Analysis and Improvement of Steady–State Response in LQ Optimal Control of Discrete–Time Systems

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이산시간 시스템의 LQ 최적제어에 있어서 정상상태 응답의 해석 및 개선

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Introduction

The dynamics of many physical systems in engineering can be modeled by the state equations. Optimal control is an important class of modern control theory based on the state-space model (Willems, 1983 and Johnson et al., 1987). Let's consider the following linear quadratic(LQ) tracking problem of a time-invariant discrete-time system;

$$x(k+1)=Ax(k)+Bu(k)+c,\quad x(0)=x_o, \qquad (1)$$

$$J=\frac{1}{2}\sum_{k=0}^{\infty}\{\|x(k)-x^d\|_Q^2+\|u(k)\|_Q^2\} \qquad (2)$$

where A is an n×n system matrix, B an n×m input matrix, c an n×1 constant input vector, x^d an n×1 constant target of the state vector, Q≥ 0 an n×n state weighting matrix and R>0 an m×m input weighting matrix.

Assumption 1 : (A, B) and (D, A) in eqns. (1) and (2) are stabilizable and detectable pairs, respectively, where $Q=D^TD$.

As is well known, the steady-state solution (Singh et al., 1978 and Friedland, 1986) to this problem is given under Assumption 1 as follows :

$$u(k)=Gx(k)+d \qquad (3)$$

where G and d are given by

$$G=-R^TB^TA^{-T}(K-Q), \qquad (4)$$

$$d=-R^{-T}B^TA^{-T}(s+Qx^d), \qquad (5)$$

Here K and s are the solutions to the discrete Riccati equation (DRE) and tracking equation (TE), respectively, given by

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$$K = Q + A^TK(I_n + BR^{-T}B^TK)^{-1}A, \quad (6)$$

$$s = -Qx^d - A^TK(I_n + BR^{-T}B^TK)^{-1}(BR^{-1}B^Ts - c) + A^Ts. \quad (7)$$

Analysis of Steady-State Tracking Error

The following theorem gives the steady-state tracking error of the above optimal tracking algorithm.

Theorem 1 : The optimal control algorithm, eqn. (3) for the LQ tracking problem has the steady-state tracking error given by

$$e_{ss} = \{I_n - A + BR^{-1}B^T(I_n - A^T)^{-1}Q\}^{-1}\{(I_n - A)x^d - c\}. \quad (8)$$

Proof : Substituting the optimal solution, eqn. (3) into the state equation, (1), we obtain the following equation.

$$x(k+1) = Fx(k) + h \quad (9)$$

where F and h are defined as

$$F = A + BG, \quad (10)$$

$$h = Bd + c. \quad (11)$$

The solution of eqn. (9) is given as

$$x(k) = F^k x(0) + \sum_{i=0}^{k-1} F^i h. \quad (12)$$

If k is large enough for the system to reach a steady-state, eqn. (12) can be expressed as follows

$$x(k) = F^k x(0) + (I_n - F)^{-1}h. \quad (13)$$

Define the state at steady-state as

$$x_s \equiv \lim_{k\to\infty} x(k). \quad (14)$$

Since F in eqn. (13) is asymptotically stable matrix, eqn. (13) can be expressed as

$$x_s = (I_n - F)^{-1}h. \quad (15)$$

Therefore, we can see that the state at steady-state is a constant vector and the control input at steady-state is also a constant vector from eqn. (3). Hence we obtain the followings from the necessary conditions for optimality (Middleton et al., 1990)

$$x_s = Ax_s + Bu_s + c, \quad (16)$$

$$u_s = -R^{-1}B^Tp_s, \quad (17)$$

$$p_s = Q(x_s - x^d) + A^Tp_s \quad (18)$$

where u_s and p_s are the steady-state control input and costate vector, respectively. From, eqns, (16), (17) and (18) we have

$$(I_n - A)x_s = -BR^{-1}B^T(I_n - A^T)^{-1}Q(x_s - x^d) + c. \quad (19)$$

Define the steady-state tracking error as

$$e_{ss} \equiv x^d - x_s. \quad (20)$$

Substituting eqn. (20) into eqn. (19) we can derive eqn. (8). This completes the proof.

Remark 1 :

(a) Theorem 1 reveals that the steady-state tracking error always exists and depends on Q and R unless $(I_n - A)\ x^d - c = 0$.

(b) An increase in $\|Q\|$ or a decrease in $\|R\|$ reduces the steady-state tracking error.

(c) The computational burden of the above optimal tracking algorithm is comparatively large.

In the following, we describe an optimal tracking algorithm which is advantageous over the conventional one in steady-state tracking error and computational burden.

Proposed Method

Let us take the performance index to reduce the steady-state tracking error as follow :

$$J_A=\frac{1}{2}\sum_{k=0}^{\infty}\{\|x(k)-x^d\|_Q^2+\|u(k)-u^n\|_R^2\} \quad (21)$$

where u^n is an $m\times 1$ pre-determined nominal control vector, which will be discussed later.

Define new state and control vectors as

$$z(k)\equiv x(k)-x^d. \quad (22\text{-a})$$
$$v(k)\equiv u(k)-u^n. \quad (22\text{-b})$$

Using eqns. (15-a) and (15-b) we can transform the tracking problem of eqns. (1) and (14) into the following regulator problem with constant input:

$$z(k+1)=Az(k)+Bv(k)+c_A, \quad (23)$$
$$J_A=\frac{1}{2}\sum_{k=0}^{\infty}\{\|z(k)\|_R^2+\|v(k)\|_Q^2\} \quad (24)$$

where

$$c_A=(A-I_n)x^d+Bu^n+c. \quad (25)$$

The following theorem gives rise to the steady-state tracking error for the proposed optimal control algorithm.

Theorem 2 : The steady-state tracking error for the proposed optimal control algorithm is given by

$$e_{ss}=-\{I_n-A+BR^{-1}B^T(I_n-A^T)^{-1}Q\}^{-1}c_A. \quad (26)$$

Proof : From the well known necessary conditions for optimality and Proof of Theorem 1, we have at the steady-state

$$z_s=Az_s+Bv_s+c_A, \quad (27)$$
$$v_s=-R^{-1}B^Tq_s, \quad (28)$$
$$q_s=Qz_s+A^Tq_s \quad (29)$$

where z_s, v_s and q_s are the steady-state state, control and costate of transformed regulator problem, respectively. From eqns. (26), (27) and (28) we obtain.

$$(I_n-A)\ z_s=-BR^{-1}B^T(I_n-A^T)^{-1}Q\ z_s+c_A. \quad (30)$$

Taking into account eqns. (20) and (22-a), we can derive eqn. (25) from eqn. (29). This completes the proof.

Corollary 1 : The sufficient condition for zero steady-state tracking error is that a vector $(I_n-A)x^d-c$ belongs to the column space of an input matrix B.

Proof : Corollary 1 can be proved directly from eqns. (25), (26) and Ref. (Brogan, 1974). Hence the proof is omitted.

Theorem 3 : Under the sufficient condition for zero steady-state tracking error the optimal control law is obtained as

$$u(k)=Gx(k)+d_A \quad (31)$$

where the compensation vector is defined as

$$d_A=-Gx^d+u^n. \quad (32)$$

Proof : Since c_A in transformed regulator problem of eqns. (23) and (24) is zero under the sufficient condition, the optimal control law of the transformed system is given by

$$v(k)=Gz(k) \tag{33}$$

where the feedback gain matrix G is obtained from eqns. (4) and (6). Substituting eqns. (22-a) and (22-b) into eqn. (33) we obtain eqns. (31) and (32). This completes the proof.

Remark 2 :

(a) Theorem 2 and Corollary 1 reveal that the steady-state tracking error does not exist regardless of Q and R if a vector $(I_n-A)x^d-c$ belongs to the column space of a input matrix B. In this case, the weighting matrices Q and R affect only on the transient responses.

(b) The optimal control law, eqns. (31) and (32) is now obtained without solving the TE, eqn. (7). Thus the computational burden is alleviated.

(c) If a vector $(I_n-A)\ x^d-c$ belongs to the column space of B, the nominal control input u^n is obtained by

$$u^n=(B^TB)^{-1}B^T\{(I_n-A)x^d-c\}. \tag{34}$$

(d) If a vector $(I_n-A)x^d-c$ doesn't belong to the column space of B, the nominal control input u^n obtained from eqn. (34) is a least-square approximate solution. In this case, the steady-state tracking error of the proposed method is given as

$$e_{ss}=\{I_n-A+BR^{-1}B^T(I_n-A^T)^{-1}Q\}^{-1}\{(I_n-B(B^TB)^{-1}B^T)[(I_n-A)x^d-c)\}. \tag{35}$$

Numerical Example

To illustrate the presented results we consider the river pollution model (Tamura, 1974 and Singh, 1975) of river Cam near Cambridge.

$$x(k+1)=Ax(k)+Bu(k)+c$$

where,

$$A=\begin{bmatrix} 0.18 & 0. & 0. & 0. \\ -0.25 & 0.27 & 0. & 0. \\ 0.55 & 0. & 0.18 & 0. \\ 0. & 0.55 & -0.25 & 0.27 \end{bmatrix}$$

$$B=\begin{bmatrix} -2.0 & 0. \\ 0. & 0. \\ 0. & -2.0 \\ 0. & 0. \end{bmatrix}$$

$$c=(4.5\quad 6.15\quad 2.0\quad 2.65)^T,$$

$$x_0=(0.\quad 0.\quad 0.\quad 1.0)^T.$$

Computer simulations are carried out for the following two cases and final time is chosen to 30 steps which are sufficiently long enough for the system to reach a steady-state.

Case I. $(I_n-A)x^d-c$ belongs to the column space of B; $x^d=(4.16\quad 7\quad 5.56\quad 7)^T$.

Case II. $(I_n-A)x^d-c$ does not belong to the column space of B; $x^d=(5\quad 7\quad 5\quad 7)^T$.

A summary of the simulation results (Kim et al., 1990) of both the conventional optimal tracking algorithm and the proposed one is given in Table 1.

The simulation results show that the steady-state tracking error of the proposed method is smaller than that of conventional method in both cases and these results are consistent with eqns. (8), (26) and (35). Especially the steady-state tracking error of the proposed method in case I does not exist irrespective of Q and R. These weighting matrices affect only on the transient responses.

Table 1. Summary of the simulation results

Method	Weighting Matrix		Steady-State Tracking Error	
	Q	R	Case I	Case II
Conventional Method	I_4	$50 I_2$	$[-1.13\ \ .39\ \ -.36\ \ -.41]^T$	$[-.34\ \ .40\ \ -.89\ \ .43]^T$
	I_4	$100 I_2$	$[-1.22\ \ .42\ \ -.45\ \ -.47]^T$	$[-.40\ \ .42\ \ -.99\ \ .47]^T$
	I_4	$500 I_2$	$[-1.30\ \ .45\ \ -.54\ \ -.52]^T$	$[-.47\ \ .45\ \ -1.09\ \ .52]^T$
Proposed Method	I_4	$50 I_2$	0.	$[0.\ \ .29\ \ 0.\ \ .02]^T$
	I_4	$100 I_2$	0.	$[0.\ \ .29\ \ 0.\ \ .02]^T$
	I_4	$500 I_2$	0.	$[0.\ \ .29\ \ 0.\ \ .02]^T$

Conclusion

In this paper, we describe an optimal tracking algorithm which is advantageous over the conventional one in steady-state tracking error and computational burden. Steady-state tracking error of both the proposed algorithm and the conventional one is derived analytically. As a results, the steady-state tracking error can be calculated by the given state equation and performance index without solving Riccati equation. Also, a sufficient condition for zero steady-state tracking error is presented.

A similar technique for the continuous-time systems is under study.

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摘 要

이산시간 시스템의 LQ 최적제어에 있어서 정상상태 추적오차를 해석적으로 유도하였으며, 그 결과 계산부담이 많은 Riccati 방정식을 풀지 않고도 주어진 상태방정식과 성능지수에 의해 정상상태 추적오차를 구할 수 있었다. 또한 추적문제를 조정기 문제로 변형함으로써 정상상태 추적오차 및 계산부담면에서 기존의 방법보다 우수한 방법을 제안하였으며, 수질오염 모델에 대한 컴퓨터 모사를 통하여 제안한 방법의 타당성을 확인하였다.