. For each $1 \le i \le n$, let $X_i = \mathbf{x}(\mathbf{e_i})^t$. Let $S = \{1, 2, \cdots, n\}$ and let $S_1 = \{j$: the jth column of $T(X_i)$ is zero for all $1 \le i \le n\}$. Then for each $i \in S - S_1$, there is a $j(i)$ such that the ith column of $T(X_{j(i)})$ is not zero. Now $T(X_1)$ has at least two nonzero columns, say columns k_1 and k_2. Let $S_2 = S - S_1 - \{k_1, k_2\}$, and let $A = X_1 + \sum_{i \in S_2} X_{j(i)}$. Note that for any $k \in S - S_1$, the kth column of $T(A)$ is nonzero. Further, since A consists of at most $n-1$ distinct summands, each of which is a column matrix, there is at least one zero column in A, say the ith. Let $B = X_i$. Since $T(A)$ has zero columns only corresponding to indices in S_1 (where $T(B)$ also must have a zero column) we can restrict our attention to those columns in $T(A)$ that are nonzero ; hence we lose no generality in assuming that $T(A)$ has no zero column. Thus, since A, and hence $T(A)$, has maximal column rank 1, $T(A) = \mathbf{u}\mathbf{a}^t$, where $\mathbf{a}^t = [a_1, \cdots, a_n]$ has all nonzero entries which are linearly dependent, and some $u_j \neq 0$. Let $T(B) = \mathbf{v}\mathbf{b}^t$ with $\mathbf{b}^t = [b_1, \cdots, b_n]$. Now we consider two cases:

Case 1) Assume that $c\mathbf{u} \neq d\mathbf{v}$ for all nonzero c, d in Z^+. Since $\alpha A + B$ has maximal column rank 1 for any positive integer α,

$$T(\alpha A + B) = [\alpha a_1 \mathbf{u} + b_1 \mathbf{v} | \alpha a_2 \mathbf{u} + b_2 \mathbf{v} | \cdots | \alpha a_n \mathbf{u} + b_n \mathbf{v}]$$

has also maximal column rank 1. Thus we have, for some fixed j,

$$\alpha a_k \mathbf{u} + b_k \mathbf{v} = \mu_k(\alpha a_j \mathbf{u} + b_j \mathbf{v})$$

for some positive integer, $\mu_k, k = 1, \cdots, n$. If $a_k \neq \mu_k a_j$ for some k, then

$$\alpha |a_k - \mu_k a_j| \mathbf{u} = |\mu_k b_j - b_k| \mathbf{v},$$

which is a contradiction to the condition that $c\mathbf{u} \neq d\mathbf{v}$ for all nonzero c, d in Z^+. Thus $a_k = \mu_k a_j$ and $b_k = \mu_k b_j, k = 1, \cdots, n$. That is, $\mathbf{a} = a_j \mathbf{w}$ and $\mathbf{b} = b_j \mathbf{w}$, where $\mathbf{w}^t = [\mu_1, \cdots, \mu_n]$ with $\psi(\mathbf{w}^t) = \psi(\mathbf{a}^t) = 1$. Then $\psi(T(\alpha A + \beta B)) = \psi((\alpha a_j \mathbf{u} + \beta b_j \mathbf{v})\mathbf{w}^t) = 1$ for arbitrary α, β in Z^+. This contradicts that T strongly preserves maximal column rank 1 since $\alpha A + \beta B$ has maximal column rank 2 for relatively prime α and β in Z^+.

Case 2) Assume that $c\mathbf{u} = d\mathbf{v}$ for some nonzero c, d in Z^+. Since $T(A) = \mathbf{u}\mathbf{a}^t$ has maximal column rank 1, all the columns of $\mathbf{a}^t$ are linearly dependent. So, without loss of generality, we can assume that $a_1 = 1$. For $T(B) = \mathbf{v}\mathbf{b}^t$, we shall show that $b_i \neq 0$ for all i.

Suppose $b_i = 0$ for some i. Choose j such that $b_j = 0$ and $a_j \leq a_h$ for all h such that $b_h = 0$. Since

$$\begin{aligned} dT(A + \beta B) &= d(\mathbf{u}\mathbf{a}^t + \beta \mathbf{v}\mathbf{b}^t) \\ &= \mathbf{u}[da_1 + \beta cb_1, da_2 + \beta cb_2, \cdots, da_n + \beta cb_n] \end{aligned}$$

has maximal column rank 1 for all β, choose β such that $da_k + \beta cb_k > da_j$ for all k with $b_k \neq 0$. Thus there exist distinct integers γ, δ such that for fixed k

with $b_k \neq 0$,

$$da_k + \beta cb_k = \gamma da_j \tag{3.7}$$

and

$$da_k + (\beta+1)cb_k = \delta da_j. \tag{3.8}$$

subtracting (3.7) from (3.8), we have

$$cb_k = (\delta - \gamma)da_j. \tag{3.9}$$

Further, if $j \neq 1$, then $b_1 \neq 0$. For, if $b_1 = 0$, then $a_1 = 1$ is the minimal entries in $\mathbf{a}$ and hence $j = 1$ from the construction of a_j. So, for $k = 1$, $da_1 + \beta cb_1 = \gamma_1 da_j$ from (3.6). Since $da_j | cb_1$ from (3.9) for $k = 1$, it follows that $da_j | da_1$, that is, $a_j | a_1 (= 1)$. Thus $a_j = 1$. So we may assume that $j = 1$.

Now,

$$\begin{aligned} cT(\alpha A + B) &= [c\alpha\mathbf{u}|c\alpha\mathbf{u}a_2 + c\mathbf{v}b_2|\cdots|c\alpha\mathbf{u}a_n + c\mathbf{v}b_n] \\ &= \mathbf{v}[\alpha d, \alpha da_2 + cb_2, \cdots, \alpha da_n + cb_n] \end{aligned}$$

must have maximal column rank 1 for all α. Thus there are γ_i such that $\alpha da_i + cb_i = \gamma_i(\alpha d)$ for all i.

It follows that $\alpha d | cb_i$ for all α, and all $i = 1, \cdots, n$, a contradiction since $b_i \neq 0$ for at least one i.

We now have shown that $b_i \neq 0$ for all i. Now, letting $A = X_i$ and $B = X_j, j = 1, \cdots, n$ and $j \neq i$, the above argument implies that $T(A)$ and $T(B)$ have no zero columns. This contradicts Lemma 3.1.

Hence the two cases show that T maps column matrices to column matrices.

Theorem 3.3. *Let $T : \mathbf{M}_{m,n}(Z^+) \to \mathbf{M}_{m,n}(Z^+)$ be a linear operator. Then T strongly preserves maximal column rank 1 if and only if there exist $Q \in \mathbf{M}_{m,m}(Z^+)$ which is nonsingular as a real matrix and a permutation matrix $P \in \mathbf{M}_{n,n}(Z^+)$ such that $T(A) = QAP$ for all $A \in \mathbf{M}_{m,n}(Z^+)$.*

Proof. Suppose there exist Q and P such that $T(A) = QAP$ for all $A \in \mathbf{M}_{m,n}(Z^+)$ and A has maximal column rank 1. Then $A = \mathbf{x}\mathbf{a}^t$ with $\psi(\mathbf{a}^t) = 1$. That is, all the columns in $\mathbf{a}^t$ are linearly dependent in each other. Let P^t correspond to a permutation $\pi \in S_n$. Then

$$QAP = Q\mathbf{x}(P^t\mathbf{a})^t$$

and the columns of $(P^t\mathbf{a})^t$ are linearly dependent in each other. Hence QAP has maximal column rank 1. Further, assume that QAP has maximal column rank

1. Since P is a permutation matrix in $\mathbf{M}_{n,n}(Z^+)$, multiplying a permutation matrix P^{-1} on the right hand of QAP does not change the maximal column rank of QAP. Hence QA has maximal column rank 1. Therefore all the columns $Q\mathbf{a}_i$ are linearly dependent with $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$. Thus for any two columns $Q\mathbf{a}_k$ and $Q\mathbf{a}_h$ of QA we have $Q\mathbf{a}_k = r_k Q\mathbf{a}_h$ or $Q\mathbf{a}_h = r_h Q\mathbf{a}_k$ with $r_k, r_h \in Z^+$. Hence $\mathbf{a}_k = r_k\mathbf{a}_h$ or $\mathbf{a}_h = r_h\mathbf{a}_k$ over real field and hence over Z^+ since Q is invertible as a real matrix. That is, $\psi(A) = 1$. Thus T strongly preserves maximal column rank 1.

Conversely, suppose T strongly preserves maximal column rank 1. Let $X_i = \mathbf{x}(\mathbf{e}_i)^t, i = 1, \cdots, n$, for some fixed $\mathbf{x} \in (Z^+)^m$. By Lemma 3.2, $T(X_i) = \mathbf{y}(\mathbf{e}_{\pi(i)})^t$ where $\pi : \{1, \cdots, n\} \to \{1, \cdots, n\}$. If π is not a permutation, then $\alpha T(X_i) + \beta T(X_j)$ has only one nonzero column for all $\alpha, \beta \in Z^+$. That is, $T(\alpha X_i + \beta X_j)$ has maximal column rank 1 for all α, β, a contradiction since $\alpha X_i + \beta X_j$ has maximal column rank 2 for relatively prime $\alpha, \beta \in Z^+$. Thus π is a permutation. So without loss of generality, we assume π is the identity permutation, so that $T(X_1) = \mathbf{u}(\mathbf{e}_1)^t$ and $T(X_2) = \mathbf{v}(\mathbf{e}_2)^t$. If $u_i \neq 0$ and $v_i = 0$, or vice versa, $\mathbf{u}(\mathbf{e}_1)^t + \mathbf{v}(\mathbf{e}_2)^t$ has maximal column rank 2, contradicting that T strongly preserves maximal column rank 1 since $X_1 + X_2$ has maximal column rank 1. Thus $u_i = 0$ if and only if $v_i = 0$. We assume without loss of generality that $0 \neq u_1 \leq v_1$. Since $X_1 + X_2$ has maximal column rank 1, $\mathbf{v} = r\mathbf{u}$ for some $r \in Z^+$. If $r \neq 1$, choose p relatively prime to r, then

$$\begin{aligned} T(pX_1 + X_2) &= [p\mathbf{u}|\mathbf{v}|\mathbf{0}|\cdots|\mathbf{0}] \\ &= [p\mathbf{u}|r\mathbf{u}|\mathbf{0}|\cdots|\mathbf{0}] \end{aligned}$$

has maximal column rank 2 while $pX_1 + X_2$ has maximal column rank 1, a contradiction. Thus $r = 1$. That is, $\mathbf{u} = \mathbf{v}$. It follows that $T(X_i) = \mathbf{u}(\mathbf{e}_i)^t$. In particular, when $X_i = E_{ji}$, there exists some vector $\mathbf{u}_j$ such that $T(E_{ji}) = \mathbf{u}_j(\mathbf{e}_i)^t$ for all i, j. Let Q be the matrix $[\mathbf{u}_1|\mathbf{u}_2|\cdots|\mathbf{u}_m]$. Then for an arbitrary $A \in \mathbf{M}_{m,n}(Z^+)$,

$$\begin{aligned} T(A) &= \sum_{j=1}^{m}\sum_{i=1}^{n} a_{ji}T(E_{ji}) \\ &= \sum_{j=i}^{m}\sum_{i=1}^{n} a_{ji}\mathbf{u}_j(\mathbf{e}_i)^t. \end{aligned}$$

So the (k,j) entry of $T(A)$ is $\sum_{i=1}^{m} a_{ij}u_{ki}$. The (k,j) entry of QA is $\sum_{i=1}^{m} u_{ki}a_{ij}$, which is the (k,j) entry of $T(A)$. Thus, $T(A) = QA$ for all $A \in \mathbf{M}_{m,n}(Z^+)$.

Further we show that Q is nonsingular as a real matrix. Suppose that $Q = (q_{ij})$ is singular. Say, $Q\mathbf{x} = \mathbf{0}$ for some nonzero real vector $\mathbf{x}$. Since $\mathbf{x}$ can be considered as a solution of the homogeneous system of linear equations with coefficients $q_{ij} \in Z^+$, we may assume, without loss of generality, that the entries of $\mathbf{x}$ are all integers. So let $\alpha = 1 + \max_{1 \le i \le m} |x_i|$, and $\mathbf{z} = \alpha\mathbf{j} + \mathbf{x}$, where $\mathbf{j}$ is the vector of all 1's. Then $\mathbf{z} \in \mathbf{M}_{m,1}(Z^+)$ and $Q\mathbf{z} = Q(\alpha\mathbf{j} + \mathbf{x}) = Q(\alpha\mathbf{j})$. Thus $T(\mathbf{z}\mathbf{e}_1^t + \alpha\mathbf{j}\mathbf{e}_2^t) = Q(\mathbf{z}\mathbf{e}_1^t) + Q(\alpha\mathbf{j}\mathbf{e}_2^t) = Q(\alpha\mathbf{j})\mathbf{e}_1^t + Q(\alpha\mathbf{j})\mathbf{e}_2^t = Q(\alpha\mathbf{j})(\mathbf{e}_1 + \mathbf{e}_2)^t$ has maximal column rank 1. Then $\mathbf{z} = k\alpha\mathbf{j}$ or $k\mathbf{z} = \alpha\mathbf{j}$ for some $k \in Z^+$. But then $Q\mathbf{z} = \mathbf{0}$ and hence $T(\mathbf{z}\mathbf{e}_1^t) = \mathbf{0}$, contradicting that T strongly preserves maximal column rank 1. Thus Q is nonsingular as a real matrix.

Corollary 3.4. *A linear operator $T : \mathbf{M}_{m,n}(Z^+) \to \mathbf{M}_{m,n}(Z^+)$ preserves maximal column rank if and only if there exist $Q \in \mathbf{M}_{m,m}(Z^+)$ which is nonsingular as a real matrix and a permutation matrix $P \in \mathbf{M}_{n,n}(Z^+)$ such that $T(A) = QAP$ for all $A \in \mathbf{M}_{m,n}(Z^+)$.*

Thus we have characterized the linear operator that preserve maximal column rank of nonnegative integer matrices.

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