濟州大學校 白鹿論叢 第1輯(1999. 2. 25)

INTERIOR TRANSITION LAYERS OF NONCONSTANT SOLUTIONS FOR THE SCALAR GINZBURG-LANDAU EQUATION *

Bong-soo Ko**

1. Introduction

We study the existence of interior transition layers for classical nonconstant solutions of the following Neumann problem if $\epsilon > 0$ is small :

$$(I_\epsilon) \qquad \begin{cases} \epsilon^2 \triangle u + u(1-u^2) = 0 \ \text{ in } \ \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where $\frac{\partial u}{\partial n}$ denotes the outward normal derivative of u on $\partial\Omega$ and $\bar{\Omega} = \Omega \cup \partial\Omega$. Throughout this paper Ω will be open bounded convex domain of $\mathbf{R}^n$, $n \geq 2$, with $\partial\Omega \in C^2$. We call the above equation (I_ϵ) a scalar Ginzburg-Landau equation. The author proved the following existence theorem of classical nonconstant solutions of (I_ϵ) for a general open bounded domain Ω with the smooth boundary.

Theorem 1.1. *[1] There is a small number $\epsilon_0 > 0$ so that for any $0 < \epsilon < \epsilon_0$, (I_ϵ) has a nonconstant classical solution u_ϵ with the property $-1 < u_\epsilon(x) < 1$ for all $x \in \bar{\Omega}$. Furthermore, for any sequence $\{\epsilon_n\}$ of positive real numbers so that $\epsilon_n \to 0$ as $n \to \infty$, there is a subsequence $\{\epsilon_{n_k}\}$ such that*

$$\lim_{k\to\infty} u_{\epsilon_{n_k}}(x) = \pm 1 \quad \text{a.e.} \quad in \quad \bar{\Omega}.$$

---

* 1991 Mathematics Subject Classification: Primary: 35J20. Secondary: 35B40
This work was supported in part by Cheju National University, Abroad Research Fund, 1997
** 수학교육과 교수, 과학교육연구소 연구원

The mountain pass theorem is used to prove Theorem 1.1 in [1]. More precisely, we proved the existence of the nonconstant critical point u_ϵ of the following functional :

$$J_\epsilon(u) = \frac{\epsilon^2}{2}\int_\Omega |\nabla u(x)|^2 dx + \frac{1}{4}\int_\Omega (u^2(x)-1)^2\, dx$$

in the Sobolev space $W^{1,2}(\Omega)$, which is the nonconstant classical solution of the problem (I_ϵ). We denote by $W^{1,2}(\Omega)$ the space of functions in $L^2(\Omega)$ whose first order generalized partial derivatives belong to $L^2(\Omega)$.

We call the nonconstant critical point u_ϵ the nonconstant solution obtained by the mountain pass theorem. This means that

$$J_\epsilon(u_\epsilon) = \inf_h \sup_{t\in[0,1]} J_\epsilon(h(t))$$

over all continuous paths h from the interval $[0,1]$ into $W^{1,2}(\Omega)$ with $h(0) = -1$ and $h(1) = 1$. Furthermore, we proved that

$$\lim_{k\to\infty} J_{\epsilon_{n_k}}(u_{\epsilon_{n_k}}) = 0.$$

For the details of the above, we see [1].

Kohn & Sternberg [2], Dancer & Guo [3] proved the existence and the interior transition layers of the nonconstant classical solutions of (I_ϵ). Their solutions u_ϵ are local minimizers of the following functional K_ϵ :

$$K_\epsilon(u) = \frac{\epsilon}{2}\int_\Omega |\nabla u(x)|^2 dx + \frac{1}{4\epsilon}\int_\Omega (u^2(x)-1)^2\, dx$$

when the following functional K_0 :

$$K_0(u) = \frac{2\sqrt{2}}{3}\mathrm{Per}_\Omega\{u=1\}$$

has an isolated L^1-local minimizer u_0, where $\{u=1\} = \{x\in\Omega \,|\, u(x)=1\}$.

Here $\mathrm{Per}_\Omega\{u=1\}$ means the perimeter of the set $\{u=1\}$ in Ω. We will see the the definition of the perimeter in Section 2. They showed

$$\lim_{\epsilon\to 0} u_\epsilon = u_0$$

and the interior transition layer occurs in the thin neighborhood of the hypersurface

$$\partial\{u_0 = 1\}\cap\Omega,$$

where $\partial\{u_0 = 1\}$ is the boundary of the set $\{u_0 = 1\}$. The existence of isolated L^1-local minimizers of K_0 depends on the geometry of $\partial\Omega$. For examples, dumbbell-shaped domains may have such minimizers. But if Ω is convex, we cannot expect the existence of those minimizers.

So we have two questions from Theorem 1.1 when Ω is not dumbbell-shaped.

(1). Does the solution $u_{\epsilon_{n_k}}$ have an interior single transition layer as $\epsilon_{n_k} \to 0$?

(2). If so, where does the behavior occur ?

Sometimes, we call those two questions a free boundary value problem. In this paper, we answer the above questions. We prove that the nonconstant solution obtained by the mountain pass theorem has an interior transition layer under some assumptions on the convex domain Ω. We see Theorem 3.4 in Section 3 for the results.

Furthermore, we can construct the location of the interior transition layer of the nonconstant solution u_ϵ as $\epsilon \to 0$ using a method of the mountain pass type for the perimeter valued functional

$$I(h(t)) = \frac{2\sqrt{2}}{3}\mathrm{Per}_\Omega\{h(t) = 1\}$$

on all continuous characteristic paths h with $h(0) = -1$, $h(1) = 1$, and $h(t) = \pm 1$ in the space $L^1(\Omega)$ of all Lebesgue integrable functions from $\bar{\Omega}$ into $\mathbf{R}^1$. To show that, we need to use the concept of the perimeter theory for functions of bounded variation.

2. Functions of Bounded Variation and Γ-Convergence

We describe some of the basic definitions and properties of functions of bounded variation. Let $C^1(\Omega; \mathbf{R}^n)$ be the set of functions from Ω into $\mathbf{R}^n$ having continuous first partial derivatives in Ω, and let $C_0^1(\Omega; \mathbf{R}^n)$ be the set of functions in $C^1(\Omega; \mathbf{R}^n)$ with compact support in Ω.

For $u \in L^1(\Omega)$, we define

$$\int_\Omega |Du| := \sup_{g \in C_0^1(\Omega;\mathbf{R}^n), |g| \le 1} \int_\Omega u(x)(\nabla \cdot g(x))\, dx,$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$ for $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$. We define the space of functions of bounded variation, $BV(\Omega)$, consists in those $u \in L^1(\Omega)$ for which $\int_\Omega |Du| < \infty$; $BV(\Omega)$ is a Banach space under the norm [4,5,6];

$$||u||_{BV(\Omega)} = \int_\Omega |u(x)|\, dx + \int_\Omega |Du|.$$

We note that $|Du|$ is not an L^1-function, but rather the total variation of vector valued measure Du. Moreover, the Sobolev space $W^{1,1}(\Omega)$ is contained in $BV(\Omega)$ and $\int_\Omega |Du|$ equals, for $u \in W^{1,1}(\Omega)$, the ordinary Lebesgue integral $\int_\Omega |\nabla u(x)|\, dx$.

If $u \in BV(\Omega)$, the integral of any positive continuous function h with respect to the measure $|Du|$ can be expressed as

$$\int_\Omega h(x)|Du| = \sup_{g \in C_0^1(\Omega;\mathbf{R}^n), |g| \le h} \int_\Omega u(x)(\nabla \cdot g(x))\, dx.$$

An important example is the case when $u = \mathcal{X}_A$, the characteristic function of a subset A of $\mathbf{R}^n$. Then

$$\int_\Omega |Du| = \sup_{g \in C_0^1(\Omega;\mathbf{R}^n), |g| \le 1} \int_A (\nabla \cdot g(x))\, dx.$$

If this supremum is finite, A is called a set of finite perimeter in Ω. If A is smooth, then by the divergence theorem

$$\int_\Omega |Du| = \mathcal{H}^{n-1}(\partial A \cap \Omega),$$

where $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure. It is therefore natural to define the perimeter of any subset of Ω by :

$$\begin{aligned} \mathrm{Per}_\Omega A &= \text{perimeter of } A \text{ in } \Omega \\ &= \int_\Omega |D\mathcal{X}_A|. \end{aligned}$$

Without loss of generality, we denote

$$\mathrm{Per}_\Omega A = \mathcal{H}^{n-1}(\partial A \cap \Omega).$$

For the details of the above results and the following theorems, we see [4,5,6].

Theorem 2.1. *(Lower Semicontinuity)* *If $u_\epsilon \to u$ in $L^1(\Omega)$, then*

$$\int_\Omega |Du| \le \liminf_{\epsilon \to 0} \int_\Omega |Du_\epsilon|$$

Theorem 2.2. *(Compactness of BV in L^1)* *Bounded sets in the BV-norm are compact in the L^1-norm.*

Theorem 2.3. *(Co-area Formula) For any continuous function f and any Lipschitz continuous function h and any differentiable function g,*

$$\int_\Omega \left[|\nabla g(h(x))|^2 + f(h(x))\right] |\nabla h(x)|\, dx$$
$$= \int_{\mathbf{R}^1} \left\{ \left[\frac{d}{ds} g(s)\right]^2 + f(s) \right\} \mathcal{H}^{n-1}\{x \in \Omega : h(x) = s\}\, ds.$$

We can find the following definitions and theorems in [11]. Let X be a topological space. The set of all neighborhood of x in X will be denoted by $\mathcal{N}(x)$. Let $\{F_n\}$ be a sequence of functions from X into $\mathbf{R}^1 \cup \{-\infty, +\infty\}$. We define the Γ-lower limit of the sequence $\{F_n\}$ the function from X into $\mathbf{R}^1 \cup \{-\infty, +\infty\}$ by

$$(\Gamma - \liminf_{n\to\infty} F_n)(x) = \sup_{U \in \mathcal{N}(x)} \liminf_{n\to\infty} \inf_{y \in U} F_n(y).$$

Theorem 2.4. *Let $\{F_n\}$ be a sequence of functions from X into $\mathbf{R}^1 \cup \{-\infty, +\infty\}$. Then*

$$\Gamma - \liminf_{n\to\infty} F_n \le \liminf_{n\to\infty} F_n.$$

Let $\{E_n\}$ be a sequence of subsets of the topological space X. We define the K-upper limit of E_n, denoted by

$$K - \limsup_{n\to\infty} E_n,$$

is the set of all points $x \in X$ with the following property: for every $U \in \mathcal{N}(x)$ and for every natural number k there exists a natural number $n \ge k$ such that $U \cap E_n \ne \emptyset$.

Remark. By the definition

$$K - \limsup_{n\to\infty} E_n = \cap_{m=1}^{\infty} \overline{\cup_{n \ge m} E_n}.$$

Theorem 2.5. *Let $\{E_n\}$ be a sequence of subsets of X, $\{\mathcal{X}_{E_n}\}$ be the corresponding sequence of characteristic functions, and let*

$$E = K - \limsup_{n\to\infty} E_n.$$

Then

$$1 - \mathcal{X}_E = \Gamma - \liminf_{n\to\infty} (1 - \mathcal{X}_{E_n}).$$

Proof. Let

$$\Gamma - \liminf_{n\to\infty}(1 - \mathcal{X}_{E_n})(x) = \mathcal{X}_F(x).$$

Then if $\mathcal{X}_F(x) = 0$, then

$$0 = \sup_{U\in\mathcal{N}(x)} \liminf_{n\to\infty} \inf_{y\in U}(1 - \mathcal{X}_{E_n})(y),$$

and so for any $U \in \mathcal{N}(x)$,

$$\liminf_{n\to\infty} \inf_{y\in U}(1 - \mathcal{X}_{E_n})(y) = 0,$$

and hence $E_n \cap U \neq \emptyset$ for infinitely many indices n. Therefore, $x \in E$. Thus we have the following;

$$x \notin F \quad \text{implies} \quad x \in E.$$

We can prove the converse of the above statement by the same method.

This completes the proof.

3. Interior Transition Layers

Let H be the collection of all continuous paths h from $[0,1]$ into $L^1(\Omega)$ with $h(0) = -1$ and $h(1) = 1$. Let $B_\chi(\Omega)$ be the set of functions $v \in L^1(\Omega)$ such that $v(x) = \pm 1$ a.e. on Ω and let

$$H_\chi = \{h \in H \mid h(t) \in B_\chi(\Omega) \quad \text{for all} \quad t \in [0,1]\}.$$

We define the functional $I : [0,1] \times L^1(\Omega) \to \mathbf{R}^1 \cup \{\infty\}$ by

$$I(h) = \begin{cases} \sup\limits_{t\in[0,1]} \dfrac{2\sqrt{2}}{3}\mathrm{Per}_\Omega\{h(t) = 1\} & \text{if} \quad h \in H_\chi, \\ \infty & \text{otherwise.} \end{cases}$$

We also define the number γ_0 as follows:

$$\gamma_0 = \inf_{h\in H_\chi} I(h).$$

We note that $0 < \gamma_0 < \infty$. Next, we choose a sequence $\{h_n\}$ in H_χ and sequences $\{\alpha_n\}$ and $\{t_n\}$ of real numbers in $[0,1]$ such that

$$\mathrm{Per}_\Omega\{h_n(t_n) = 1\} + \alpha_n = \sup_{t\in[0,1]} \mathrm{Per}_\Omega\{h_n(t) = 1\},$$

$$\lim_{n\to\infty} \alpha_n = 0,$$

and

$$\lim_{n\to\infty} \frac{2\sqrt{2}}{3}\mathrm{Per}_\Omega\{h_n(t_n) = 1\} = \gamma_0.$$

Let

$$\Omega' = \cap_{m=1}^{\infty}\overline{\cup_{n\geq m}\{h_n(t_n) = 1\}}$$

and

$$\Gamma_0 = \partial\Omega' \cap \Omega.$$

Proposition.

$$\int_\Omega |D\mathcal{X}_{\Omega'}| = \frac{3}{2\sqrt{2}}\gamma_0$$

To show the equality, we need the following lemma. First of all, we let

$$E_m = \{h_m(t_m) = 1\}.$$

It follows from the definition that

$$K - \limsup_{n\to\infty} E_m = \Omega'.$$

Lemma.

$$\Gamma - \liminf_{m\to\infty}(1 - \mathcal{X}_{E_m}) = \liminf_{m\to\infty}(1 - \mathcal{X}_{E_m})$$

a.e. *in* Ω

Proof of Lemma. ¿From Theorem 2.4 and 2.5 it suffices to show that

$$\Gamma - \liminf_{m\to\infty}(1 - \mathcal{X}_{E_m}) \geq \liminf_{m\to\infty}(1 - \mathcal{X}_{E_m})$$

a.e. in Ω. Suppose that there is a Lebesque measurable set E whose Lebesque measure is positive such that

$$\liminf_{m\to\infty}(1 - \mathcal{X}_{E_m})(x) > \sup_{U\in\mathcal{N}(x)} \liminf_{m\to\infty} \inf_{y\in U}(1 - \mathcal{X}_{E_m})(y)$$

for all $x \in E$. Hence, for any $x \in E$

$$\liminf_{m\to\infty}(1 - \mathcal{X}_{E_m})(x) = 1$$

and

$$\Gamma - \liminf_{m\to\infty}(1 - \mathcal{X}_{E_m})(x) = (1 - \mathcal{X}_{\Omega'})(x) = 0.$$

Thus, x is in Ω' almost everyehere. Therefore, the Lebesque measure of $\Omega' \cap E$ is positive. But we have a natural number n such that for all $k \geq n$,

$$(1 - \mathcal{X}_{E_k})(x) = 1.$$

This means that $x \notin E_k$ for all $k \geq n$. From the definition of Ω', the Lebesque measure of $\Omega' \cap E$ is zero. This leads to a contradiction for the above result.

This completes the proof of Lemma.

The Proof of Proposition. By Theorem 2.1, 2.5 and the above Lemma, we note that

$$\int_\Omega |D\mathcal{X}_{\Omega'}| = \int_\Omega |D(1 - \mathcal{X}_{\Omega'})| \leq \frac{3}{2\sqrt{2}}\gamma_0.$$

To show the equality, we assume that

$$\int_\Omega |D\mathcal{X}_{\Omega'}| < \frac{3}{2\sqrt{2}}\gamma_0.$$

Then since $\liminf_{m\to\infty} \mathcal{X}_{E_m} = \mathcal{X}_{\Omega'}$, so

$$\sup_{g\in C_0^1(\Omega;\mathbf{R}^n),|g|\leq 1} \liminf_{m\to\infty} \int_{E_m} \nabla \cdot g(x)dx < \frac{3}{2\sqrt{2}}\gamma_0.$$

This implies that there is a positive number r such that

$$\liminf_{m\to\infty} \int_{E_m} \nabla \cdot g(x)dx \leq r < \frac{3}{2\sqrt{2}}\gamma_0$$

for all $g \in C_0^1(\Omega;\mathbf{R}^n)$, $|g| \leq 1$. Hence, for any $\eta > 0$ we assume without loss of generality there is a natural number m_η, which is independent of g, such that

$$\int_{E_m} \nabla \cdot g(x)dx \leq r + \eta < \frac{3}{2\sqrt{2}}\gamma_0$$

for all $m > m_\eta$ and for all $g \in C_0^1(\Omega;\mathbf{R}^n)$, $|g| \leq 1$, and hence

$$\sup_{g\in C_0^1(\Omega;\mathbf{R}^n),|g|\leq 1} \int_{E_m} \nabla \cdot g(x)dx = \mathrm{Per}_\Omega\{h_m(t_m) = 1\} \leq r + \eta$$

for all $m > m_\eta$. Since η was arbitrary, so

$$\lim_{m\to\infty} \frac{2\sqrt{2}}{3}\mathrm{Per}_\Omega\{h_m(t_m) = 1\} < \gamma_0,$$

which leads to a contradiction.

This completes the proof of Proposition.

Example. In many cases we can explain the set

$$\Gamma_0 = \partial\Omega' \cap \Omega.$$

If we consider $\Omega \subset \mathbf{R}^2$ as the ellipse of semi-major axis a and semi-minor axis b, by the fact that $C_0(\Omega; \mathbf{R}^n)$ is dense in $L^1(\Omega)$,

$$\Gamma_0 = \{(0, y) \in \mathbf{R}^2 \mid -b < y < b\}.$$

We also note that uniqueness, connectedness, and the regularity of the set Γ_0 depend on the geometry of Ω.

Assumption. *Throughout this paper, we assume that Ω is convex, Γ_0 is connected, and $\Gamma_0 \in C^2$.*

Definition. That Γ_0 is unique means that if $h \in H_\chi$ with

$$\sup_{t\in[0,1]} \frac{2\sqrt{2}}{3}\mathrm{Per}_\Omega\{h(t) = 1\} = \frac{2\sqrt{2}}{3}\mathrm{Per}_\Omega\{h(t^*) = 1\} \le \frac{2\sqrt{2}}{3}\mathcal{H}^{n-1}(\Gamma_0)$$

for some $t^* \in [0, 1]$, then

$$\Gamma_0 = \partial\{x \in \Omega \mid h(t^*) = 1\} \cap \Omega$$

a.e. in Ω.

Example. If Ω is the ellipse

$$\Omega = \{(x, y) \in \mathbf{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1, 0 < b < a\},$$

then $\Gamma_0 = \{(x, y) \in \mathbf{R}^2 \mid -b < y < b\}$ is unique.

We choose two connected disjoint nonempty open subsets Ω_1 and Ω_2 of Ω such that

$$\bar{\Omega}_1 \cup \bar{\Omega}_2 = \bar{\Omega}$$

and

$$\Gamma_0 = \partial\Omega_1 \cap \partial\Omega_2.$$

If we denote $d(x, \Gamma_0)$ the distance from x to Γ_0 and define

$$d(x) = \begin{cases} d(x, \Gamma_0) & \text{if } x \in \Omega_1 \\ -d(x, \Gamma_0) & \text{if } x \in \Omega_2, \end{cases}$$

and if $\mathcal{H}^{n-1}(\Gamma_0 \cap \partial\Omega) = 0$, then we note that d is a C^2-function in the set $\{|d(x)| < s\}$ for some $s > 0$ with $|\nabla d| = 1$. Furthermore,

$$\lim_{s \to 0} \mathcal{H}^{n-1}\{d(x) = s\} = \mathcal{H}^{n-1}(\Gamma_0).$$

For the above facts, we see [4,5,6,7,8]. Let

$$Z(t) = \tanh \alpha t = \frac{e^{\alpha t} - e^{-\alpha t}}{e^{\alpha t} + e^{-\alpha t}}$$

and $\alpha > 0$. Then

$$1 - Z(t) \le 2e^{-2\alpha t} \qquad (0 < t < \infty) \tag{3}$$

and

$$1 + Z(t) \le 2e^{2\alpha t} \qquad (-\infty < t < 0). \tag{4}$$

Now we define a function $g_\epsilon : \mathbf{R}^1 \to \mathbf{R}^1$ which effects the transition between -1 and 1 as follows :

$$g_\epsilon(s) = \begin{cases} 1 & \text{if } s > 2\sqrt{\epsilon}, \\ \left(\dfrac{1 - Z(\frac{1}{\sqrt{\epsilon}})}{\sqrt{\epsilon}}\right)(s - 2\sqrt{\epsilon}) + 1 & \text{if } \sqrt{\epsilon} \le s \le 2\sqrt{\epsilon}, \\ Z(\frac{s}{\epsilon}) & \text{if } |s| \le \sqrt{\epsilon}, \\ \left(\dfrac{Z(-\frac{1}{\sqrt{\epsilon}}) + 1}{\sqrt{\epsilon}}\right)(s + 2\sqrt{\epsilon}) - 1 & \text{if } -2\sqrt{\epsilon} \le s \le -\sqrt{\epsilon}, \\ -1 & \text{if } s < -2\sqrt{\epsilon}. \end{cases}$$

Replacing s by $d(x)$, we obtain a function v_ϵ from $\bar{\Omega}$ into $\mathbf{R}^1$ is given by

$$v_\epsilon(x) = g_\epsilon(d(x)).$$

Since Ω is convex, we can find a directed line segment in Ω with the direction $\mathbf{v}$ so that for any $\tau \in [-1, 1]$, the set

$$\Gamma_\tau = \{y + \tau\mathbf{v} \in \mathbf{R}^n \,|\, y \in \Gamma_0\}$$

has the following properties:

∘ There are two numbers $-1 < \tau_1 < 0 < \tau_2 < 1$ so that $\Gamma_{\tau_1} \cap \bar{\Omega}$ and $\Gamma_{\tau_2} \cap \bar{\Omega}$ are singleton sets.

∘ $\Omega \setminus \Gamma_\tau$ is the union of disjoint two components if $\tau_1 < \tau < \tau_2$.

∘ $\Gamma_\tau \cap \bar{\Omega} = \emptyset$ if either $-1 \le \tau < \tau_1$ or $\tau_2 < \tau \le 1$.

∘ $\mathcal{H}^{n-1}(\Gamma_\tau \cap \Omega) \le \mathcal{H}^{n-1}(\Gamma_0)$ for all $\tau \in [-1, 1]$.

In that case we define the following sets:

$$\Omega_1^\tau = \Omega \cap \{y + r\mathbf{v} \mid -1 \le r < \tau, \quad y \in \Gamma_0\},$$

$$\Omega_2^\tau = \Omega \cap \{y + r\mathbf{v} \mid \quad \tau < r \le 1, \quad y \in \Gamma_0\},$$

$$\Omega_1^0 = \Omega_1 \quad \text{and} \quad \Omega_2^0 = \Omega_2.$$

Using the previous function v_ϵ, we define the special continuous path $h_\epsilon : [0,1] \to W^{1,2}(\Omega)$ with $h_\epsilon(0) = -1$ and $h_\epsilon(1) = 1$ as follows: For any $\tau \in [-1, 1]$ and $x \in \bar{\Omega}$, we let

$$\bar{h}_\epsilon(\frac{1}{2}(\tau+1))(x) = \begin{cases} 1 & \text{if} \quad d(x, \Gamma_\tau) > 2\sqrt{\epsilon}, \\ g_\epsilon(d^\tau(x)) & \text{if} \quad -2\sqrt{\epsilon} \le d(x, \Gamma_\tau) \le 2\sqrt{\epsilon}, \\ -1 & \text{if} \quad d(x, \Gamma_\tau) < -2\sqrt{\epsilon} \end{cases}$$

where

$$d^\tau(x) = \begin{cases} d(x, \Gamma_\tau) & \text{if } x \in \Omega_1^\tau, \\ -d(x, \Gamma_\tau) & \text{if } x \in \Omega_2^\tau. \end{cases}$$

And let

$$h_\epsilon(t) = \bar{h}_\epsilon(\frac{1}{2}(\tau+1)), \tag{5}$$

where $t = \frac{1}{2}(\tau + 1)$. If ϵ satisfies the inequalities

$$d(\Gamma_{-1}, \bar{\Omega}) \ge 2\sqrt{\epsilon} \quad \text{and} \quad d(\Gamma_1, \bar{\Omega}) \ge 2\sqrt{\epsilon},$$

then h_ϵ becomes a continuous path from $[0,1]$ into $W^{1,2}(\Omega)$ with $h_\epsilon(0) = -1$ and $h_\epsilon(1) = 1$. Let

$$v_\epsilon^\tau = h_\epsilon(t)$$

for $t \in [0, 1]$ with $\tau = 2t - 1$.

The main idea in the proof of the following theorem can be found in [5,6].

Theorem 3.1. *Let h_ϵ be the previous special curve* (5) *from* $[0,1]$ *into* $W^{1,2}(\Omega)$ *with* $h_\epsilon(0)=-1$ *and* $h_\epsilon(1)=1$. *Then for any* $t\in[0,1]$ *with* $\tau=2t-1$,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}J_\epsilon(h_\epsilon(t))=\frac{2\sqrt{2}}{3}\mathcal{H}^{n-1}(\Gamma_\tau\cap\Omega).$$

Proof. We first note that

$$\int_{\{|d^\tau(x)|\geq 2\sqrt{\epsilon}\}}\left[\frac{\epsilon}{2}|\nabla v_\epsilon^\tau(x)|^2+\frac{1}{4\epsilon}\left((v_\epsilon^\tau(x))^2-1\right)^2\right]dx=0.$$

Hence

$$\begin{aligned}
&\frac{1}{\epsilon}J_\epsilon(v_\epsilon^\tau)\\
&=\int_{\{|d^\tau(x)|\leq 2\sqrt{\epsilon}\}}\left[\frac{\epsilon}{2}|\nabla v_\epsilon^\tau(x)|^2+\frac{1}{4\epsilon}\left((v_\epsilon^\tau(x))^2-1\right)^2\right]dx\\
&=\int_{\{|d^\tau(x)|\leq\sqrt{\epsilon}\}}\left[\frac{\epsilon}{2}|\nabla v_\epsilon^\tau(x)|^2+\frac{1}{4\epsilon}\left((v_\epsilon^\tau(x))^2-1\right)^2\right]|\nabla d^\tau(x)|\,dx\\
&+\int_{\{\sqrt{\epsilon}\leq|d^\tau(x)|\leq 2\sqrt{\epsilon}\}}\left[\frac{\epsilon}{2}|\nabla v_\epsilon^\tau(x)|^2+\frac{1}{4\epsilon}\left((v_\epsilon^\tau(x))^2-1\right)^2\right]|\nabla d^\tau(x)|\,dx\\
&=\int_{\{|d^\tau(x)|\leq\sqrt{\epsilon}\}}\left[\frac{\epsilon}{2}\left|\nabla\tanh\frac{\alpha d^\tau(x)}{\epsilon}\right|^2+\frac{1}{4\epsilon}\left(\left(\tanh\frac{\alpha d^\tau(x)}{\epsilon}\right)^2-1\right)^2\right]\\
&|\nabla d^\tau(x)|\,dx+\int_{\{\sqrt{\epsilon}\leq|d^\tau(x)|\leq 2\sqrt{\epsilon}\}}\left[\frac{\epsilon}{2}\left|\nabla g_\epsilon\left(d^\tau(x)\right)\right|^2+\frac{1}{4\epsilon}((g_\epsilon(d^\tau(x)))^2-1)^2\right]\\
&|\nabla d^\tau(x)|\,dx.
\end{aligned}$$

By the Co-area Formula (Theorem 2.3),

$$\begin{aligned}
&\frac{1}{\epsilon}J_\epsilon(v_\epsilon^\tau)=\\
&\int_{-\sqrt{\epsilon}}^{\sqrt{\epsilon}}\left[\frac{\epsilon}{2}\left(\frac{d}{ds}\left(\tanh\frac{\alpha s}{\epsilon}\right)\right)^2+\frac{1}{4\epsilon}\left(\left(\tanh\frac{\alpha s}{\epsilon}\right)^2-1\right)^2\right]\mathcal{H}^{n-1}\{d^\tau(x)=s\}\,ds\\
&+\int_{\sqrt{\epsilon}}^{2\sqrt{\epsilon}}\left[\frac{\epsilon}{2}\left(\frac{d}{ds}g_\epsilon(s)\right)^2+\frac{1}{4\epsilon}\left((g_\epsilon(s))^2-1\right)^2\right]\mathcal{H}^{n-1}\{d^\tau(x)=s\}\,ds\\
&+\int_{-2\sqrt{\epsilon}}^{-\sqrt{\epsilon}}\left[\frac{\epsilon}{2}\left(\frac{d}{ds}g_\epsilon(s)\right)^2+\frac{1}{4\epsilon}\left((g_\epsilon(s))^2-1\right)^2\right]\mathcal{H}^{n-1}\{d^\tau(x)=s\}\,ds.
\end{aligned}$$

First :

$$\int_{\sqrt{\epsilon}}^{2\sqrt{\epsilon}} \left[\frac{\epsilon}{2}\left(\frac{d}{ds}g_\epsilon(s)\right)^2 + \frac{1}{4\epsilon}\left((g_\epsilon(s))^2-1\right)^2\right]\mathcal{H}^{n-1}\{d^\tau(x)=s\}\,ds$$

$$= \int_{\sqrt{\epsilon}}^{2\sqrt{\epsilon}} \left[\frac{\epsilon}{2}\left(\frac{1-Z(\frac{1}{\sqrt{\epsilon}})}{\sqrt{\epsilon}}\right)^2\right]\mathcal{H}^{n-1}\{d^\tau(x)=s\}\,ds$$

$$+ \int_{\sqrt{\epsilon}}^{2\sqrt{\epsilon}} \left[\frac{1}{4\epsilon}\left(\left(\frac{1-Z(\frac{1}{\sqrt{\epsilon}})}{\sqrt{\epsilon}}(s-2\sqrt{\epsilon})+1\right)^2-1\right)^2\right]\mathcal{H}^{n-1}\{d^\tau(x)=s\}\,ds$$

From the decay estimate (3), the above integral approaches zero if $\epsilon \to 0$.

A similar approach leads to the same conclusion concerning the following :By the decay estimate (4)

$$\int_{-2\sqrt{\epsilon}}^{-\sqrt{\epsilon}} \left[\frac{\epsilon}{2}\left(\frac{d}{ds}g_\epsilon(s)\right)^2 + \frac{1}{4\epsilon}\left((g_\epsilon(s))^2-1\right)^2\right]\mathcal{H}^{n-1}\{d^\tau(x)=s\}\,ds \to 0$$

if $\epsilon \to 0$.

Second :

$$\int_{-\sqrt{\epsilon}}^{\sqrt{\epsilon}} \left[\frac{\epsilon}{2}\left(\frac{d}{ds}\left(\tanh\frac{\alpha s}{\epsilon}\right)\right)^2 + \frac{1}{4\epsilon}\left(\left(\tanh\frac{\alpha s}{\epsilon}\right)^2-1\right)^2\right]\mathcal{H}^{n-1}\{d^\tau(x)=s\}\,ds$$

$$= \int_{-\sqrt{\epsilon}}^{\sqrt{\epsilon}} \frac{2\alpha^2+1}{4\epsilon}\left(\left(\tanh\frac{\alpha s}{\epsilon}\right)^2-1\right)^2\mathcal{H}^{n-1}\{d^\tau(x)=s\}\,ds$$

$$\le \left(\sup_{|s|\le\sqrt{\epsilon}}\mathcal{H}^{n-1}\{d^\tau(x)=s\}\right)\frac{2\alpha^2+1}{4\epsilon}\int_{-\sqrt{\epsilon}}^{\sqrt{\epsilon}}\left(\left(\tanh\frac{\alpha s}{\epsilon}\right)^2-1\right)^2 ds$$

$$= \left(\sup_{|s|\le\sqrt{\epsilon}}\mathcal{H}^{n-1}\{d^\tau(x)=s\}\right)\frac{2\alpha^2+1}{4\alpha}\int_{\tanh(-\frac{\alpha}{\sqrt{\epsilon}})}^{\tanh\frac{\alpha}{\sqrt{\epsilon}}}(1-t^2)\,dt$$

$$= \left(\sup_{|s|\le\sqrt{\epsilon}}\mathcal{H}^{n-1}\{d^\tau(x)=s\}\right)\frac{2\alpha^2+1}{4\alpha}\left[\tanh\frac{\alpha}{\sqrt{\epsilon}}-\frac{1}{3}\left(\tanh\frac{\alpha}{\sqrt{\epsilon}}\right)^3\right.$$

$$\left.-\tanh\left(-\frac{\alpha}{\sqrt{\epsilon}}\right)+\frac{1}{3}\left(\tanh\left(-\frac{\alpha}{\sqrt{\epsilon}}\right)\right)^3\right].$$

Hence, combining the first and the second,

$$\limsup_{\epsilon\to 0}\frac{1}{\epsilon}J_\epsilon(v_\epsilon^\tau) \le \frac{2\alpha^2+1}{3\alpha}\,\mathrm{Per}_\Omega\{v^\tau(x)=1\}$$

for all $\alpha > 0$. Here $v^\tau(x) = 1$ for $x \in \Omega_1^\tau$ and $v^\tau(x) = -1$ for $x \in \Omega_2^\tau$. The right hand side of the above inequality has the minimum value at $\alpha = \frac{1}{\sqrt{2}}$. Hence,

$$\limsup_{\epsilon\to 0} \frac{1}{\epsilon} J_\epsilon(v_\epsilon^\tau) \le \frac{2\sqrt{2}}{3}\mathcal{H}^{n-1}(\Gamma_\tau \cap \Omega). \qquad (*)$$

We note that

$$\begin{aligned}\frac{1}{\epsilon} J_\epsilon(v_\epsilon^\tau) &= \frac{\epsilon}{2}\int_\Omega |\nabla v_\epsilon^\tau(x)|^2\,dx + \frac{1}{4\epsilon}\int_\Omega ((v_\epsilon^\tau(x))^2 - 1)^2\,dx \\ &\ge \frac{1}{\sqrt{2}}\int_\Omega |\nabla v_\epsilon^\tau(x)|(1-(v_\epsilon^\tau(x))^2)\,dx\end{aligned}$$

by the Cauchy-Schwarz inequality. Then

$$\liminf_{\epsilon\to 0} \frac{1}{\epsilon} J_\epsilon(v_\epsilon^\tau) \ge \liminf_{\epsilon\to 0} \frac{1}{\sqrt{2}}\int_\Omega \left|\nabla \int_{-1}^{v_\epsilon^\tau(x)} (1-t^2)\,dt\right|\,dx.$$

Since $\lim_{\epsilon\to 0} v_\epsilon^\tau = v^\tau$ a.e. on Ω,

$$\lim_{\epsilon\to 0}\int_{-1}^{v_\epsilon^\tau(x)} (1-t^2)\,dt = \int_{-1}^{v^\tau(x)} (1-t^2)\,dt.$$

By the lower semicontinuity (Theorem 2.1),

$$\begin{aligned}&\liminf_{\epsilon\to 0} \frac{1}{\epsilon} J_\epsilon(v_\epsilon^\tau) \\ &\ge \frac{1}{\sqrt{2}}\int_\Omega \left|\nabla \liminf_{\epsilon\to 0}\int_{-1}^{v_\epsilon^\tau(x)} (1-t^2)\,dt\right|\,dx \\ &= \frac{1}{\sqrt{2}}\int_\Omega \left|\nabla \int_{-1}^{v^\tau(x)} (1-t^2)\,dt\right|\,dx.\end{aligned}$$

Now

$$\int_{-1}^{v^\tau(x)} (1-t^2)\,dt = \begin{cases} 0 & \text{if } \{v^\tau = -1\} \\ \frac{3}{4} & \text{if } \{v^\tau = 1\}. \end{cases}$$

By the definition of the perimeter,

$$\int_\Omega \left|\nabla \int_{-1}^{v^\tau(x)} (1-t^2)\,dt\right|\,dx = \frac{4}{3}\operatorname{Per}_\Omega\{v^\tau = 1\}.$$

Hence

$$\liminf_{\epsilon\to 0} \frac{1}{\epsilon} J_\epsilon(v_\epsilon^\tau) \ge \frac{2\sqrt{2}}{3}\operatorname{Per}_\Omega\{v^\tau = 1\}. \qquad (**)$$

From $(*)$ and $(**)$, the proof is completed.

Theorem 3.2. *For any sequence $\{\epsilon_n\}$ of positive real numbers so that $\epsilon_n \to 0$ as $n \to \infty$, let u_{ϵ_n} be the nonconstant solution of (I_{ϵ_n}) obtained by the mountain pass theorem. Then there are a sequence $\{k_{\epsilon_n}\}$ of continuous paths from $[0,1]$ into $W^{1,2}(\Omega)$ with $k_{\epsilon_n}(0) = -1$ and $k_{\epsilon_n}(1) = 1$ and a sequence $\{t_n\}$ of real numbers in $[0,1]$ such that*

$$\lim_{n\to\infty} \|k_{\epsilon_n}(t_n) - u_{\epsilon_n}\|_{W^{1,2}(\Omega)} = 0,$$

$$\limsup_{\epsilon_n\to 0} \sup_{t\in[0,1]} \frac{1}{\epsilon_n} J_{\epsilon_n}(k_{\epsilon_n}(t)) \le \frac{2\sqrt{2}}{3}\mathcal{H}^{n-1}(\Gamma_0),$$

and

$$\lim_{n\to\infty} k_{\epsilon_n}(t) = \pm 1$$

for all $t \in [0,1]$ and a.e. in Ω, where $\|\cdot\|_{W^{1,2}(\Omega)}$ is the usual norm on the Sobolev space $W^{1,2}(\Omega)$.

Proof. Let $\{\delta_n\}$ be a sequence of positive real numbers with $\delta_n = o(\epsilon_n)$ as $\epsilon_n \to 0$. Since

$$J_{\epsilon_n}(u_{\epsilon_n}) = \inf_h \sup_{t\in[0,1]} J_{\epsilon_n}(h(t)),$$

by the well-known Deformation Lemma [12] there is a continuous path k_{ϵ_n} from $[0,1]$ into $W^{1,2}(\Omega)$ with $k_{\epsilon_n}(0) = -1$ and $k_{\epsilon_n}(1) = 1$ so that

$$\sup_{t\in[0,1]} J_{\epsilon_n}(k_{\epsilon_n}(t)) \le J_{\epsilon_n}(u_{\epsilon_n}) + \delta_n,$$

$$\|k_{\epsilon_n}(t_n) - u_{\epsilon_n}\|_{W^{1,2}(\Omega)} < \delta_n,$$

and

$$J_{\epsilon_n}(k_{\epsilon_n}(t_n)) = \sup_{t\in[0,1]} J_{\epsilon_n}(k_{\epsilon_n}(t))$$

for some $t_n \in [0,1]$. For the details, we show the existence of the path. Let ϵ_n be fixed. ¿From the definition of $J_{\epsilon_n}(u_{\epsilon_n})$, we have a continuous path k_{ϵ_n} from $[0,1]$ into $W^{1,2}(\Omega)$ with $k_{\epsilon_n}(0) = -1$ and $k_{\epsilon_n}(1) = 1$ so that

$$(*) \qquad \sup_{t\in[0,1]} J_{\epsilon_n}(k_{\epsilon_n}(t)) \le J_{\epsilon_n}(u_{\epsilon_n}) + \delta_n$$

and

$$(**) \qquad J_{\epsilon_n}(k_{\epsilon_n}(t_n)) = \sup_{t\in[0,1]} J_{\epsilon_n}(k_{\epsilon_n}(t))$$

for some $t_n \in [0,1]$. Then we prove the existence of a continuous path k_{ϵ_n} from $[0,1]$ into $W^{1,2}(\Omega)$ such that k_{ϵ_n} satisfies the above (*), (**), and

$$\|k_{\epsilon_n}(t_n) - u_{\epsilon_n}\|_{W^{1,2}(\Omega)} < \delta_n.$$

Suppose that there is no such any continuous paths. Then

$$\|k_{\epsilon_n}(t_n) - u_{\epsilon_n}\|_{W^{1,2}(\Omega)} \geq \delta_n$$

for all such paths. From the definition of $J_{\epsilon_n}(u_{\epsilon_n})$, there exists a sequence of continuous path p_k from $[0,1]$ into $W^{1,2}(\Omega)$ such that $p_k(0) = -1$, $p_k(1) = 1$,

$$\|p_k(t_k) - u_{\epsilon_n}\|_{W^{1,2}(\Omega)} \geq \delta_n$$

and

$$\lim_{k\to\infty} J_{\epsilon_n}(p_k(t_k)) = J_{\epsilon_n}(\epsilon_n),$$

where $J_{\epsilon_n}(p_k(t_k)) = \sup_{t\in[0,1]} J_{\epsilon_n}(p_k(t))$ for some $t_k \in [0,1]$. Let $\beta = J_{\epsilon_n}(u_{\epsilon_n})$ and $N = B(u_{\epsilon_n}, \delta_n)$ be the open ball centered at u_{ϵ_n} with its radius δ_n in $W^{1,2}(\Omega)$. Since $J_{\epsilon_n} \in C^1(W^{1,2}(\Omega))$ and satisfies the Palais-Smale condition, the well-known deformation lemma [pp.75, 12] implies that for any $\bar{\delta} > 0$ there exist a number δ with $0 < \delta < \bar{\delta}$ and a continuous 1-parameter family of homeomorphisms $\Phi(\cdot, t)$ of $W^{1,2}(\Omega)$, $0 \leq t < \infty$ with the properties

(i). $\Phi(u,t) = u$ if $t = 0$ or $\frac{d}{du}J_{\epsilon_n}(u) = 0$, or $|J_{\epsilon_n}(u) - \beta| \geq \bar{\delta}$;

(ii). $J_{\epsilon_n}(\Phi(u,t))$ is non-increasing in t for any $u \in W^{1,2}(\Omega)$;

(iii). $J_{\epsilon_n}(\Phi(v,1)) < \beta - \delta$ or $\Phi(v,1) \in N$ if $J_{\epsilon_n}(v) < \beta + \delta$.

Since $\lim_{k\to\infty} \sup_{t\in[0,1]} J_{\epsilon_n}(p_k(t)) = J_{\epsilon_n}(u_{\epsilon_n})$, there is a path p_k such that

$$\sup_{t\in[0,1]} J_{\epsilon_n}(p_k(t)) < \beta + \delta.$$

From the above deformation and the assumption that the maxima of $J_{\epsilon_n}(\Phi(p_k(t),1))$ is not contained in N,

$$J_{\epsilon_n}(\Phi(p_k(t),1)) < \beta - \delta$$

for all $t \in [0,1]$. This leads to a contradiction for the definition of β.

For the special path h_{ϵ_n} in Theorem 3.1, we note that

$$J_{\epsilon_n}(u_{\epsilon_n}) \leq \sup_{t\in[0,1]} J_{\epsilon_n}(h_{\epsilon_n}(t)).$$

Hence, by Theorem 3.1,

$$\begin{aligned}\limsup_{\epsilon_n\to 0}\frac{J_{\epsilon_n}(u_{\epsilon_n})}{\epsilon_n} &\le \limsup_{\epsilon_n\to 0}\sup_{t\in[0,1]}\frac{1}{\epsilon_n}J_{\epsilon_n}(h_{\epsilon_n}(t))\\ &\le \frac{2\sqrt{2}}{3}\mathcal{H}^{n-1}(\Gamma_0)<\infty.\end{aligned}$$

Thus,

$$\limsup_{\epsilon_n\to 0}\sup_{t\in[0,1]}\frac{1}{\epsilon_n}J_{\epsilon_n}(k_{\epsilon_n}(t))\le\frac{2\sqrt{2}}{3}\mathcal{H}^{n-1}(\Gamma_0).$$

This implies that

$$\begin{aligned}0&=\limsup_{\epsilon_n\to 0}\sup_{t\in[0,1]}J_{\epsilon_n}(k_{\epsilon_n}(t))\\ &=\limsup_{\epsilon_n\to 0}\sup_{t\in[0,1]}\left(\frac{\epsilon_n^2}{2}\int_\Omega|\nabla k_{\epsilon_n}(t)|^2dx+\frac{1}{4}\int_\Omega(1-k_{\epsilon_n}^2(t))^2dx\right)\end{aligned}$$

Therefore,

$$\lim_{\epsilon_n\to 0}k_{\epsilon_n}^2(t)=1$$

a.e. in Ω and for all $t\in[0,1]$.

Theorem 3.3. *Assume Γ_0 is unique. Let u_{ϵ_n} be a nonconstant solution of (I_{ϵ_n}) obtained by the mountain pass theorem and let $\gamma_{\epsilon_n}=J_{\epsilon_n}(u_{\epsilon_n})$. Then*

$$\liminf_{n\to\infty}\frac{\gamma_{\epsilon_n}}{\epsilon_n}=\gamma_0.$$

Proof. Since Ω is convex and smooth and Γ_0 is unique, we can choose a point $x_0\in\Gamma_0$ so that

$$\frac{2\sqrt{2}}{3}\mathcal{H}^{n-1}(\Gamma'\cap\Omega)\ge\gamma_0$$

for every connected hypersurface Γ' passing through x_0 and $\Omega\backslash\Gamma'$ is disconnected. Then we can also choose the sequence of continuous paths $k_{\epsilon_n}(t)$ in Theorem 3.2 and a sequence $\{t_n\}$ of real numbers in $[0,1]$ so that $k_{\epsilon_n}(0)=-1$, $k_{\epsilon_n}(1)=1$, $x_0\in\{k_{\epsilon_n}(t_n)=0\}$ and

$$\begin{aligned}&\frac{1}{\sqrt{2}}\int_\Omega|1-k_{\epsilon_n}^2(t_n)||\nabla k_{\epsilon_n}(t_n)|dx\\ (*)\qquad &\le\sup_{t\in[0,1]}\frac{1}{\epsilon_n}J_{\epsilon_n}(k_{\epsilon_n}(t))\\ &\le\frac{\gamma_{\epsilon_n}}{\epsilon_n}+\frac{\delta_n}{\epsilon_n}.\end{aligned}$$

We can assume $\delta_n = o(\epsilon_n)$ as in the proof of Theorem 3.2. Let Γ_{ϵ_n} be a connected subset of the level set $\{x \in \Omega : k_{\epsilon_n}(t_n) = 0\}$ with $x_0 \in \Gamma_{\epsilon_n}$. We denote that $k_{\epsilon_n}(t)$ by $k_{\epsilon_n}(\cdot, t)$ which means a function from $\bar{\Omega}$ into $\mathbf{R}^1$. Then since

$$\lim_{\epsilon_n \to 0} k^2_{\epsilon_n}(x,t) = 1$$

a.e. in Ω, we let

$$\Omega_0 = \{x \in \Omega : \liminf_{\epsilon_n \to 0} k_{\epsilon_n}(x, t_n) = 1\}.$$

Then from the above $(*)$

$$\begin{aligned}
&\liminf_{\epsilon_n \to 0} \frac{\gamma_{\epsilon_n}}{\epsilon_n} \\
&\geq \liminf_{\epsilon_n \to 0} \sup_{t \in [0,1]} \frac{1}{\epsilon_n} J_{\epsilon_n}(k_{\epsilon_n}(x,t)) \\
&\geq \liminf_{\epsilon_n \to 0} \frac{1}{\sqrt{2}} \int_\Omega |1 - k^2_{\epsilon_n}(x,t_n)||\nabla k_{\epsilon_n}(x,t_n)| dx \\
&= \liminf_{\epsilon_n \to 0} \frac{1}{\sqrt{2}} \int_\Omega \left| \nabla \int_{-1}^{k_{\epsilon_n}(x,t_n)} (1 - s^2) ds \right| dx \\
&\geq \frac{1}{\sqrt{2}} \int_\Omega \left| \nabla \liminf_{\epsilon_n \to 0} \int_{-1}^{k_{\epsilon_n}(x,t_n)} (1 - s^2) ds \right| dx \\
&= \frac{2\sqrt{2}}{3} \mathrm{Per}_\Omega \Omega_0.
\end{aligned}$$

Let

$$\Gamma_2 = \cap_{m=1}^\infty \overline{\cup_{n \geq m} \Gamma_{\epsilon_n}}.$$

Then Γ_2 is connected and contains x_0. By the definition of K-upper limit of the sequence Γ_{ϵ_n}, we have

$$\Gamma_2 \subset \partial\Omega_0 \cap \Omega \quad \text{a.e.}$$

with respect to $\mathcal{H}^{n-1}$. Hence by Proposition

$$\begin{aligned}
&\frac{2\sqrt{2}}{3} \mathrm{Per}_\Omega \Omega_0 \\
&\geq \frac{2\sqrt{2}}{3} \mathcal{H}^{n-1}(\Gamma_2) \\
&= \liminf_{\epsilon_n \to 0} \frac{2\sqrt{2}}{3} \mathcal{H}^{n-1}(\Gamma_{\epsilon_n}) \\
&\geq \gamma_0
\end{aligned}$$

We note that

$$\liminf_{\epsilon_n \to 0} \frac{\gamma_{\epsilon_n}}{\epsilon_n} \leq \frac{2\sqrt{2}}{3} \sup_{\tau \in [-1,1]} \mathcal{H}^{n-1}(\Gamma_\tau \cap \Omega) = \gamma_0.$$

This completes the proof.

The following theorem is the main result of this paper.

Theorem 3.4. *Let Γ_0 be unique. If $\{\epsilon_n\}$ is any sequence of positive numbers with $\epsilon_n \to 0$ as $n \to \infty$ and if $\{u_{\epsilon_n}\}$ is the sequence of the nonconstant solutions of (I_{ϵ_n}) obtained by the mountain pass theorem, then there is a subsequence $\{\epsilon_{n_k}\}$ of $\{\epsilon_n\}$ so that*

$$\lim_{\epsilon_{n_k} \to 0} u_{\epsilon_{n_k}}(x) = \begin{cases} 1, & \text{if} \quad x \in \Omega_1 \\ -1, & \text{if} \quad x \in \Omega_2. \end{cases}$$

uniformly on every compact subset in $\Omega_1 \cup \Omega_2$, where $\Gamma_0 = \partial\Omega_1 \cap \partial\Omega_2$.

Proof. From Theorem 1.1, we assume that there is a sequence $\{u_{\epsilon_n}\}$ of the nonconstant solutions of (I_{ϵ_n}) obtained by the mountain pass theorem so that $\lim_{\epsilon_n \to 0} u_{\epsilon_n} = \pm 1$ a.e. in Ω, and let h_{ϵ_n} be the sequence of continuous paths in Theorem 3.1, and let $\{k_{\epsilon_n}\}$ be the sequence of continuous paths in Theorem 3.2. By reparametrization of $k_{\epsilon_n}(t)$ about t, we can assume that the existence of the continuous path, we again denote it k_{ϵ_n}, from $[0,1]$ into $W^{1,2}(\Omega)$ and that the existence of the sequence $\{t_n\}$ of real numbers in $[0,1]$ such that

$$\lim_{\epsilon_n \to 0} \|k_{\epsilon_n}(t_n) - u_{\epsilon_n}\|_{W^{1,2}(\Omega)} = 0,$$

$$\limsup_{\epsilon_n \to 0} \sup_{t \in [0,1]} \frac{1}{\epsilon_n} J_{\epsilon_n}(k_{\epsilon_n}(t)) \leq \frac{2\sqrt{2}}{3} \mathcal{H}^{n-1}(\Gamma_0),$$

$$\frac{1}{\epsilon_n} J_{\epsilon_n}(k_{\epsilon_n}(t)) \leq \frac{1}{\epsilon_n} J_{\epsilon_n}(h_{\epsilon_n}(t)),$$

and

$$\lim_{\epsilon_n \to 0} k_{\epsilon_n}(x,t) = \pm 1$$

for all $t \in [0,1]$ and a.e. on Ω. From the following inequality

$$\begin{aligned} &\frac{1}{\epsilon_n} J_{\epsilon_n}(k_{\epsilon_n}(t)) \\ &\geq \frac{1}{\sqrt{2}} \int_\Omega |1 - k_{\epsilon_n}^2(x,t)| |\nabla k_{\epsilon_n}(x,t)| dx \\ &= \frac{1}{\sqrt{2}} \int_\Omega \left| \nabla \int_{-1}^{k_{\epsilon_n}(x,t)} (1 - s^2) ds \right| dx, \end{aligned}$$

we obtain

$$\begin{aligned}
&\liminf_{\epsilon_n\to 0}\frac{1}{\epsilon_n}J_{\epsilon_n}(k_{\epsilon_n}(t))\\
&\geq \liminf_{\epsilon_n\to 0}\frac{1}{\sqrt{2}}\int_\Omega\left|\nabla\int_{-1}^{k_{\epsilon_n}(x,t)}(1-s^2)ds\right|dx\\
&=\liminf_{\epsilon_n\to 0}\frac{1}{\sqrt{2}}\sup_{g\in C_0^1(\Omega;\mathbf{R}^n),|g|\leq 1}\int_\Omega\left(\int_{-1}^{k_{\epsilon_n}(x,t)}(1-s^2)ds\right)(\nabla\cdot g(x))dx.
\end{aligned}\tag{6}$$

From the following inequality

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\sup_{g\in C_0^1(\Omega;\mathbf{R}^n),|g|\leq 1}\int_\Omega\left(\int_{-1}^{k_{\epsilon_n}(x,t)}(1-s^2)ds\right)(\nabla\cdot g(x))dx\\
&\geq \frac{1}{\sqrt{2}}\int_\Omega\left(\int_{-1}^{k_{\epsilon_n}(x,t)}(1-s^2)ds\right)(\nabla\cdot g(x))dx
\end{aligned}$$

for all $g\in C_0^1(\Omega;R^n)$ with $|g|\leq 1$, we have the following:

$$\begin{aligned}
&\liminf_{\epsilon_n\to 0}\frac{1}{\sqrt{2}}\sup_{g\in C_0^1(\Omega;\mathbf{R}^n),|g|\leq 1}\int_\Omega\left(\int_{-1}^{k_{\epsilon_n}(x,t)}(1-s^2)ds\right)(\nabla\cdot g(x))dx\\
&\geq \sup_{g\in C_0^1(\Omega;\mathbf{R}^n),|g|\leq 1}\liminf_{\epsilon_n\to 0}\frac{1}{\sqrt{2}}\int_\Omega\left(\int_{-1}^{k_{\epsilon_n}(x,t)}(1-s^2)ds\right)(\nabla\cdot g(x))dx.
\end{aligned}\tag{7}$$

Then by combining (6) and (7) and using Fatou's lemma, we have

$$\begin{aligned}
&\liminf_{\epsilon_n\to 0}\frac{1}{\epsilon_n}J_{\epsilon_n}(k_{\epsilon_n}(t))\\
&\geq \frac{1}{\sqrt{2}}\sup_{g\in C_0^1(\Omega;\mathbf{R}^n),|g|\leq 1}\int_\Omega\liminf_{\epsilon_n\to 0}\left[\int_{-1}^{k_{\epsilon_n}(x,t)}(1-s^2)ds\right](\nabla\cdot g(x))dx.
\end{aligned}\tag{8}$$

Now, we define the set

$$\Omega(t)=\{x\in\Omega\mid \liminf_{\epsilon_n\to 0}\int_{-1}^{k_{\epsilon_n}(x,t)}(1-s^2)ds=\frac{4}{3}\}.$$

Next, we define the characteristic function $\mathcal{X}_{\Omega(t)}$ on Ω by $\mathcal{X}_{\Omega(t)}(x)=1$ if $x\in\Omega(t)$ and $\mathcal{X}_{\Omega(t)}(x)=0$ if $x\neq\Omega(t)$. Then by the definition of the perimeter of a subset of Ω and by (8),

$$\begin{aligned}
&\liminf_{\epsilon_n \to 0} \frac{1}{\epsilon_n} J_{\epsilon_n}(k_{\epsilon_n}(t)) \\
&\geq \frac{1}{\sqrt{2}} \sup_{g \in C_0^1(\Omega;\mathbf{R}^n),|g|\leq 1} \int_\Omega \frac{4}{3} \mathcal{X}_{\Omega(t)}(x)(\nabla \cdot g(x))dx \\
&= \frac{2\sqrt{2}}{3} \sup_{g \in C_0^1(\Omega;\mathbf{R}^n),|g|\leq 1} \int_\Omega \mathcal{X}_{\Omega(t)}(x)(\nabla \cdot g(x))dx \\
&= \frac{2\sqrt{2}}{3} \mathrm{Per}_\Omega \Omega(t).
\end{aligned} \tag{9}$$

Claim For any sequence $\{t_m\}$ in $[0,1]$ with $t_m \to t$ as $m \to \infty$,

$$\limsup_{m\to\infty} \lambda(\Omega(t_m)\triangle\Omega(t)) = 0,$$

where λ is the Lebesgue measure on $\mathbf{R}^n$ and $\triangle$ means that $A\triangle B = (A \setminus B) \cup (B \setminus A)$.

Proof of Claim Suppose that

$$\limsup_{m\to\infty} \lambda(\Omega(t_m)\triangle\Omega(t)) > 0.$$

Then we let

$$\Omega^* = \cap_{m=1}^{\infty} \overline{\cup_{k\geq m}(\Omega(t_k)\triangle\Omega(t))}.$$

Then

$$\begin{aligned}
\lambda(\Omega^*) &= \lim_{m\to\infty} \lambda(\overline{\cup_{k\geq m}(\Omega(t_k)\triangle\Omega(t))}) \\
&\geq \limsup_{m\to\infty} \lambda(\Omega(t_m)\triangle\Omega(t)) > 0.
\end{aligned}$$

Then there is a measurable set $E \subset \Omega^* \cap \Omega$ with the following properties: $\lambda(E) > 0$ and for any m there is $k \geq m$ such that

$$E \subset (\Omega(t_k) \setminus \Omega(t)) \cup (\Omega(t) \setminus \Omega(t_k)).$$

Then we let

$$E_k' = E \cap (\Omega(t_k) \setminus \Omega(t))$$

and

$$E_k'' = E \cap (\Omega(t) \setminus \Omega(t_k))$$

for such k. First, suppose that

$$\liminf_{k\to\infty} \lambda(E_k') > 0.$$

Then

$$\mathcal{X}_{\Omega(t)} \neq \liminf_{m\to\infty} \mathcal{X}_{\Omega(t_m)}.$$

But we note that

$$\Omega(t) = \cap_{m=1}^{\infty} \overline{\cup_{n\geq m}\Omega(t_n)}.$$

Then, Theorem 2.5 and Lemma imply

$$\begin{aligned}\mathcal{X}_{\Omega(t)} &= \Gamma - \liminf_{n\to\infty} \mathcal{X}_{\Omega(t_n)}\\ &= \liminf_{n\to\infty} \mathcal{X}_{\Omega(t_n)}.\end{aligned}$$

This leads to a contradiction. Hence,

$$\liminf_{k\to\infty} \lambda(E_k'') > 0.$$

Then by the similar method, we can also see a contradiction. Thus,

$$\lambda(E) = \liminf_{k\to\infty} \lambda(E_k') + \liminf_{k\to\infty} \lambda(E_k'') = 0.$$

This also leads to a contradiction.

This completes the proof of Claim.

Therefore, the characteristic function $\mathcal{X}_{\Omega(t)} : [0,1] \to L^1(\Omega)$ is continuous. We hence define the continuous path k from $[0,1]$ into $L^1(\Omega)$ by

$$k(t) = 2\mathcal{X}_{\Omega(t)} - 1 \in \mathcal{H}_{\chi}.$$

From (9) and Theorem 3.1

$$\begin{aligned}&\frac{2\sqrt{2}}{3}\mathrm{Per}_{\Omega}\{k(t)=1\}\\ &= \frac{2\sqrt{2}}{3}\mathrm{Per}_{\Omega}\Omega(t)\\ &\leq \liminf_{\epsilon_n\to 0} \frac{1}{\epsilon_n} J_{\epsilon_n}(k_{\epsilon_n}(t))\\ &\leq \liminf_{\epsilon_n\to 0} \frac{1}{\epsilon_n} J_{\epsilon_n}(h_{\epsilon_n}(t))\\ &= \frac{2\sqrt{2}}{3}\mathcal{H}^{n-1}(\Gamma_\tau \cap \Omega)\end{aligned}$$

for all t with $\tau = 2t-1$. From the uniqueness of Γ_0, we have

$$\Gamma_0 = \partial\{k(t^*) = 1\} \cap \Omega$$

for some $t^* \in [0,1]$. Since

$$\|k_{\epsilon_n}(t_n) - u_{\epsilon_n}\|_{W^{1,2}(\Omega)} \to 0$$

as $\epsilon_n \to 0$, by Hölder inequality

$$\|k_{\epsilon_n}(t_n) - u_{\epsilon_n}\|_{L^1(\Omega)} \to 0.$$

Here $\|u\|_{L^1(\Omega)} = \int_\Omega |u(x)|dx$. Therefore, if

$$\limsup_{n\to\infty} \|k_{\epsilon_n}(t_n) - k(t^*)\|_{L^1(\Omega)} = 0,$$

we are done.

So we show the last limit. Since $\mathcal{H}^{n-1}(\Gamma_\tau \cap \Omega)$ has only one maxima in τ, Theorem 3.3 implies that $t_n \to t^*$ as $n \to \infty$.

Clearly, we note that

$$\liminf_{\epsilon_n\to 0} k_{\epsilon_n}(t) = k(t)$$

a.e. in Ω. Let a positive number δ be given. Since k is continuous at t^*, there is a positive number η such that

$$|s - t^*| < \eta \quad \text{implies} \quad \|k(s) - k(t^*)\|_{L^1(\Omega)} < \delta.$$

This means that

$$\|k_{\epsilon_n}(s) - k(t^*)\|_{L^1(\Omega)} < \delta$$

for sufficiently small ϵ_n. If n is so large that $|t_n - t^*| < \eta$, then

$$\|k_{\epsilon_n}(t_n) - k(t^*)\|_{L^1(\Omega)} < \delta$$

and so

$$\limsup_{\epsilon_n\to 0} \|k_{\epsilon_n}(t_n) - k(t^*)\|_{L^1(\Omega)} \le \delta.$$

Since δ was arbitrary, so

$$\lim_{\epsilon_n\to 0} \|k_{\epsilon_n}(t_n) - k(t^*)\|_{L^1(\Omega)} = 0.$$

Therefore,

$$\lim_{\epsilon_n\to 0} u_{\epsilon_n} = \mathcal{X}_{\Omega(t^*)} = \mathcal{X}_{\Omega'}$$

a.e. in Ω.

The uniform convergence of u_{ϵ_n} on every compact subset of $\Omega_1 \cup \Omega_2$ as $\epsilon_n \to 0$ follows from the well known linear variational method, the uniqueness and the maximum principles. See [1,9,10]

This completes the proof of Theorem 3.4.

Theorem 3.5. *Let Ω be an open ball in $\mathbf{R}^n$, and let P be a $n-1$ dimensional hyperplane passing through the center of the ball, and let $\Gamma = \Omega \cap P$ and $\Omega \setminus \Gamma = \Omega_1 \cup \Omega_2$. Then there is a sequence $\{u_{\epsilon_n}\}$ of nonconstant solutions of (I_{ϵ_n}) such that*

$$\lim_{\epsilon_n \to 0} u_{\epsilon_n}(x) = \begin{cases} 1, & \text{if} \quad x \in \Omega_1 \\ -1, & \text{if} \quad x \in \Omega_2. \end{cases}$$

uniformly on every compact subset of $\Omega_1 \cup \Omega_2$.

Proof. Without loss of generality, we assume that the center of Ω is the origin. We consider a sequence $\{v_{\epsilon_n}\}$ of nonconstant solutions of (I_{ϵ_n}) which are obtained by the mountain pass theorem and $\lim_{n\to\infty} v_{\epsilon_n} = \pm 1$ a.e. in Ω. If we consider Γ as Γ_0, by Theorem 3.2 and 3.3, we have a sequence $\{k_{\epsilon_n}\}$ of continuous paths from $[0,1]$ into $W^{1,2}(\Omega)$ with $k_{\epsilon_n}(0) = -1$ and $k_{\epsilon_n}(1) = 1$ and a sequence $\{t_n\}$ of real numbers in $[0,1]$ such that

$$\lim_{\epsilon_n \to 0} \|k_{\epsilon_n}(t_n) - v_{\epsilon_n}\|_{W^{1,2}(\Omega)} = 0,$$

$$\limsup_{\epsilon_n \to 0} \sup_{t \in [0,1]} \frac{1}{\epsilon_n} J_{\epsilon_n}(k_{\epsilon_n}(t)) \le \frac{2\sqrt{2}}{3} \mathcal{H}^{n-1}(\Gamma_0),$$

$$\frac{1}{\epsilon_n} J_{\epsilon_n}(k_{\epsilon_n}(t)) \le \frac{1}{\epsilon_n} J_{\epsilon_n}(h_{\epsilon_n}(t))$$

and

$$\lim_{\epsilon_n \to 0} k_{\epsilon_n}(x,t) = \pm 1$$

for all $t \in [0,1]$ and a.e. on Ω, where h_{ϵ_n} is the continuous path in Theorem 3.1. Let

$$\Omega(t) = \{x \in \Omega \mid \liminf_{\epsilon_n \to 0} \int_{-1}^{k_{\epsilon_n}(x,t)} (1 - s^2) ds = \frac{4}{3}\}$$

and let

$$k(t) = 2\mathcal{X}_{\Omega(t)} - 1.$$

Then k is a continuous path form $[0,1]$ into $L^1(\Omega)$ with $k(0) = -1$ and $k(1) = 1$, and for some $t^* \in [0,1]$

$$\begin{aligned}
&\frac{2\sqrt{2}}{3}\mathrm{Per}_\Omega\{k(t^*)=1\}\\
&= \sup_{t\in[0,1]} \frac{2\sqrt{2}}{3}\mathrm{Per}_\Omega\{k(t)=1\}\\
&\le \sup_{t\in[0,1]} \liminf_{\epsilon_n\to 0} \frac{1}{\epsilon_n} J_{\epsilon_n}(k_{\epsilon_n}(t))\\
&= \liminf_{\epsilon_n\to 0} \frac{1}{\epsilon_n} J_{\epsilon_n}(k_{\epsilon_n}(t_n))\\
&\le \frac{2\sqrt{2}}{3}\mathcal{H}^{n-1}(\Gamma_0).
\end{aligned}$$

Hence,

$$\liminf_{\epsilon_n\to 0} k_{\epsilon_n}(t_n) = k(t^*)$$

and

$$\partial\{k(t^*)=1\}\cap\Omega = \Omega\cap P'$$

for some hyperplane P' passing through the origin. Let

$$\Gamma' = \Omega\cap P' \quad \text{and} \quad \Omega\setminus\Gamma' = \Omega_1'\cup\Omega_2'.$$

Then we note that

$$\liminf_{\epsilon_n\to 0} v_{\epsilon_n}(x) = \begin{cases} 1 & \text{if} \quad x\in\Omega_1', \\ -1 & \text{if} \quad x\in\Omega_2'. \end{cases}$$

Let r_θ be the rotation through the angle θ about the origin so that $r_\theta(P') = P$. Let $u_{\epsilon_n} = v_{\epsilon_n}(r_\theta)$. Then u_{ϵ_n} is also a nonconstant solution of (I_{ϵ_n}) and

$$\liminf_{\epsilon_n\to 0} u_{\epsilon_n}(x) = \begin{cases} 1 & \text{if} \quad x\in\Omega_1, \\ -1 & \text{if} \quad x\in\Omega_2. \end{cases}$$

Acknowledgements: I am grateful to Professor Wei-Ming Ni and Professor Walter Kelley for their encouragement and helpful advice.

REFERENCES

1. Ko, B., *The Existence of nonconstant solutions of a class of semilinear elliptic Neumann problems with a small parameter*, Nonlinear Analysis **28(7)** (1997).
2. Kohn, R. and Sternberg, P, *Local minimizers and singular perturbations*, Proc. Roy. Soc. Edinburgh Sect. A 111 (1989), 69–84.
3. Dancer, E. and Guo, Z., *Some remarks on the stability of sign changing solutions*, Tôhoku Math, J. **47** (1995), 199–225.
4. Giusti, E., *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser (1984).
5. Sternberg, P., *The effect of a singular perturbation on nonconvex variational problems*, Arch. Rational Mech. Anal. **101(3)** (1988).
6. Modica, L., *The gradient theory of phase transitions and the minimal interface criterion*, Arch. Rational Mech. Anal. **98(2)** (1987).
7. Gilbarg, D. and Trudinger, N., *Elliptic Partial Differential Equations of Second Order*, Springer -Verlarg, Berlin, 1983.
8. Krantz, S. and Parks, H., *Distance to C^k hypersurfaces*, J. Diff. Eqns. **40(1)** (1981), 116-120.
9. Schmitt, K., *Boundary value problems for quasilinear second order elliptic equations*, Nonlinear Analysis **2(3)** (1978), 263-309.
10. Angenent, S., *Uniqueness of the solutions of a semilinear elliptic boundary value problem*, Math. Annln **272** (1985), 129-138.
11. Dal Maso, G., *An introduction to Γ-convergence*, Birkhäuser, Boston, 1993.
12. Struwe, M., *Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer-Verlag, Berlin, 1990.

<국문초록>

스칼라 긴즈버그-란다우 방정식에서 비상수 해들에 대한 내부 전이 현상층 연구

고 봉 수

볼록영역상에서 정의된 특이섭동 노이만 문제의 해들의 극한상황을 연구하게 된다. 노이만 경계치 문제는 두 개의 안정해를 가지면 마운틴 패스 정리를 사용하여 얻은 비 상수해들은 영역 내부에서 상전이 현상을 일으킨다. 곡면의 면적 개념으로 정의되는 페리메터 범함수를 사용하여 영역내의 상전이 발생 장소를 추정한다.