The Duality between 0–dimensional Spaces and 2–regular Semigroups

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0－차원 위상공간과 2－정규 반군과의 관계

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Summary

We introduce 2–regular semigroup and endow p–topology on it, and we have two functor H and C. We prove that π and p are H, C–universal map and H(Y, C(X)) is topological isomorphic C(X, H(Y)) for any 2–regular semigroup X and 0–dimensional space Y.

I. Introductin

It is well known ([4]) that a compact (realcompact, resp.) space X can be completely determined by the homomorphisms on the ring $C^*(X, R)$ (C(X, R), resp.).

In this paper, we will introduce a concept of 2–regular semigroup and show that C(X) is 2–regular for any topological space. Next, we are concerned with the analogous problem between 0–dimensional spaces and 2–regular semigroups

1. 2–regular semigroups

In this section, we can introduce the concepts of **prime ideal and 2–regular semigroup** which will be used throught this thesis.

Definition 1.1. A proper ideal I of a semigroup is said to be prime if whenever $xy \in I$, $x \in I$ or $y \in I$.

Let $2=\{0,1\}$ be the two point semigroup such that $xy=x$ if $x=y$ and $xy=0$ if $x \neq y$ for any $x, y \in 2$, then $\{0\}$ is the unique prime ieal of 2. For any semigroup X, let's denote P(X) for the set of all prime ideals of X, ϕ and X. Then there is an one-to-one correspondence between H(X), the set of all homomorphisms on X into 2, and P(X). Let $T: H(X) \rightarrow P(X)$ be defined by $T(f)=f^{-1}(0)$ for any $f \in H(X)$ and define $G: P(X) \rightarrow H(X)$ by G(I) the characteristic function for $\complement I$, the complement of I, for any $I \in P(X)$. Then $T \cdot G = 1_{P(X)}$ and $G \cdot T = 1_{H(X)}$. In the fllowing, we may assume that H(X)=

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P(X), i.e., $f=f^{-1}(0)$ for any $f\in H(X)$, for any semigroup X. Moreover, for any f, $g\in H(X)$, $fg=(fg)^{-1}(0)=f^{-1}(0)\cup g^{-1}(0)$. Hence $\phi=\underline{1}$, $\underline{1}(x)=1$ for any $x\in X$, is the identity and $X=\underline{0}$, $\underline{0}(x)=0$ for any $x\in X$, is the zero element of H(X), and any element except ϕ has no inverse.

Proposition 1.2. Let X be a semigroup and $I\subset X$, then I is an ideal and $\complement I$ is a subsemigroup of X if and only if I is a prime ideal of X.

Proof. Using the contraposition of the definition 1.1, we have an equivalence statement.

Proposition 1.3. Let X and Y be semigroups, $f:X\rightarrow Y$ a homomorphism and let J be a prime ideal of Y, then $f^{-1}(J)$ is a prime ideal of X.

Proof. Clearly $J=g$ for some $g\in H(Y)$. Hence $g\cdot f\in H(X)$ and $f^{-1}(J)=g\cdot f$. Thus $f^{-1}(J)$ is prime.

Definition 1.4. We say that a semigroup X is 2-regular if the family H(X) is a $\underline{SG}$ mono-source, equivalently, for any $x,y\in X$ with $x\neq y$, there is a prime ideal I of X such that I contains either x or y.

In the following, $\underline{SG}$($\underline{2-reg}$) is the category of semigroups (2-regular semigroups, resp.) and homomorphisms. In the above definition, the concept that H(X) is a $\underline{SG}$ mono-source means that $H(X)\subset\underline{SG}$ and whenever $x\neq y$ in X, there is a $f\in H(X)$ with $f(x)\neq f(y)$, i.e., $f(x)=f(y)$ for all $f\in H(X)$ implies $x=y$.

Theorem 1.5. Let X be a semigroup and let $(X_i)_{i\in I}$ be a family of 2-regular semigroups. If $\{f_i:X-X_i \mid f_i$ is a homomorphism, $i\in I\}$ is a $\underline{SG}$ mono-source, then X is a 2-regular semigroup.

Proof. Take any $x,y\in X$ with $x=y$. Since $\{f_i \mid i\in I\}$ is a $\underline{SG}$ mono-source, $f_j(x)\neq f_j(y)$ for some $j\in I$. Since X_j is 2-regular, $g(f_j(x))\neq g(f_j(x))$ for some $g\in H(X_j)$, clearly $f\cdot g_j\in H(X)$, and hence H(X) is a $\underline{SG}$ mono-source, i.e., X is 2-regular.

Corollary 1.6. 1) $\underline{2-reg}$ is productive and hereditary.

2) Let X be a Hausdroff topological space and $2=\{0,1\}$ be the two points discrete space and let C(X) be the set of all continuous function of X into 2. Let's define an associative binary operation on C(X) for any Hausdroff topological space X using the pointwise multiplication, i.e., for any f, $g\in C(X)$ and $x\in X$, $(fg)(x)=f(x)g(x)$. For any $x\in X$, define a map $\pi_x: C(X)\rightarrow 2$ by $\pi_x(f)=f(x)$ for any $f\in C(X)$, then $\{\pi_x \mid x\in X\}$ is a $\underline{SG}$ mono-source. Hence for any Hausdroff topology X, C(X) is a 2-regular semigroup.

Lemma 1.7. Let X be a semigroup, then the following are equivalent:

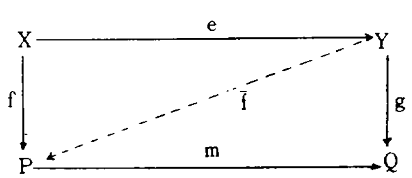
1) X is 2-regular.

2) There is a $\underline{SG}$ mono-source $(f_i:X\rightarrow 2)_{i\in I}$, where I is an index set.

3) X is isomorphic with a subsemigroup of a power of 2.

4) For any $x,y\in X$ with $x\neq y$, there is a prime ideal I of X such that I contains either x or y.

Lemma 1.8. For any semigroups X, Y, P and Q, let $e:X\rightarrow Y$ be an onto homomorphism, $f:X\rightarrow P$ and $g:Y\rightarrow Q$ are homomorphisms and $m:P\rightarrow Q$ is an one-to-one homomorphism with $g\cdot e=m\cdot f$, then there is a unique homomorphism $\bar{f}:Y\rightarrow P$ such that the following diagram commutes:



Proof. Note that if $K(e)=\{(x,y)\in X\times X \mid e(1x)=e(y)\}\subset K(f)=\{(a,b)\in X\times X \mid f(a)=f(b)\}$, then there exists a unique homomorphism $\bar{f}: Y\to P$ with $\bar{f}\cdot e=f$ by the Indeced Homomorphism Theorem Hence it suffices to show that $K(e)\subset K(f)$.
Take any $(x,y)\in K(e)$, then $e(x)=e(y)$, and hence $g(e(x)).=g(e(y))$ i.e., $m(f(x))=m(f(y))$. Since m is ont-to-one, $f(x)=f(y)$: $(x,y)\in K(f)$.

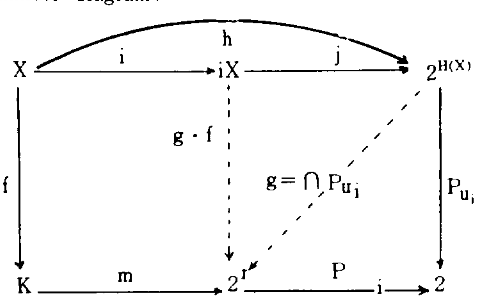
Theorem 1.9. 2-reg is epireflective on SG, i.e., for any semigroup, there is an onto homomorphism $i: X\to iX$ such that

1) iX is 2-regular; and

2) for any homomorphism $f: X\to K$, where K is a 2-regular semigroup, there is a unique homomorphism $\bar{f}: iX\to K$ with $\bar{f}\cdot i=f$.

Proof. Let $h=\cap H(X): X\to 2^{H(X)}$ be defined by $h(x)=Pr_x$, where $Pr_x(f)=f(x)$ for any $f\in H(X)$, then h is a homomorphism.

Let iX be the subsemigroup of $2^{H(X)}$ whose underlying set is h(X) and let i be the correstriction of h by h(X). Then clearly i is an onto homomorphism. Since 2-regular semigroup is productive and hereditary, and 2 is 2-regular, iX is a 2-regular semigroup. Now, take any homomouphism $f: X\to K$ such that K is a 2-regular semigroup, and hence there is a one-to-one homomorphism $m: K\to 2^I$ for some index set I, by Lemma 1.4. For sach $i\in I$, $Pr_i\cdot m\cdot f\in H(X)$, and let $u_i=Pr_i\cdot m\cdot f$, then $Pr_i\cdot h=u_i$. Consider a commute diagram:



(i ∈ I),

where j is the inclusion map on iX to $2^{H(X)}$ and $g=\cap Pu_i: 2^{H(X)}\to 2^I$ is the map with $P_i\cdot g=Pu_i$ for all $i\in I$. Since $(P_i)_{i\in I}$ is a SG mono-source and esch Pu_i is a homomorphism, g is a homomorphism. Thus $P_i\cdot g\cdot j\cdot i=Pu_i\cdot j\cdot i=P_i\cdot m\cdot f$ for all $i\in I$, and so $g\cdot j\cdot i=m\cdot f$, for $(P_i)_{i\in I}$ is a mono-source. By the above Lemma 1.8, there is a unique homomorphism $\bar{f}: iX\to K$ with $\bar{f}\cdot i=f$. This completes the proof.

Definition 1.10. For any semigroup X, $i: X\to iX$ or iX is called the 2-regular reflection of X.

If X is a 2-regular semigroup then i is an isomorhism, i.e., we consider $x=iX=\{P_x: H(X)\to 2;\ x\in X\}$. Now, define a functor i: SG → 2-reg as the follow: for any semigroup X, Y and Z and any homomorphisms $f: X\to Y$, and $g: Y\to Z$, define $f^i: iX\to iY$ by $f^i(P_x)=P_{f(x)}$ for any $x\in X$. Then for any x, $Y\in X$, $f^i(P_xP_y)=f^i(P_{xy})=P_{f(xy)}=P_{f(x)f(y)}=P_{f(x)}P_{f(y)}=f^i(P_x)f^i(P_y)$, $1_{iX}(P_x)=P_x$, and $(g\cdot f)^i(P_x)=P_{(g\cdot f)(x)}=P_{g(f(x))}=g^i(P_{f(x)})=g^i(f^i(P_x))=(g^if^i)(P_x)$. Moreover, we have

Theorem 1. 11. The functor i: SG → 2-reg is a full functor, i.e., when to every semigroups X and Y and to any homomorphism $g: iX\to iY$, there is homomorhism $f: X\to Y$ with $g=f^i$.

Proof. Since g is a homomorphism, for any $x\in X$ there is unique $y_x\in Y$ with $g(P_x)=P_y$. Define $f: X\to Y$ by $f(x)=y_x$, then for any $a,b\in X$, $g(P_{ab})=g(P_a P_b)=g(P_a)g(P_b)=P_{y_a}P_{y_b}=f(a)f(b)$: and hence f is a homomorphism. Moreover, for any $x\in X$, $f^i(P_x)=P_{f(x)}=Py_x=g(P_x)$: $f^i=g$.

2. Constructing a p-topology on semigroups

Let's endow topology on a semigroup X using

its prime iedals. Let $\& = \{J : J$ or $\not{c}J$ is a prime iedal of $X\}$, and let
$£ = \{ \&' ; \&'$ is a finite subfamily of $\&\}$. Then $X \in £$ and £ is closed under finite intersectin and $\& \subset £$, and hence £ is a base for a topology on X. Let σ be the Topology on X generated by it.once again, we will simply denote(X, σ) by X and we will say (X, σ) is the <u>p-topology</u> on a semigroup X with its prime ideal. In particular the two point semigroup 2 has a discrete space with its p-tokpology.

Proposition. 2.1. For any semigroup X and Y with p-topology, every homorphisms on X to Y is continuous.

Proof. It follows immediately from the proposition 1.3.

Theorem. 2.2. For any semigroup X with p-topology H(X) is an initial source, and if for any topological semigroup, X, H(X) is initial in
<u>Top</u> the qiven topolpgy on X coincides with the p-topology.

Proof. By the above proposition 2.1, every element of H(X) is continuous. Moreover, for any prime ideal J of X there exists a homomorphism g on X to 2 with $J = g^{-1}(0)$. let Y be a space and let $h : Y \to X$ be a map such that $f \cdot h$ is continuous for any $f \in H(X)$. Hence for any prime ideal J of X, $h^{-1}(J) = (g \cdot h)^{-1}(0)$ is clopen in Y, and so h is continuous. Thus H(X) is initial. Let X′ be the space on a semigroup X with p-topology. Then H(X′) is initial. Hence $f = 1_X : X \to X'$ is continuous.

Theorem 2.3. Every binary operation on a semigroup with p-topology is continuous.

Proof. We have a commute diagram:

$$\begin{array}{ccc} X \times X & \xrightarrow{\quad m_X \quad} & X \\ \Big\downarrow{\scriptstyle f \times f} & & \Big\downarrow{\scriptstyle f} \\ 2 \times 2 & \xrightarrow{\quad m_2 \quad} & 2 \end{array}$$

for any $f \in H(X)$, where m_X, m_2 are the binary operation on X and 2, respectively. By the above Theorem 2.2, H(X) is initial, m_X is continuous.

Proposition. 2.4. Every 2-regular semigroup X with p-topology is Hausdroff and vice versa.

Proof. Take any $x, y \in X$ with $x \neq y$. Then there is a $f \in H(X)$ with $f(x) \neq f(y)$. We may assume $f(x) = 1$ and $f(y) = 0$. Then $f^{-1}(0)$ and $f^{-1}(1)$ are disjoint open neigrhborhoods of x and y, resp.

Take any $x, y \in X$ with $x \neq y$. Then there exist open neighborhood U and V of x and y, resp. Hence there is a $I \in £$ with $x \in I$ but $y \in I$. We may assume I is a prime ideal of X. Hence $I = f^{-1}0)$ for some $f \in H(X)$, and hence $f(x) = f(y)$. Thus X is 2-regular.

Corollary 2.5. 1) Every 2-regular semigroup X with p-topology is a topological semigroup.

2) Take any prime ieal I of X and $x, y \in X$ with $xy \in \not{c}I$, then there are basic open neighborhood U and V of x and y, resp. with $UV \quad \not{c}I$.

Remark 2.6. 1) Every 2-regular semigroup X with p-topology is 0-dimensional, i.e., its space is Hausdroff and it has a base consisting of clopen subsets of X.

2) For any semigroup X, H(X) with p-topology is 0-dimensional compact semigroup.

Proof. 1) follow immediately frome the constructing. By 2.2. H(H(X)) is an initial monosource in Top, and hence we can consider H(X) as a subspace, subsemigroup with p-topology, of a power of 2, i.e., a 0-dimensional semigroup. Moreover, $H(X)=\cap K(x,y)$, where $K(x,y)=\{f=2^X; f(xy)=f(x)f(y)\}$. Since 2 is Hausdroff, K(x,y) is closed and hence H(X) is closed. Hence H(X) is a 0-dimensional compat semigroup.

3. Adjoint functor

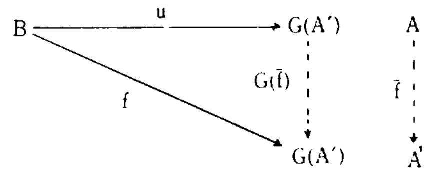
To each category C we also associate the opposite category C^{op}. The objects of C^{op} are the objects of C, the arrows of C^{op} are arrows f^{op} in one-to-one correspondence $f \to f^{op}$ with the arows of C. For each arrow $f: \to b$ of C, f^{op}: $b \to a$(the direction is reversed).The composite $f^{op} \cdot g^{op} = (gh)^{op}$ is defined in C^{op} exactly with the composite of defined in C. Let's denote o-dim for the category of all 0-dimensional spaces and all continuous maps.

From the section 1, 2, we have two functor

1) $H: 2\text{-}reg^{op} \to 0\text{-}dim$ defined by H(f): $H(X) \to H(Y)$ for any arrows $f: X \to Y$ in $2\text{-}reg^{op}$, where $H(f)(g)=g \cdot f^{op}$ for any $g \in H(X)$, and

2) $C: 0\text{-}dim \to 2\text{-}reg^{op}$ defined by $C(f): C(X) \to C(Y)$ for any arrows $f: X \to Y$ in 0-dim, where $C(f)(g)=g \cdot f^{op}$, for any $g \in H(X)$. The following definition is due to H. Herrilich [2].

Definition 3.1. Let $G: \mathcal{A} \to \mathcal{B}$ be a functor and let $B \in Ob(\mathcal{B})$, a pair (u, A) with $A \in Ob(\mathcal{A})$ and $u: B \to G(A)$ is called a universal map for B with respect to G(or a G-universal map for B) if for each $A'=Ob(\mathcal{A})$ and each $f: B \to G(A')$ there exists a uniqu $\mathcal{A}$-morphism $\bar{f}A \to g(A')$ such that the triangle



commutes.

Lemma 3.2. Let X be a topological space and let π be defined by $\pi(x)=\pi_x$ for any $x \in X$. Then X is 0-dimensional if and only if π is one-to-one.

Proof. Clearly a topological space X is 0-dimensional if and only if C(X) is a mono-source. Take any $x,y \in X$with $\pi(x)=\pi(y)$. Then $f(x)=f(y)$ for any $f \in C(X)$, thus if X is 0-dimensional $x=y$, conversely,take any $x,y \in X$ with $f(x)=f(y)$ for any $f \in C(X)$. Then $\pi(x)=\pi(y)$. Since π is one-to-one, $x=y$; X is 0-dimensional.

A functor $T: C \to B$ is faithful hen to every pair X, Y of Ob(C) and to every pair $f,g: X \to Y$ of mor(C) the equality $T(f)=T(g): T(x) \to T(Y)$ implies $f=g$. (mac Lane [3]).

Lemma 3.3. The functor H and C are faithful.

Proof. Let $f,g: Y \to X$ be in 2-reg with $H(f)=H(g): H(X) \to H(Y)$. Then $h \cdot f=h \cdot g$ for any $h \in H(X)$. suppose that $f \neq g(x)$ gor some $x \in X$. Since X is 2-regular, there is a $k \in H(X)$ with $k(f(x))=k(g(x))$. This contradicts the fact $h \cdot f=h \cdot g$ for any $h \in H(X)$. Hence H is faithful. The case C is similar.

Theorem 3.4. π is a H-universal map.

Proot. Take any $Y \in Ob(2\text{-}reg)$, and let $f: X \to H(Y)$ be in 0-dim. Define $\bar{f}: Y \to C(X)$ by $\bar{f}(y)(x)=f(x)(y)$ kfor any $x \in X$ and $y \in Y$. On the otherhand, the map $p: Y \to C(H(Y))$ defined by

$P(y)=P_y$ for any $y \in Y$, where $P_y(f)=f(y)$ for any $f \in H(Y)$, is well-defined, for every homomorphisms between semigroups with p-topology is continuous.

Now, let $y \in Y$ be fixed and take ay $x \in X$. Then clearly $\bar{f}(y)(x)=f(x)(y)=P_y(f(x))=(P_y \cdot f)(x)$, and so $\bar{f}(y)^{-1}(i)=(P_y \cdot f)^{-1}(i)$ for any $i=0, 1$. Thus $\bar{f}(y)$ is continuous on X to 2. And let $a, b \in Y$ and $x \in X$, then $\bar{f}(ab)(x)=f(x)(ab)=f(x)(a)f(x)(b)=[\bar{f}(a)\bar{f}(a)](x)$, thus $\bar{f}$ is a homomorphism. In all, $\bar{f}$ is well-defined. Moreover, for any $x \in X$ and $y \in Y$ $[H(\bar{f}) \cdot \pi](x)(y)=[H(\bar{f})(\pi_x)](y)$

$=(\pi_x \cdot \bar{f})(y)=\bar{f}(y)(x)$; $H(\bar{f}) \cdot \pi=f$.

From the above Lemma 3.2, $H(\bar{f})$ is unique. Once again, from the above lemma 3.3, $\bar{f}$ is unique. In all, π is a H-universal map.

Corollary 3.5. Let X Ob(2-reg) and let $P: X \rightarrow C(H(X))$ be defined by $P(y)=P_y$ for any $x \in X$, where $P_y(f)=f(x)$ for any $f \in H(X)$. Then P is a C-univeral map.

Remark 3.6. 1) From the above Theorem 3.4 and the above corollary 3.5, we have $H \dashv C$ and$C \dashv H$.

2) From the fact $H \dashv C$ and $C \dashv H$, we have: $\pi: H(X) \rightarrow H(C(H(X))$ and $P: C(Y) \rightarrow C(H(C)(Y)))$ are bijective for any $X \in$ 2-reg and $Y \in$ 0-dim, respectively. Since P is a homomorphism, P is an isomorphism in 2-reg. Moreover, π and π^{-1} are homomorphisms, π is a homeomorphism in 0-dim.

Theorem 3.7. For any 0-dimensional topological space X and 2-regular semigroup Y, C(X, H(Y)) is topological isomorphic with H((Y), C(X)).

Proof. Define a map $T: C(X, H(Y)) \rightarrow H(Y, C(X))$ by $T(f)(y)(x)=f(x)(y)$ for any $f \in C(X, H(Y))$, $x \in X$ and $y \in Y$. Then $T(f)=T(g)$ implies $f(x)(y)=g(x)(y)$ for any $x \in X$ and $y \in Y$. Then $T(f)=T(g)$ implies $f(x)(y)=g(x)(y)$ for any $x \in Y$, and so $f=g$, i.e., T is ont-to-one. Again we define another map $G: H(Y, C(X)) \rightarrow C(X, H(Y))$ by $G(g)(x)(y)=g(y)(x)$ for any $g \in H(Y, C(X))$, $x \in X$ and $y \in Y$. Then for any $g \in H(Y, H(X))$, $(T \cdot G)(g)(y)(x)=G(g)(x)(y)=g(y)(x)$; $T \cdot G=1_{H(Y, C(Y))}$. Similary $G \cdot T+1_{C(X, H(T))}$. Hence T is onto. Take $f, g \in C(X, H(Y))$, $x \in X$ and $y \in Y$, then $T(fg)(y)(x)=(fg)(x)(y)=f(x)(y)g(x)(y)=[T(f)T(g)](y)(x)$; $T(fg)= T(f)T(g)$. Thus T is a homomorphism. Thus T is a semigroup isomorphism.

Clearly G is also a homomorphism. By the proposition 2.1, T and G are ccntinuous. Thus C(X, H(Y)) is topological isomorphic with H(Y, C(X)).

Let 1 be the trivial singleton simigroup, then $H(1)=\{0,1\}$ and it has the discrete topclogy as a p-topology. Moreover, 1 id 2-regular. Hence we have.

Corollary 3.7. For any 0-diminsional space X, C(X) is a compact.

Proof. Clearly C(X) is topological isomorphic with H(1, C(X)). By the similar method of Remark 2.6, H(1, C(X))[C(X)]. By the similar method of remark 2.6, H(1, C(X))[C(X)] is closed subspace of a power of 2. Hence C(X) is compact.

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국 문 초 록

0–차원 위상공간과 2–정규 반군과의 관계

이 논문에서는 2–정규 반군을 도입하고 그 위에 p–위상공간을 부여하여 두 functor H, C을 도입하였다. 그리고 π, p는 각각 H, C–universal map임을 보이고, H(Y. C(x))와 C(X. H(y)는 위상적 동형관계에 있음을 보였다.