Static Output Feedback Linear System Modeling in Real Grassmann Space

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Abstract

It is shown how the pole-assignment problem in m-input, p-output, nth order linear (strictly proper) systems by real output feedback gains can be modeled in real Grassmanns pace. For the parametrization in real Grassmann space, the so-called Plücker matrix formula $L\boldsymbol{k} = \boldsymbol{a}$ is applied (where L indicates Plücker matrix, $\boldsymbol{k}$ indicates extended static output feedback (SOF) vector whose elements are defined in the Grassmann coordinates constrained in some nonlinear equations, called quadratic Plücker relations ($QPRs$), and $\boldsymbol{a}$ indicates arbitrary real coefficient vector of closed-loop characteristic polynomial). It is shown that under full-rank of some Plücker sub-matrix, $rank(L_{sub}) = n$, as a necessary condition of exact SOF pole-assignment, a row-reduced unity diagonal formula (symbolized, $L_{sub}'\boldsymbol{k}_{sub}' = \boldsymbol{a}_{sub}'$) of $L\boldsymbol{k} = \boldsymbol{a}$ is formulated, and the row-reduced unity diagonal formula associated with $QPRs$ plays an essential role for complete parametrization of the SOF pole-assignment problem. An exemplar of a 2-input, 2-ouput, 4th order system in this area is illustrated.

Index Terms

SOF pole-assignment, Plücker matrix formula for closed-loop characteristic polynomial in SOF systems, necessary condition of exact pole-assignment (EPA), real Grassmann space, complete parametrization in real Grassmann space.

1. Introduction

The loop connections of SOF controller for m-input, p-output linear MIMO system, $G(s)$, can be figured like Fig. 1.

In time domain analysis under the coordinates of state space,

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{1.1}$$

with real coefficient matrices $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{p\times n}$, the pole-assignment problem via SOF law $\boldsymbol{u}(t) = -K\boldsymbol{y}(t)$ is usually analyzed in eigenspace formula of closed-loop characteristic polynomial [1],

$$p_c(s) = det\ (sI - A - BKC).$$

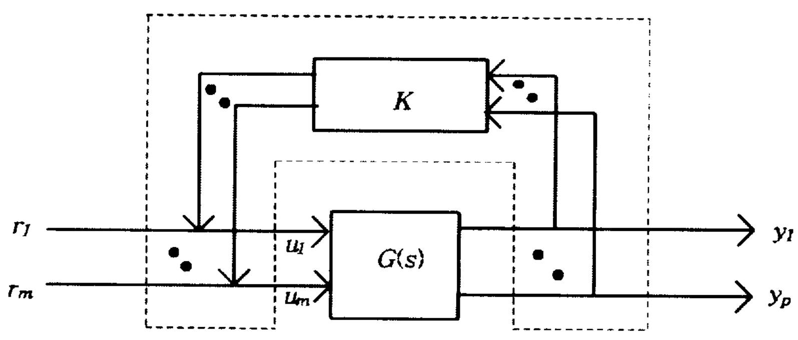

Fig. 1. Outlook of SOF controller for MIMO system

In this state space framework, the complete feature of SOF pole-assignability is hardly obtained in MIMO system case, but only generic feature has been known whose generic pole-assignability covers almost all systems, even if mathematical algorithm like Schubert enumerous calculus is applied [2-4]. For rigorous examination of this genericity problem, "a Grassmann invariant" as a complete system invariant was developed in [5, 6]. They observed that the genericity problem in pole-assignment would be reflected (or checked) in some formula of Grassmann invariant, because it is associated with the closed-loop characteristic polynomial, pc(s). As expected, they could derive "a real matrix form" (called, Plücker matrix) from the Grassmann invariant, and showed in strictly proper systems that the full-rank condition of a sub-matrix of Plücker matrix becomes a new necessary condition of "exact" pole-assignment (EPA) in addition to the well-known necessary condition of EPA, mp $\geq$ n [6, Theorem 5.3]. In other words, using exterior algebra in [5, 6], it is concretely exhibited that the closed-loop characteristic polynomial pc(s) can be described under the coordinates of Grassmann space by

$$p_c(s) = \boldsymbol{b}(s)L\boldsymbol{k} \qquad (1.2)$$

where $\boldsymbol{b}(s) = [s^n \; s^{n-1} \; \cdots \; s \; 1]$ is a basis vector, $L \in \mathbf{R}^{(n+1)\times(\sigma+1)}$ indicates *Plücker matrix*, and $\boldsymbol{k} = [1 \; k_{(1)} \; \ldots \; k_{(mp)} \; k_{(mp+1)} \; \ldots \; k_{(\sigma)}]^t$ is extended SOF vector whose elements are defined in "inhomogenized coordinates of Grassmann space, *Grass* (*m*, *m*+*p*)" where $\sigma = \binom{m+p}{m} - 1$.

From (1.2), *n* number of SOF equations for pole-assignment (P-A) in characteristic polynomial $p_c(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$ are transformed into "a SOF vector equation" like

$$L\boldsymbol{k} = \boldsymbol{a} \qquad (1.3)$$

where $\boldsymbol{a} = [1 \; a_1 \; \ldots \; a_n]^t$. In (1.3), it is observed that the real matrix L of $\boldsymbol{k}$ presents "a real Grassmann parameter" for P-A [9], because the elements of $\boldsymbol{k}$ are defined on the coordinates of Grassmann space associated with constraints of *QPRs*. A followed natural question of numerical construction of L (i.e., numerical construction algorithm of $L\boldsymbol{k}$) is outlined in next section 2.

Remark 1: The Plücker matrix L is constructed on the base of minimal transfer function matrix $G(s)$ having McMillan degree n [6, section 4]. Therefore in the controllable and observable system with n states of $A \in \mathbf{R}^{n\times n}$, the dimension of Plücker matrix is obtained by $L \in \mathbf{R}^{(n+1)\times(\sigma+1)}$.

2. Signal flow graph analysis of SOF loops and its application to *Lk* = *a*

In signal flow graph analysis in frequency s-domain [7, 8], the gain M between $U_i(s)$ and $Y_j(s)$ in SOF linear systems over (negative) SOF law $\boldsymbol{U}(s) = -\boldsymbol{KY}(s)$, called Mason's gain formula, is given by

$$M = \frac{Y_j(s)}{U_i(s)} = \sum_{k=1}^{\nu} \frac{P_k \Delta_k}{\Delta} \tag{2.1}$$

where $\boldsymbol{U}(s) = [U_1(s)\ U_2(s)\ \cdots\ U_m(s)]'$ is control input vector,

$\boldsymbol{Y}(s) = [Y_1(s)\ Y_2(s)\ \cdots\ Y_p(s)]'$ is output vector,

ν = total number of forward paths between $U_i(s)$ and $Y_j(s)$,

P_k = gain of the k-th forward paths between $U_i(s)$ and $Y_j(s)$,

Δ = 1 - (sum of the gains of *all* individual loops) + (sum of products of gains of all possible combinations of *two* nontouching loops) - (sum of products of gains of all possible combinations of *three* nontouching loops) + ⋯

Δ_k = the cofactor value of Δ that is nontouching with the k-th forward path.

Let's symbolize the loop determinant Δ composed by constant '1' and all SOF loop gains by

$$\Delta = 1 - \sum_{\alpha} \ell_{\alpha(1)} + \sum_{\beta} \ell_{\beta(2)} + \sum_{\delta} \ell_{\delta(3)} + \cdots \tag{2.2a}$$

Then the sum of *all* individual loop gains $\sum_{\alpha} \ell_{\alpha(1)}$ can be divided into two parts by "linear terms" and "nonlinear (multiplicative) terms" over the variables $k_{11}, \cdots, k_{mp}$: Single-path loops like $\{-G_{ij}(s)k_{ji}$ for all i and $j\}$ and multi-path loops like $\{-G_{ij}(s)k_{js}G_{st}(s)k_{ti}, -G_{ij}(s)k_{js}G_{st}(s)k_{tu}G_{uv}(s)k_{ui}$, etc.$\}$. So we can rewrite the (2.5a) by

$$\Delta = 1 - \sum_{\alpha} \ell_{\alpha(1)}{}^{single} - \sum_{\alpha} \ell_{\alpha(1)}{}^{multi} + \sum_{\beta} \ell_{\beta(2)} + \sum_{\delta} \ell_{\delta(3)} + \cdots \tag{2.2b}$$

As seen in (2.5b), the loop determinant Δ is grossly divided into 3 parts [9]:

i) One constant term '1',

ii) mp number of linear terms, $\left(-\sum_{\alpha} \ell_{\alpha(1)}{}^{single}\right)$,

iii) $(\sigma - mp)$ number of nonlinear multiplicative $\left(-\sum_{\alpha} \ell_{\alpha(1)}{}^{multi} + \sum_{\beta} \ell_{\beta(2)} + \sum_{\delta} \ell_{\delta(3)} + \cdots\right)$ terms on the variables $k_{11}, \cdots, k_{mp}$.

Hence the linear vector equation $\boldsymbol{Lk} = \boldsymbol{a}$ in (1.3) can be numerically constructed on the foundation of the 3 divisions of (2.2b) with multiplication of the open-loop characteristic polynomial $p(s)$, through the equality.

$$p_c(s) = p(s)\Delta = \boldsymbol{b}(s)\boldsymbol{Lk} \tag{2.3a}$$

Recall that in matrix fraction description (MFD) of transfer function $G(s) = D_L(s)^{-1}N_L(s)$, the closed-loop characteristic polynomial $p_c(s)$ is presented by

$$\begin{aligned} p_c(s) &= det\,[D_L(s) + N_L(s)K] = det\,[D_L(s)]\ det\,[I_p + G(s)K] \\ &= p(s)\ det\,[I_p + G(s)K] \\ &:= p(s)\ det\,[T(s)F] \end{aligned} \tag{2.3b}$$

where $T(s) = [I_p \; G(s)] \in \mathbf{R}(s)^{p \times (m+p)}$, and $F = [I_p \; K]^t \in \mathbf{R}^{(m+p) \times p}$. Applying "Binet-Cauchy Theorem" to *det* $[T(s) \; F]$, the loop determinant Δ in (2.3b) is re-written by

$$det\,[T(s)\;F] = 1 + \sum_{(j,i)=(1,1)}^{(p,m)} G_{ji}(s)k_{ij} + \sum_{i=1}^{\binom{m}{2}\times\binom{p}{2}} (i\text{-th } 2\times 2 \text{ minor of } G(s))(\text{corresp. } 2\times 2 \text{ minor of } K)$$

$$+ \cdots + \sum_{i=1}^{\binom{\max(m,p)}{z}} (i\text{-th } z\times z \text{ minor of } G(s))(\text{corresp. } z\times z \text{ minor of } K) \quad (2.4)$$

where $z := \min(m, p)$, and it is obtained from the equivalency (2.3a) = (2.3b) that

$$\Sigma\,(i\text{-th } 2\times 2 \text{ minor of } G(s)) \cdot (\text{corresp. } z\times z \text{ minor of } K) + \cdots + \Sigma\,(i\text{-th } z\times z \text{ minor of } G(s))$$

$$\cdot\,(\text{corresp. } z\times z \text{ minor of } K) = -\sum \ell_{\alpha 1}{}^{multi} + \sum \ell_{\beta 2} + \sum \ell_{\delta 3} + \cdots, \quad (2.5)$$

From (2.4) and (2.5), we can numerically construct the SOF vector equation $\boldsymbol{Lk} = \boldsymbol{a}$ by filling the ingredients like Fig.2, according to the descending orders of a, where l = 2, ... , z (= min{m, p}).

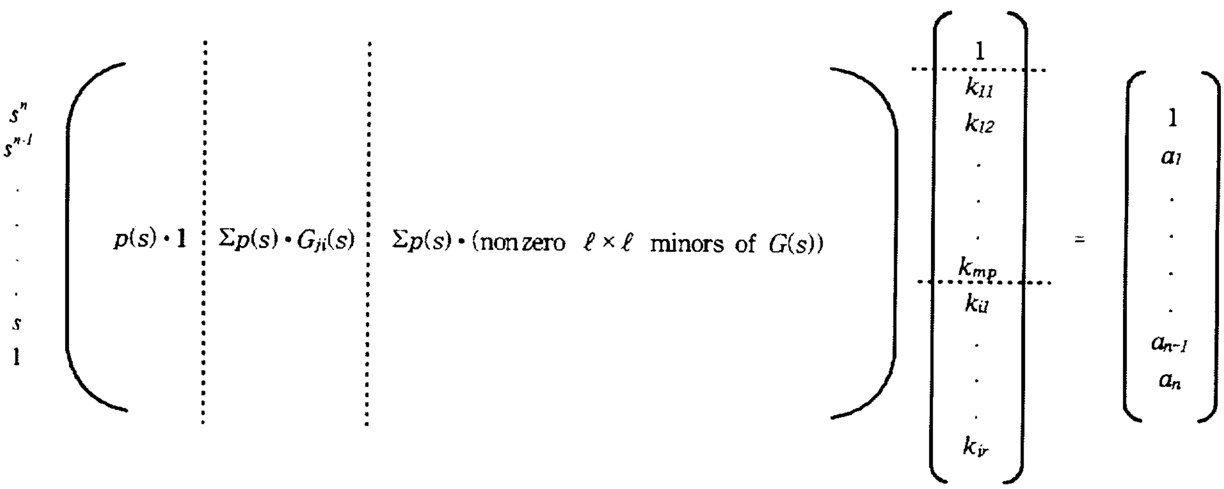

Fig. 2. Internal ingredients of $Lk = a$

Remark 2 : From (2.5), the "interacting factors" $k_{i1}, \cdots, k_{ir}$ of $\boldsymbol{k}$ formulate the inhomogenized arbitrary-order nonlinear equations (*NEs*) in equality forms

$$k_{i1} = \begin{vmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{vmatrix}, \quad k_{i2} = \begin{vmatrix} k_{11} & k_{13} \\ k_{21} & k_{23} \end{vmatrix}, \quad \cdots, \quad k_{ir} = \begin{vmatrix} k_{m-p\cdot 1,1} & k_{m-p\cdot 1,2} & \cdots & k_{m-p\cdot 1,p} \\ k_{m-p\cdot 2,1} & k_{m-p\cdot 2,2} & \cdots & k_{m-p\cdot 2,p} \\ \vdots & \vdots & \ddots & \vdots \\ k_{m1} & k_{m2} & \cdots & k_{mp} \end{vmatrix}$$

where $r = \sigma - mp$ in $m \geq p$ systems. In [9, remark 2], it is also shown that these *NEs* are transformed into inhomogenized quadratic equations (*QEs*). In other words, it is exposed that these inhomogenized *NEs* (or *QEs*) are localized formulas of the so-called, *homogeneous* quadratic Plücker relations (*QPRs*) in $\boldsymbol{k} = [k_{(0)} \; k_{(1)} \; \ldots \; k_{(mp)} \; k_{(mp+1)} \; \ldots \; k_{(\sigma)}]^t$, through specifying (i.e., inhomogenizing) the SOF loops in Δ of Mason's formula in (2.1).

3. Illustrations

Example 1. Consider a strictly proper system given by

$$A=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&1&0\end{bmatrix},\quad B=\begin{bmatrix}0&0\\1&0\\0&0\\0&1\end{bmatrix},\quad C=\begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix}$$

Step 1 (*check pole-assignability (P-A) in* $\boldsymbol{Lk}$ = $\boldsymbol{a}$). $G(s)$ (= $C(sI - A)^{-1}B$) is obtained by

$$G(s) = \begin{bmatrix}\dfrac{s^2-1}{s^4-s^2-1} & \dfrac{1}{s^4-s^2-1}\\ \dfrac{s^3-s}{s^4-s^2-1} & \dfrac{s}{s^4-s^2-1}\end{bmatrix}$$

From Fig. 2, $\boldsymbol{Lk} = \boldsymbol{a}$ is constructed by

$$\left[\begin{array}{c|ccccc}1&0&0&0&0&0\\ \hline 0&0&1&0&0&0\\-1&1&0&0&0&0\\0&0&-1&0&1&0\\-1&-1&0&1&0&0\end{array}\right]\begin{bmatrix}1\\k_{11}\\k_{12}\\k_{21}\\k_{22}\\k_t\end{bmatrix}=\begin{bmatrix}1\\a_1\\a_2\\a_3\\a_4\end{bmatrix}$$

L_{sub}

without constraint of $k_t - k_{11}k_{22} + k_{21}k_{12} = 0$. In the rank test, $rank(L_{sub})$ = 4, and the last column of L_{sub} is zero.

Step 2 (*computation of K*). From arbitrary desired pole positions of (s+1)(s+1)(s+2)(s+2) = 0, the real coefficients of the closed-loop characteristic polynomial $p_c(s)$ are obtained by a_1 = 6, a_2 = 13, a_3 = 12, a_4 = 4.

From $rank(L_{sub})$ = 4, the row-reduced unity diagonal form $L_{sub}'\boldsymbol{k}_{sub}' = \boldsymbol{a}_{sub}'$ is obtained by

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}k_{11}\\k_{12}\\k_{21}\\k_{22}\end{bmatrix}=\begin{bmatrix}14\\6\\19\\18\end{bmatrix}$$

From (3.2), k_t is calculated with k_t = 14 x 18 - 6 x 19 and the real solution K (for negative feedback law, $U(s) = -KY(s)$) is directly obtained by

$$K=\begin{bmatrix}k_{11}&k_{12}\\k_{21}&k_{22}\end{bmatrix}=\begin{bmatrix}14&6\\19&18\end{bmatrix}$$

Remark 3: This example was given in [10] to demonstrate an eigenvalue-generalized eigenvector assignment over some multiple eigenvalues, under necessary and sufficient condition of eigenstructure assignment by real SOF. In our Grassmannian parametrization method within $L_{sub}'\boldsymbol{k}_{sub}' = a_{sub}'$, it is revealed that this system has intrinsically the exact pole-assignment (EPA) feature over any closed-loop poles as rank-one system, and whose real SOF gains are algebraically computable in deterministic way.

4. Conclusions

In this paper, the numerical construction algorithm of Plücker matrix form $\boldsymbol{Lk} = \boldsymbol{a}$ is presented for modeling SOF linear systems in real Grassmann space. It is also illustrated how the pole-assignment problem of a 2-input, 2-ouput, 4th order linear system by real SOF gains can be completely parametrized in real Grassmann space.

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