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Linear Operator Preserving Zero-term Rank of Nonnegative Integer Matrices

Hee-Jeong Han, Kyung-Tae Kang and Seok-Zun Song

Department of Mathematics, Cheju National University, Jeju 690-756, Korea

Abstract

Zero-term rank of an $m \times n$ matrix A is the minimum number of lines(rows or columns) needed to cover all the zero entries of A. In this thesis, we obtain characterizations the linear operators preserving zero-term rank on the set of $m \times n$ matrices over the nonnegative integer semiring.

Keywords: Zero-term rank; Term rank; (P, Q, B)-operator; Zero-term rank preserver

AMS Subject Classifications: 15A03, 15A04, 15A33

1 Introduction

There are many papers on the ranks of matrices and their preservers. Boolean matrices also have been the subject of research by many authors. Beasley and Pullman characterized those linear operators that preserve Boolean rank in [3] and term rank of matrices over semirings in [5]. But there are few papers on the linear operators that preserve zero-term rank of the matrices.

Recently, Beasley, Song and Lee obtained characterizations of the linear operators that preserve zero-term rank of Boolean matrices in [7]. They gave us the motivation to research on the zero-term rank of the nonnegative integer matrices.

In this thesis, we investigate the zero-term rank of the nonnegative integer matrices. We characterize the linear operators that preserve zero-term rank of $m \times n$ matrices over the nonnegative integer semiring $\mathbb{Z}^+$.

2 Preliminary

A *semiring* ([5]) consists of a set $\mathbf{S}$, and two binary operations on $\mathbf{S}$, addition(+) and multiplication($\cdot$), such that

(1) $(\mathbf{S}, +)$ is an Abelian monoid under addition (identity denoted by 0);
(2) $(\mathbf{S}, \cdot)$ is a monoid under multiplication (identity denoted by 1);
(3) multiplication distributes over addition ;
(4) $s0 = 0s = 0$ for all $s \in \mathbf{S}$; and
(5) $0 \neq 1$.

Let $\mathbb{Z}^+$ be the set of all nonnegative integers. Then $\mathbb{Z}^+$ with usual addition and multiplication becomes a semiring. Usually $\mathbb{Z}^+$ denotes both the semiring and the set. Let $M_{m,n}(\mathbb{Z}^+)$ denote the set of all $m \times n$ matrices with entries in $\mathbb{Z}^+ = \{0, 1, 2, 3...\}$, the nonnegative integers. The zero matrix and the $n \times n$ identity matrix I_n are defined as if $\mathbb{Z}^+$ were a field. Addition, multiplication by scalar, and the product of matrices are also defined as if $\mathbb{Z}^+$ were a field. The $m \times n$ matrix of $1's$ is denoted $J_{m,n}$. The $m \times n$ matrix of whose entries are zero except its (i, j)th, which is 1, is denoted E_{ij}. We call E_{ij} a *cell*. The set of all cells is denoted $\triangle = \{E_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ and the set of its indices is denoted $\mathcal{E} = \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq n\}$. The set of all cells spans $M_{m,n}(\mathbb{Z}^+)$.

If A and B are in $M_{m,n}(\mathbb{Z}^+)$, we say A *dominates* B (written $A \geq B$ or $B \leq A$) if $a_{ij} \geq b_{ij}$ for all i, j. This provides a reflexive, transitive relation on $M_{m,n}(\mathbb{Z}^+)$.

The *zero-term rank* of a matrix X in $M_{m,n}(\mathbb{Z}^+)$, $z(X)$, is the minimum number of lines(rows or columns) needed to cover all the zero entries in X. And the *term rank* of a matrix X, $t(X)$, is the minimum number of lines(rows or columns) needed to cover all the nonzero entries in X.

LEMMA 2.1. *For $A, B \in M_{m,n}(\mathbb{Z}^+)$, $A \geq B$ implies that $z(A) \leq z(B)$.*

Proof. If $z(B) = k$, then there are k lines which cover all zero entries in B. Since $A \geq B$, this k lines can also cover all zero entries in A. Hence $z(A) \leq k = z(B)$. □

3 Zero-term rank preserver of matrices over nonnegative integer

In this chapter, we obtain the properties of zero-term rank of matrices over nonnegative integers and also have the characterizations of the linear operators that preserve the zero-term rank of the matrices. We extend the results over Boolean matrices of Beasley, Song and Lee in [7] into matrices over nonnegative integer.

A function T mapping $M_{m,n}(\mathbb{Z}^+)$ into itself is called an *operator* on $M_{m,n}(\mathbb{Z}^+)$. The operator T is *linear* if $T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$ for all $\alpha, \beta \in \mathbb{Z}^+$ and all $A, B \in M_{m,n}(\mathbb{Z}^+)$.

DEFINITION 3.1. Let T be a linear operator on $M_{m,n}(\mathbb{Z}^+)$. If $z(T(X)) = k$ whenever $z(X) = k$ for all X in $M_{m,n}(\mathbb{Z}^+)$, we say T *preserves zero-term rank* k. If T preserves zero-term rank k for every $k \leq min\{m, n\}$, then we say T *preserves zero-term rank.*

Consider the semiring $\mathbb{Z}^+$. Which linear operators on $M_{m,n}(\mathbb{Z}^+)$ preserve zero-term rank? The operations of (1) permuting rows, (2) permuting columns, and (3) (if $m = n$) transposing the matrices in $M_{m,n}(\mathbb{Z}^+)$ are all linear operators that preserve zero-term rank of the matrices on $M_{m,n}(\mathbb{Z}^+)$.

If we take a fixed $m \times n$ matrix B in $M_{m,n}(\mathbb{Z}^+)$, then its *Schur product operator* on $M_{m,n}(\mathbb{Z}^+)$ is defined by $B \circ X = [b_{ij}x_{ij}]$ for all $X \in M_{m,n}(\mathbb{Z}^+)$.

LEMMA 3.2. *Suppose that T is an operator on $M_{m,n}(\mathbb{Z}^+)$ such that $T(X) = B \circ X$, where B is fixed in $M_{m,n}(\mathbb{Z}^+)$, none of whose entries is zero in $\mathbb{Z}^+$. Then T is a linear operator which preserves zero-term rank.*

Proof. For all α, $\beta \in \mathbb{Z}^+$ and A, $B \in M_{m,n}(\mathbb{Z}^+)$, we have the following equality;

$$\begin{aligned} T(\alpha X + \beta Y) &= B \circ (\alpha X + \beta Y) = B \circ (\alpha X) + B \circ (\beta Y) \\ &= \alpha(B \circ X) + \beta(B \circ Y) = \alpha T(X) + \beta T(Y). \end{aligned}$$

Hence T is linear. The fact T preserves zero-term rank follows the definition of Schur product operator. □

DEFINITION 3.3. ([5]) If P and Q are $m \times m$ and $n \times n$ permutation matrices, respectively and B is an $m \times n$ matrix, none of whose entries is zero in $\mathbb{Z}^+$, then T is a (P, Q, B)-*operator* on $M_{m,n}(\mathbb{Z}^+)$ if

(1) $T(X) = P(B \circ X)Q$ for all X in $M_{m,n}(\mathbb{Z}^+)$ or
(2) $m = n$, and $T(X) = P(B \circ X^t)Q$ for all X in $M_{m,n}(\mathbb{Z}^+)$.

THEOREM 3.4. ([5]) *If* $\mathbf{S}$ *is any semiring, then the following are equivalent for any linear operator* T *on* $M_{m,n}(\mathbf{S})$*;*

(1) T *is a* (P, Q, B)*-operator;*
(2) T *preserves term rank;*
(3) T *preserves term ranks 1 and 2.*

We will show that the zero-term rank preservers on $M_{m,n}(\mathbb{Z}^+)$ are also of the same form as the term rank preservers on $M_{m,n}(\mathbb{Z}^+)$. For this purpose, we define a mapping $T' : \mathcal{E} \to \mathcal{E}$ by $T'(i,j) = (u,v)$ whenever $T(E_{ij}) = b_{ij}E_{uv}$ with $0 < b_{ij} \leq 1$, where $\mathcal{E} = \{(i,j) : 1 \leq i \leq m, 1 \leq j \leq n\}$ is the set of all indices.

Now we have some Lemmas which are need to obtain the main Theorem 3.9.

LEMMA 3.5. *Suppose that* T *preserves zero-term rank* 1 *on* $M_{m,n}(\mathbb{Z}^+)$ *and* $T(J) \geq J$. *Then* T *maps a cell onto a cell with a scalar multiple and hence* T' *is a bijection on* $\mathcal{E}$.

Proof. If $T(E_{ij}) = 0$ for some $E_{ij} \in \Delta$, then we can choose $mn - 1$ cells

$E_1, E_2, \cdots, E_{mn-1}$ which are different from E_{ij}. Thus we have

$$\begin{aligned} J \leq T(J) &= T(E_{ij} + \sum_{k=1}^{mn-1} E_k) \\ &= T(E_{ij}) + T(\sum_{k=1}^{mn-1} E_k) \\ &= 0 + T(\sum_{k=1}^{mn-1} E_k) \\ &= T(\sum_{k=1}^{mn-1} E_k). \end{aligned}$$

But $z(J) = 0$ and $z(\sum_{k=1}^{mn-1} E_k) = 1$. Since T preserves zero-term rank 1, we have $z(T(\sum_{k=1}^{mn-1} E_k)) = 1$. Since $J \leq T(J) = T(\sum_{k=1}^{mn-1} E_k)$, we have $0 = z(J) \geq z(T(\sum_{k=1}^{mn-1} E_k)) = 1$ by Lemma 2.1. Then we have a contradiction. Therefore $T(E_{ij})$ dominates at least one cell with a scalar multiple. That is, $T(E_{ij})$ is of the form $T(E_{ij}) \geq b_{ij} E_{uv}$ for some $E_{uv} \in \Delta$ with some nonzero integer b_{ij}.

For some cell $E_{ij} \in \Delta$, suppose $T(E_{ij})$ dominates two cells, that is, $T(E_{ij}) \geq b_{ij} E_{kl} + b'_{ij} E_{uv}$ with some nonzero integers b_{ij}, b'_{ij}. For each cell E_{rs} except for both E_{kl} and E_{uv}, we can choose one cell E_h such that $T(E_h)$ dominates E_{rs} because $T(J) \geq J$. Since the number of cells except for both E_{kl} and E_{uv} is mn-2, there exist at most mn-1 cells $E_1, E_2, \cdots, E_{mn-1}$ containing E_{ij} such that $T(\sum_{h=1}^{mn-1} E_h)) \geq J$. Since T preserves zero-term rank 1, we have $z(T(\sum_{h=1}^{mn-1} E_h)) = z(\sum_{h=1}^{mn-1} E_h) = 1$. But $1 = z(\sum_{h=1}^{mn-1} E_h) \leq z(J) = 0$ by Lemma 2.1. Thus we have a contradiction. Hence $T(E_{ij})$ dominates only one cell with a scalar multiple. That is, T maps a cell into a cell with a scalar multiple.

Now, we show that T' is a bijection on $\mathcal{E}$. If $T'(i,j) = T'(k,l) = (u,v)$ for some distinct indices (i,j) and (k,l), then we have $T(E_{ij}) = a_{ij} E_{uv}$ and $T(E_{kl}) = b_{kl} E_{uv}$ with some nonzero integers a_{ij} and b_{kl}. Thus we have

$$\begin{aligned} J \leq T(J) &= T(J - (E_{ij} + E_{kl}) + (E_{ij} + E_{kl})) \\ &= T(J - (E_{ij} + E_{kl})) + T(E_{ij} + T(E_{kl}) \\ &= T(J - (E_{ij} + E_{kl})) + a_{ij} E_{uv} + b_{kl} E_{uv} \\ &= T(J - (E_{ij} + E_{kl})) + (a_{ij} + b_{kl}) E_{uv} \end{aligned}$$

But we have

$$\begin{aligned}
& z(T(J-(E_{ij}+E_{kl}))+(a_{ij}+b_{kl})E_{uv}) \\
= \; & z(T(J-(E_{ij}+E_{kl}))+a_{ij}E_{uv}) \\
= \; & z(T(J-(E_{ij}+E_{kl}))+T(E_{ij})) \\
= \; & z(T(J-(E_{ij}+E_{kl})+E_{ij})) \\
= \; & z(T(J-E_{kl}))
\end{aligned}$$

Since T preserves zero-term rank 1 and $z(J-E_{kl})=1, z(T(J-E_{kl})=1$. Since $J \le T(J-(E_{ij}+E_{kl}))+(a_{ij}+b_{kl})E_{uv}$, we have

$$0=z(J)\ge z(T(J-(E_{ij}+E_{kl}))+(a_{ij}+b_{kl})E_{uv})=z(T(J-E_{kl}))=1.$$

This is a contradiction. Therefore T' is an injection on $\mathcal{E}$ and hence a bijection on $\mathcal{E}$. □

LEMMA 3.6. *If T preserves zero-term rank 1 on $M_{m,n}(\mathbf{Z}^+)$ and $T(J)\ge J$, then T preserves term rank 1.*

Proof. Suppose that T does not preserve term rank 1. Then there exist some cells E_{ij} and E_{il} on the same row(or column) such that $T(E_{ij}+E_{il})=T(E_{ij})+T(E_{il})= b_{ij}E_{pq}+b_{il}E_{rs}$ with some nonzero integers b_{ij} and b_{il}, where $T'(i,j)=(p,q)$ and $T'(i,l)=(r,s)$ with $p\ne r$ and $q\ne s$. Since T preserves zero-term rank 1 and $T(J)\ge J$, we have that T' is a bijection on $\mathcal{E}$ by Lemma 3.5. Thus we have $T(J)= B=(b_{uv})_{m\times n}$, for some $B\in M_{m,n}(\mathbf{Z}^+)$ with $b_{uv}\ge 1$. Since T preserve zero-term rank 1 and $z(J-E_{ij}-E_{il})=1$, we have $z(T(J-E_{ij}-E_{il}))=1$. But $T(J-E_{ij}-E_{il})$ has zeros in the (p,q) and (r,s) positions because $T(E_{ij}+E_{il})=b_{ij}E_{pq}+b_{il}E_{rs}$. Then $z(T(J-E_{ij}-E_{il}))=2$. This is impossible. Therefore T preserves term rank 1. □

LEMMA 3.7. *If T preserves zero-term rank 1 on $M_{m,n}(\mathbf{Z}^+)$ and $T(J)\ge J$, then T maps a row of a matrix onto a row(or column if m=n) with a scalar multiple in $\mathbf{Z}^+$.*

Proof. Suppose T does not map a row into a row(or column if m=n) with a scalar multiple. Then T does not preserve term rank 1. This contradicts to Lemma 3.6. Hence T maps a row into a row(or column if m=n) with a scalar multiple. Lemma 3.5 implies that T' is a bijection on $\mathcal{E}$. Then the bijectivity of T' implies that T maps a row onto a row(or column if m=n) with a scalar multiple. □

LEMMA 3.8. *For the case m=n, suppose that T preserves zero-term rank 1 on $M_{m,n}(\mathbb{Z}^+)$ and $T(J) \geq J$. If T maps a row onto a row(or column) with a scalar multiple, then all rows of a matrix must be mapped some rows(or columns, respectively) with a scalar multiple.*

Proof. Lemma 3.5 implies that T' is a bijection on $\mathcal{E}$. Let $R_i = \sum_{j=1}^{n} E_{ij}$, $C^{(j)} = \sum_{i=1}^{n} E_{ij}$ for $i, j = 1, 2, \cdots, n$. Suppose T maps a row, say R_1 , onto an ith row R_i with a scalar multiple B_i and another row, say R_2 , onto a jth column $C^{(j)}$ with a scalar multiple $B^{(j)}$. That is, $T(R_1) = B_i \circ R_i$ and $T(R_2) = B^{(j)} \circ C^{(j)}$. Then $R_1 + R_2$ has $2n$ cells but $B_i \circ R_i + B^{(j)} \circ C^{(j)}$ has $2n-1$ cells. This contradicts to the bijectivity of T' on $\mathcal{E}$. Hence all row must be mapped some rows(or columns, respectively) with a scalar multiple. □

We have the following characterization theorem for zero-term rank preserver on $M_{m,n}(\mathbb{Z}^+)$.

THEOREM 3.9. *Suppose that T is a linear operator on $M_{m,n}(\mathbb{Z}^+)$. Then the following statements are equivalent;*

(i) *T is a (P, Q, B)-operator;*

(ii) *T preserves zero-term rank;*

(iii) *T preserves zero-term rank 1 and $T(J) \geq J$.*

Proof. (i)$\Longrightarrow$ (ii): Suppose T is a (P, Q, B)-operator and $X \in M_{m,n}(\mathbb{Z}^+)$. Then $T(X) = P(B \circ X)Q$(or $m = n$, and $T(X) = P(B \circ X^t)Q$), where P and Q are $m \times m$ and $n \times n$ permutation matrices and B is an $m \times n$ matrix in $M_{m,n}(\mathbb{Z}^+)$, none of whose entries is zero. Hence $z(T(X)) = z(P(B \circ X)Q) = z(X)$ or $z(T(X)) =$

$z(P(B \circ X^t)Q) = z(X)$. Since X is arbitrary, T preserves zero-term rank.

(ii)$\Longrightarrow$ (iii): clearly.

(iii)$\Longrightarrow$ (i): Suppose T preserves zero-term rank 1 and $T(J) \geq J$. Lemmas 3.7 and 3.8 imply that T maps all rows of a matrix onto rows(or columns if m=n) with a scalar multiple. Thus T is of the form $T(X) = P(B \circ X)Q$ or $T(X) = P(B \circ X^t)Q$, where P and Q are permutation matrices and B is a fixed $m \times n$ matrix in $M_{m,n}(\mathbb{Z}^+)$, none of whose entries is zero. Hence T is a (P, Q, B)-operator. □

LEMMA 3.10. *For A, B in $M_{m,n}(\mathbb{Z}^+)$, $A \geq B$ implies $T(A) \geq T(B)$.*

Proof. By definition of $A \geq B$, we have $a_{ij} \geq b_{ij}$ for all i, j. Using the forms of $A = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} E_{ij}$ and $B = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} E_{ij}$, we have

$$\begin{aligned} T(A) &= T(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}E_{ij}) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}T(E_{ij}) \\ &\geq \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}T(E_{ij}) \\ &= T(\sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}E_{ij}) \\ &= T(B). \end{aligned}$$

because of linearity and $a_{ij} \geq b_{ij}$. Hence $T(A) \geq T(B)$. □

We say that a linear operator T on $M_{m,n}(\mathbb{Z}^+)$ *strongly preserves zero-term rank* k provided that $z(T(A)) = k$ if and only if $z(A) = k$.

LEMMA 3.11. *If T strongly preserves zero-term rank 1 on $M_{m,n}(\mathbb{Z}^+)$, then we have $T(J) \geq J$.*

Proof. Since T strongly preserves zero-term rank 1 and $z(J) \neq 1$. we have $z(T(J)) = 0$ or $z(T(J)) \geq 2$. Suppose $z(T(J)) \geq 2$. Since $J \geq J - E_{ij}$ for any cell E_{ij} in $\mathcal{E}$, we have $T(J) \geq T(J - E_{ij})$ by Lemma 3.10. But Lemma 2.1

implies $z(T(J)) \leq z(T(J - E_{ij}))$. Since T strongly preserves zero-term rank 1 and $z(J - E_{ij}) = 1$, we have $z(T(J - E_{ij})) = 1$. This is a contradiction because $z(T(J)) \geq 2$ and $z(T(J)) \leq z(T(J - E_{ij})) = 1$. Thus $z(T(J)) = 0$ and hence $T(J) \geq J$. □

THEOREM 3.12. *Suppose that T is a linear operator on $M_{m,n}(\mathbf{Z}^+)$. Then T preserves zero-term rank if and only if it strongly preserves zero-term rank* 1.

Proof. Suppose T strongly preserves zero-term rank 1. Then Lemma 3.11 implies that $T(J) \geq J$. By Theorem 3.9, T preserves zero term rank. Conversely, suppose T preserves zero-term rank. If $z(T(X)) = 1$ and $z(X) \neq 1$, then $z(X) = 0$ or $z(X) \geq 2$. If $z(X) = 0$, then $z(T(X)) = 0$ by assumption. If $z(X) \geq 2$, then $z(T(X)) \geq 2$ by assumption. Those contradict to $z(T(X)) = 1$. Hence T strongly preserves zero-term rank 1. □

Thus we have characterized the linear operators that preserve the zero-term rank on $M_{m,n}(\mathbf{Z}^+)$, which extend the results on Boolean case in [7]. It turns out that the linear operator is a (P, Q, B)-operator, which equals term rank preserver. Also, we obtained several kinds of conditions that are equivalent to a (P, Q, B)-operator.

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비음인 정수행렬의 영향계수를 보존하는 선형연산자

한희정, 강경태, 송석준

제주도 제주시 아라동 제주대학교 정보수학과 (690-756)

요 약

주어진 행렬의 영항 계수는 그 행렬에 나타나는 모든 영 원소들을 덮을 수 있는 행과 열의 최소수로 정의된다. 본 논문에서는 비음의 정수들로 이루어진 반환에서 원소를 갖는 행렬들을 생각한다. 이 행렬들의 영항계수를 보존하는 선형연산자를 연구하여 그 형태를 규명하였고, 또 이 선형연산자와 필요충분조건이 되는 명제들을 찾아서 그 동치성을 증명하였다.