Asymptotic Stability in Functional Differential Equations

Bang Eun-sook, Koh Youn-hee***

함수 미분 방정식에서의 점근 안정성

방은숙*, 고윤희**

1. Introduction

This paper is concerned with the asymptotic stability of certain functional differential equations. The equation is investigated by means of Lyapunov's direct method.

In this discussion, $(C, \| \ \|)$ is the Banach space of continuous functions $\phi : [-h,0] \to R^n$, $\|\phi\| = \sup_{-h \le s \le 0} |\phi(s)|$, and $| \ |$ is any convenient norm in R^n. For a positive constant H, C_H denotes the set of $\phi \in C$ with $\|\phi\| < H$.

If $x : [t_0-h, T) \to R^n$ $(0 \le t_0 < T \le \infty)$ is continuous and $t \in [t_0, T)$, we define $x_t(s) = x(t+s)$ for $s \in [-h, 0]$. Let $x'(t)$ denote the right-hand derivative at x if it exists and is finite.

Consider the system

$$\text{(A)} \qquad x'(t) = F(t, x_t)$$

where $F : R_+ \times C_H \to R^n$ is continuous, $R_+ \equiv [0, \infty)$ and takes bounded sets into bounded sets. It is then known [4] that for each $t_0 \in R_+$ and each $\phi \in C_H$ there is at least one solution $x(t_0, \phi)$ satisfy (A) on an interval $[t_0, t_0+\alpha)$ with $x_{t_0}(t_0, \phi) = \phi$ and with a value at t denoted by $x(t, t_0, \phi)$. Moreover, if there is an $H_1 < H$ and if $|x(t, t_0, \phi)| \le H_1$ for all $t \ge t_0$ for which $x(t_0, \phi)$ can be defined, then $\alpha = \infty$.

Generalizing Lyapunov's classical stability theory on an ordinary differential equations to functional differential equations, Krasovskii [5] replaced the Lyapunov function $V : R_+ \times R^n \to R$, with a continuous functional $V : R_+ \times C_H \to R$, whose derivative V' with respect to (A) was defined by

$$V'(t, \phi) = \lim_{\delta \to 0^+} \sup [V(t+\delta, x_{t+\delta}(t, \phi)) - V(t, \phi)] / \delta .$$

Throughout this disscussion we work with wedges, denoted by W_i, which are continuous

* 자연과학대학 수학과

** Dept. of Math. Memphis State Univ.

functions from R_+ to R_+, which are strictly increasing, and which are satisfy $W_i(0)=0$. These wedges are related to properties of the Lyapunov functionals $V: R_+ \times C_H \to R$. We suppose that $F(t,0)\equiv 0$ so that $x=0$ is a solution of (A) and is called the *zero solution*.

1.1 Definition.

(a) The zero solution of (A) is *stable* if for each $\epsilon>0$ there exists $\delta>0$ such that $[\phi\in C_\delta$ and $t\geq t_0]$ imply $|x(t,t_0,\phi)|<\epsilon$.

(b) The zero solution of (A) is *asymptotic stable* if it is stble and if for each $t_0\geq 0$ there is a $\gamma=\gamma(t_0)>0$ such that $\phi\in C_\gamma$ implies that $|x(t,t_0,\phi)|\to 0$ as $t\to\infty$.

2. Main Results

2.1 Theorem Let $V: R_+\times C_H \to R_+$ be continuous and let τ be a measurable function from R_+ t_0 R_+ such that

$$\lim_{t\to\infty} \inf \int_t^{t+\epsilon} \tau(s)\,ds) > 0 \text{ for each } \epsilon>0$$

Suppose that there are wedges W_1, W_2 and W_3 such that the inequalities :

i) $0\leq V(t,\phi)\leq W_1(|\phi(0)|)+W_2(\|\phi\|)$

and

ii) $V'(t,\phi)\leq -\tau(t)W_3(|\phi(0)|)$

hold for all $t\in R_+$ and $\phi\in C_H$.

Then for every bounded solution $x:[t_0-h,\infty)\to R^n$ of (A), $\lim_{t\to\infty} V(t,x_t)=0$

Proof. Suppose not. Then there exist $T>0$ and $\epsilon>0$ such that $\epsilon\leq V(t,x_t)$ for all $t\geq T$.

This implies

$W_1(|x(t)|)\geq \frac{\varepsilon}{2}$ or

$W_2(\|x_t\|)\geq\frac{\varepsilon}{2}$ for any $t\geq T$.

case 1) $W_1(|x(t)|)\geq\frac{\varepsilon}{2}$ for any $t\geq T$.

By inequality ii), $\int_{t_0}^t V'(s,x_s)ds\leq -\int_{t_0}^t \tau(s)W_3(|x(s)|)ds$ holds. Therefore we have

$$V(t,x_t)-V(t_0,\phi)\leq -\int_T^t \tau(s)W_3(W_1^{-1}(\frac{\varepsilon}{2}))ds,$$

and hence

$$V(t,x_t)\leq V(t_0,\phi)-\int_T^t \tau(s)W_3(W_1^{-1}(\frac{\varepsilon}{2}))ds,$$

Now the right hand side of this inequality approaches to $-\infty$ as $t\to\infty$. This a contradiction to $V(t,\phi)\geq 0$.

case 2) $W_2(\|x_t\|)\geq\frac{\varepsilon}{2}$ for all $t\geq T$.

Since W_2 is strictly increasing, $\|x_t\|\geq\delta$, where $\delta\equiv W_2^{-1}(\frac{\varepsilon}{2})$. Now we note that each interval of length h contains an s such that $|x(s)|\geq\delta$.

Thus there exist a sequence $\{t_n\}\uparrow\infty$ as $n\to\infty$ such that for each $n=1,2,\cdots$,

$$T+(2n-1)h\leq t_n\leq T+2nh \text{ and } |x(t_n)|\geq\delta.$$

On the other hand, by the assumption on F, there exists a constant L such that

$$|x'(t)| < L \text{ for all } t\geq T.$$

Then

$$t_n-\frac{\delta}{2L}\leq t\leq t_n-\frac{\delta}{2L}.$$

So we have

$$|x(t)| \geq \frac{\delta}{2}.$$

That is,

$$\int_{t_0}^{t} V'(s,x_s)\,ds \leq -\int_{t_0}^{t} \tau(s) W_3(|x(s)|)\,ds$$

$\leq -\sum_{n=1}^{\infty}\int_{I_n} \tau(s) W_3(|x(s)|)\,ds$, where $I_n = [t_n - \frac{\delta}{2L},\ t_n + \frac{\delta}{2L}]$

$$\leq -W_3\left(\frac{\delta}{2}\right)\sum_{n=1}^{\infty}\int_{I_n} \tau(s)\,ds \to -\infty \text{ as } t\to\infty,$$

which is a contradiction. Hence we completes the proof.

2.2 Corollary Let $V: R_+ \times C_H \to R_+$ be continuous and let τ be a measurable function from R_+ to R_+ such that

$$\lim_{t\to\infty}\inf\int_{t}^{t+\varepsilon} \tau(s)\,ds > 0 \text{ for each } \varepsilon > 0$$

Suppose that there are wedges $W_i\,(i=1,2,3,4)$ and a constant K, where $0<K<H$ such that the inequalities :

i) $W_1(|\phi(0)|) \leq V(t,\phi) \leq W_2(|\phi(0)|) + W_3(\|\phi\|)$

and

ii) $V'(t,\phi) \leq -\tau(t) W_4(|\phi(0)|)$

hold for all $t \in R_+$ and $\phi \in C_K$.

Then the zero solution of (A) is asymptotically stable.

Proof. Clealy, there is a wedge W_5 with $V(t,\phi) \leq W_5(\|\phi\|)$, so the zero solution is stable ([4. Theorem 5.2.1]). Let $t_0 \in R_+$ be given and define $r = r(t_0) = r(K, t_0) > 0$ where $r(K, t_0)$ is chosen from the Lyapunov stability. Let $\phi \in C_r$. We will show that $V(t, x_t(\cdot\, t_0, \phi)) \to 0$ as $t\to\infty$, yielding $|x(t,t_0,\phi)| \to 0$ as $t\to\infty$. By way of contradiction, we have the similar proof of the preceding theorem.

3. An Example

Consider the scalar quation [2],[3]

(B) $\quad x'(t) = -a(t)x(t) + b(t)x(t-h)$

where $a, b: R_+ \to R$ are continuous such that for $t \in R_+$

$$a(t) \geq (1+K)|b(t+h)| \text{ for some } K>0$$

Assume that $\tau(t) = |b(t+h)|$ has a property such that

$$\lim_{t\to\infty}\inf\int_{t}^{t+\varepsilon} |b(s)|\,ds > 0 \text{ for each } \varepsilon > 0$$

and

$$\int_{t-h}^{t} |b(u+h)|\,du \leq C \text{ for some } C>0$$

Then the zero solution of (B) is asymptotically stable.

Proof. Consider the Lyapunov functional

$$V(t,x_t) = |x(t)| + \int_{t-h}^{t} |b(u+h)|\,|x(u)|\,du$$

Then

$$V(t,x_t) \le |x(t)| + \int_{t-h}^{t} |b(u+h)||x(u)|du$$
$$\le |x(t)| + \int_{t-h}^{t} |b(u+h)| \|x_t\| du$$
$$\le |x(t)| + C\|x_t\|$$

Let $W_1(t) = W_2(t) = W_3(t) = t$.
We have

$$W_1(|x(t)|) \le V(t,x_t) \le W_2(|x(t)|) + W_3(C\|x_t\|)$$

Moreover,

$$V'(t,x_t) = (|x(t)|)' + |b(t+h)||x(t)| - |b(t)||x(t-h)|$$
$$= -a(t)|x(t)| + |b(t)||x(t-h)| + |b(t+h)||x(t)| - |b(t)||x(t-h)|$$
$$\le -a(t)|x(t)| + |b(t+h)||x(t)|$$
$$= -[a(t) - |b(t+h)|]|x(t)|$$
$$\le -K|b(t+h)||x(t)|$$

Let $W_4(t) = Kt$. Then

$$V'(t,x_t) \le -\tau(t) W_4(|x(t)|)$$

By the corollary 2.2, the zero solution of (B) is asymptotically stable.

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〈國文抄錄〉

함수 미분 방정식에서의 점근 안정성

본 논문에서는 어떤 함수 미분방정식의 해, 영의 점근적 안정성을 직접적인 Lyapunov 방법을 이용하여 연구하고 그 예를 하나 들었다.