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The properties of the transversal Killing spinor on a Riemannian foliation

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Abstract. We study the properties of the transversal Killing spinors on a foliated Riemannian manifold with a transverse spin structure.

1 Introduction

Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and a bundle-like metric g_M. In [9], the author introduced the transversal Killing spinor which is given by the solution of the equation

$$\nabla_X \Psi + f\pi(X) \cdot \Psi = 0 \quad \text{for } X \in TM, \tag{1.1}$$

where f is a basic function and $\pi : TM \to Q$ is a projection (see (2.1)). It is well known [9] that any eigenvalue λ of the basic Dirac operator D_b satisfies the inequality

$$\lambda^2 \geq \frac{q}{4(q-1)} \inf_M (\sigma^\nabla + |\kappa|^2) \tag{1.2}$$

where $q = codim\mathcal{F}$, σ^∇ is the transversal scalar curvature and κ is the mean curvature form of $\mathcal{F}$. And in the limiting case, M admits a transversal Killing spinor.

In this paper, we study the properties of the transversal Killing spinor which occurs in the limiting case in (1.2).

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2 Preliminaries and known facts

In this section, we review the basic properties of the Riemannian foliation, which are studied in [11,18]. Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$. We recall the exact sequence

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0 \tag{2.1}$$

determined by the tangent bundle L and the normal bundle $Q = TM/L$ of $\mathcal{F}$. The assumption of g_M to be a bundle-like metric means that the induced metric g_Q on the normal bundle $Q \cong L^\perp$ satisfies the holonomy invariance condition $\overset{\circ}{\nabla} g_Q = 0$, where $\overset{\circ}{\nabla}$ is the Bott connection in Q.

For a distinguished chart $\mathcal{U} \subset M$ the leaves of $\mathcal{F}$ in $\mathcal{U}$ are given as the fibers of a Riemannian submersion $f : \mathcal{U} \to \mathcal{V} \subset N$ onto an open subset $\mathcal{V}$ of a model Riemannian manifold N.

For overlapping charts $U_\alpha \cap U_\beta$, the corresponding local transition functions $\gamma_{\alpha\beta} = f_\alpha \circ f_\beta^{-1}$ on N are isometries. Further, we denote by ∇ the canonical connection of the normal bundle Q of $\mathcal{F}$. It is defined by

$$\begin{cases} \nabla_X s = \pi([X, Y_s]) & \text{for } X \in \Gamma L, \\ \nabla_X s = \pi(\nabla^M_X Y_s) & \text{for } X \in \Gamma L^\perp, \end{cases} \tag{2.2}$$

where $s \in \Gamma Q$, and $Y_s \in \Gamma L^\perp$ corresponding to s under the canonical isomorphism $L^\perp \cong Q$. The connection ∇ is metric and torsion free. It corresponds to the Riemannian connection of the model space N. The curvature R^∇ of ∇ is defined by

$$R^\nabla(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \quad \text{for } X,\ Y \in TM.$$

Since $i(X)R^\nabla = 0$ for any $X \in \Gamma L$([11]), we can define the (transversal) Ricci curvature $\rho^\nabla : \Gamma Q \to \Gamma Q$ and the (transversal) scalar curvature σ^∇ of $\mathcal{F}$ by

$$\rho^\nabla(s) = \sum_a R^\nabla(s, E_a)E_a, \quad \sigma^\nabla = \sum_a g_Q(\rho^\nabla(E_a), E_a),$$

where $\{E_a\}_{a=1,\cdots,q}$ is an orthonormal basic frame for Q. $\mathcal{F}$ is said to be (transversally) *Einsteinian* if the model space N is Einsteinian, that is,

$$\rho^{\nabla} = \frac{1}{q}\sigma^{\nabla} \cdot id \tag{2.3}$$

with constant transversal scalar curvature σ^{∇}.

The *mean curvature vector field* of $\mathcal{F}$ is then defined by

$$\tau = \sum_i \pi(\nabla^M_{E_i} E_i), \tag{2.4}$$

where $\{E_i\}_{i=1,\cdots,p}$ is an orthonormal basis of L. The dual form κ, the *mean curvature form* for L, is then given by

$$\kappa(X) = g_Q(\tau, X) \quad \text{for } X \in \Gamma Q. \tag{2.5}$$

The foliation $\mathcal{F}$ is said to be *minimal* (or *harmonic*) if $\kappa = 0$.

Let $\Omega^r_B(\mathcal{F})$ be the space of all *basic r-forms*, i.e.,

$$\Omega^r_B(\mathcal{F}) = \{\psi \in \Omega^r(M) |\ i(X)\phi = 0,\ \theta(X)\phi = 0, \text{ for } X \in \Gamma L\}.$$

The foliation $\mathcal{F}$ is said to be *isoparametric* if $\kappa \in \Omega^1_B(\mathcal{F})$. We already know that κ is closed, i.e., $d\kappa = 0$ if $\mathcal{F}$ is isoparametric ([18]). Since the exterior derivative preserves the basic forms (that is, $\theta(X)d\phi = 0$ and $i(X)d\phi = 0$ for $\phi \in \Omega^r_B(\mathcal{F})$), the restriction $d_B = d|_{\Omega^*_B(\mathcal{F})}$ is well defined. Its cohomology

$$H_B(M/\mathcal{F}) = H(\Omega^*_B(\mathcal{F}), d_B) \tag{2.6}$$

is called the *basic cohomology* of $\mathcal{F}$. Let δ_B the adjoint operator of d_B. Then it is well-known([1,9]) that

$$d_B = \sum_a \theta_a \wedge \nabla_{E_a}, \quad \delta_B = -\sum_a i(E_a)\nabla_{E_a} + i(\kappa^\sharp_B), \tag{2.7}$$

where $\kappa^\sharp_B$ is the g_Q-dual vector field of the basic component κ_B of κ, $\{E_a\}$ is a local orthonormal basic frame in Q and $\{\theta_a\}$ its g_Q-dual 1-form.

The *basic Laplacian* acting on $\Omega_B^*(\mathcal{F})$ is defined by ([16])

$$\Delta_B = d_B\delta_B + \delta_B d_B. \tag{2.8}$$

If $\mathcal{F}$ is the foliation by points of M, the basic Laplacian is the ordinary Laplacian.

3 Transversal Dirac operator

Let $S(\mathcal{F})$ be a foliated spinor bundle on a transverse spin foliation $\mathcal{F}$ and $<\cdot,\cdot>$ a hermitian scalar product on $S(\mathcal{F})$.

By the Clifford multiplication in the fibers of $S(\mathcal{F})$ for any vector field X in Q and any transversal spinor field Ψ, the Clifford product $X\cdot\Psi$, which is also a trnasversal spinor field, is defined. This product has the following properties: for any $X, Y \in \Gamma Q$ and $\Phi, \Psi \in \Gamma S(\mathcal{F})$,

$$(X\cdot Y + Y\cdot X)\Psi = -2g_Q(X,Y)\Psi \tag{3.1}$$

$$< X\cdot\Psi, \Phi > + < \Psi, X\cdot\Phi >= 0 \tag{3.2}$$

$$\nabla_Y(X\cdot\Psi) = (\nabla_Y X)\cdot\Psi + X\cdot(\nabla_Y\Psi), \tag{3.3}$$

where ∇ is a metric covariant derivation on $S(\mathcal{F})$, i.e., for all $X \in \Gamma Q$, and all $\Psi, \Phi \in \Gamma S(\mathcal{F})$, it holds

$$X < \Psi, \Phi >=< \nabla_X\Psi, \Phi > + < \Psi, \nabla_X\Phi > . \tag{3.4}$$

Moreover if we define the Clifford product $\xi\cdot\Psi$ of a 1-form $\xi \in Q^*$ and a transversal spinor field Ψ as

$$\xi\cdot\Psi \equiv \xi^\sharp\cdot\Psi, \tag{3.5}$$

where $\xi^\sharp \in \Gamma Q$ is a g_Q-dual vector of ξ, then any basic r-form can be considered as an endomorphism of $S(\mathcal{F})$. Namely, for any basic r-form $\omega = \sum_{i_1<\cdots<i_r}\omega_{i_1\cdots i_r}\theta^{i_1}\wedge\cdots\wedge\theta^{i_r} (\in \Omega_B^r(\mathcal{F}))$, we define the Clifford product $\omega\cdot\Phi$ locally by

$$\omega\cdot\Phi = \sum\omega_{i_1\cdots i_r}\theta_{i_1}\cdots\theta_{i_r}\cdot\Phi. \tag{3.6}$$

On the other hand, the transversal Dirac operator D_{tr} acting on sections of the foliated spinor bundle $S(\mathcal{F})$ is locally given by [3,6,9]

$$D_{tr}\Psi = \sum_a E_a \cdot \nabla_{E_a}\Psi - \frac{1}{2}\kappa \cdot \Psi, \tag{3.7}$$

where $\{E_a\}_{a=1,\cdots,q}$ is a local orthonormal basic frame in Q. At any point $x \in M$, we choose normal coordinates at this point so that $(\nabla E_a)(x) = 0$, for all a. From now on, all the computations in this paper will be made in such charts.

Now, we define the subspace $\Gamma_B S(\mathcal{F})$ of *basic* or *holonomy invariant* sections of $S(\mathcal{F})$ by

$$\Gamma_B S(\mathcal{F}) = \{\Psi \in \Gamma S(\mathcal{F}) |\ \nabla_X \Psi = 0 \quad \text{for } X \in \Gamma L\}.$$

Then we see that D_{tr} leaves $\Gamma_B S(\mathcal{F})$ invariant if and only if the foliation $\mathcal{F}$ is isoparametric, i.e., $\kappa \in \Omega_B^1(\mathcal{F})$. Let $D_b = D_{tr}|_{\Gamma_B S(\mathcal{F})} : \Gamma_B S(\mathcal{F}) \to \Gamma_B S(\mathcal{F})$. This operator D_b is called the *basic Dirac operator* on (smooth) basic sections. It is well-known([6]) that D_b and D_b^2 have the discrete spectrums on M. On an isoparametric transverse spin foliation $\mathcal{F}$ with $\delta\kappa = 0$, we have the Lichnerowicz type formular ([6,9])

$$D_{tr}^2\Psi = \nabla_{tr}^*\nabla_{tr}\Psi + \frac{1}{4}K^\sigma\Psi, \tag{3.8}$$

where $K^\sigma = \sigma^\nabla + |\kappa|^2$ and

$$\nabla_{tr}^*\nabla_{tr}\Psi = -\sum_a \nabla^2_{E_a,E_a}\Psi + \nabla_{\kappa^\sharp}\Psi. \tag{3.9}$$

The operator $\nabla_{tr}^*\nabla_{tr}$ is non-negative and formally self-adjoint ([9]) such that

$$\int_M < \nabla_{tr}^*\nabla_{tr}\Phi, \Psi >= \int_M < \nabla_{tr}\Phi, \nabla_{tr}\Psi > \tag{3.10}$$

for all $\Phi, \Psi \in \Gamma S(\mathcal{F})$. Moreover, the curvature transform R^S on $S(\mathcal{F})$ is given ([9,12]) as

$$R^S(X,Y)\Psi = \frac{1}{4}\sum_{a,b} g_Q(R^\nabla(X,Y)E_a, E_b)E_a \cdot E_b \cdot \Psi \quad \text{for } X, Y \in \Gamma TM. \tag{3.11}$$

Then we have the following lemma.

Lemma 3.1 ([9]) *On the foliated spinor bundle $S(\mathcal{F})$, we have the following equations*

$$\sum_{a<b} E_a \cdot E_b \cdot R^S(E_a, E_b)\Psi = \frac{1}{4}\sigma^\nabla \Psi, \tag{3.12}$$

$$\sum_a E_a \cdot R^S(X, E_a)\Psi = -\frac{1}{2}\rho^\nabla(X) \cdot \Psi \quad \textit{for } X \in \Gamma Q. \tag{3.13}$$

4 Transversal Killing spinor

For a basic function f, the spinor field $\Psi \in \Gamma S(\mathcal{F})$ satisfies the *transversal Killing equation* if

$$\nabla_X^f \Psi \equiv \nabla_X \Psi + f\pi(X) \cdot \Psi = 0 \quad \text{for any } X \in TM. \tag{4.1}$$

In this case, Ψ is called the *transversal Killing spinor* on $\mathcal{F}$.

Lemma 4.1 *If Ψ is a transversal Killing spinor, then the associate vector field X_Ψ defined by*

$$X_\Psi = i\sum_a < \Psi, E_a \cdot \Psi > E_a$$

is a transversal Killing vector field, i.e., $\theta(X_\Psi)g_Q = 0$.

Proof. Generally, we have that for any $Y, Z \in \Gamma Q$

$$(\theta(X)g_Q)(Y,Z) = g_Q(\nabla_Y \pi(X), Z) + g_Q(Y, \nabla_Z \pi(X)).$$

Let $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ with the property that $(\nabla E_a)_x = 0$ for all a. Then we have at x that for any transversal Killing spinor Ψ with $\nabla_X \Psi = -f\pi(X) \cdot \Psi$

$$\begin{aligned}
\nabla_Y X_\Psi &= i\sum_a Y < \Psi, E_a \cdot \Psi > E_a \\
&= i\sum_a \{< \nabla_Y \Psi, E_a \cdot \Psi > + < \Psi, E_a \cdot \nabla_Y \Psi >\} E_a \\
&= -if\sum_a \{< Y \cdot \Psi, E_a \cdot \Psi > + < \Psi, E_a \cdot Y \cdot \Psi >\} E_a.
\end{aligned}$$

Hence we have

$$g_Q(\nabla_Y X_\Psi, Z) = -if\{< Y\cdot\Psi, Z\cdot\Psi > + < \Psi, Z\cdot Y\cdot\Psi >\}.$$

Similarly,

$$g_Q(Y, \nabla_Z X_\Psi) = -if\{< Z\cdot\Psi, Y\cdot\Psi > + < \Psi, Y\cdot Z\cdot\Psi >\}.$$

Hence we have

$$(\theta(X_\Psi)g_Q)(Y,Z) = g_Q(\nabla_Y X_\Psi, Z) + g_Q(Y, \nabla_Z X_\Psi) = 0.$$

This implies that X_Ψ is a transversal Killing vector field. □

Lemma 4.2 *If Ψ is a transversal Killing spinor, then $|\Psi|^2$ is constant.*

Proof. Let Ψ be a transversal Killing spinor, i.e., for some basic function f $\nabla_X\Psi = -f\pi(X)\cdot\Psi$. For any $X \in TM$

$$\begin{aligned} X|\Psi|^2 &= < \nabla_X\Psi, \Psi > + < \Psi, \nabla_X\Psi > \\ &= -f\{< \pi(X)\cdot\Psi, \Psi > + < \Psi, \pi(X)\cdot\Psi >\} \\ &= 0. \end{aligned}$$

So $|\Psi|^2$ is constant. □

Theorem 4.3 ([9]) *If M admits a transversal Killing spinor Ψ with $\nabla^f_X\Psi = 0$, then*

(1) *f is constant and $f^2 = \frac{\sigma^\nabla}{4q(q-1)}$*

(2) *$\mathcal{F}$ is transversally Einsteinian with constant transversal scalar curvature σ^∇.*

Proof. By direct calculation, we have

$$\sum_a E_a\cdot R^f_{XE_a}\Psi = -\frac{1}{2}\rho^\nabla(X)\cdot\Psi + 2(q-1)f^2X\cdot\Psi - qX(f)\Psi - grad_\nabla(f)\cdot X\cdot\Psi$$

for $X \in \Gamma Q$. Since $\nabla^f \Psi = 0$, we have

$$0 = -\frac{1}{2}\rho^\nabla(X)\cdot\Psi + 2(q-1)f^2 X\cdot\Psi - qX(f)\Psi - grad_\nabla(f)\cdot X\cdot\Psi. \tag{4.2}$$

If we put $X = grad_\nabla(f)$, then

$$< -\frac{1}{2}\rho^\nabla(X)\cdot\Psi + 2(q-1)f^2 X\cdot\Psi, \Psi >= (q-1)|grad_\nabla(f)|^2|\Psi|^2. \tag{4.3}$$

Since the left hand side is pure imaginary and right hand side is real, we have

$$|grad_\nabla(f)| = 0.$$

Since f is a basic function, f is constant. Hence (4.2) implies that

$$-\frac{1}{2}\rho^\nabla(X)\cdot\Psi + 2(q-1)f^2 X\cdot\Psi = 0.$$

Hence we have

$$\rho^\nabla(X) = 4(q-1)f^2 X.$$

This implies that $\mathcal{F}$ is transversally Einsteinian. From (2.3), we have $\sigma^\nabla = 4q(q-1)f^2$. □

Theorem 4.4 *If Ψ is a transversal Killing spinor, then*

$$|D_{tr}\Psi|^2 = \frac{1}{4}(\frac{q}{q-1}\sigma^\nabla + |\kappa|^2)|\Psi|^2 \tag{4.4}$$

$$\mathrm{Re} < D_{tr}\Psi, \kappa\cdot\Psi >= -\frac{1}{2}|\kappa|^2|\Psi|^2. \tag{4.5}$$

Proof. Let Ψ be the transversal Killing spinor with $\nabla^f_X\Psi = 0$. From Theorem 4.3, we have

$$\nabla_X\Psi = -fX\cdot\Psi, \quad D_{tr}\Psi = fq\Psi - \frac{1}{2}\kappa\cdot\Psi, \tag{4.6}$$

where $f^2 = \frac{\sigma^\nabla}{4q(q-1)}$. From the second equation in (4.6), we get

$$\begin{aligned} < D_{tr}\Psi, D_{tr}\Psi > &= < fq\Psi - \frac{1}{2}\kappa\cdot\Psi, fq\Psi - \frac{1}{2}\kappa\cdot\Psi > \\ &= (f^2q^2 + \frac{1}{4}|\kappa|^2) < \Psi, \Psi > . \end{aligned}$$

Hence we have

$$|D_{tr}\Psi|^2 = \frac{1}{4}(\frac{q}{q-1}\sigma^\nabla + |\kappa|^2)|\Psi|^2.$$

Since $< X \cdot \Psi, \Psi >$ is pure imaginary, the equation (4.5) follows from (4.6). □

Corollary 4.5 *If there exists an eigenspinor Ψ of D_b with $\nabla^f\Psi = 0$, then $\mathcal{F}$ is minimal.*

Corollary 4.6 *On the minimal foliation $\mathcal{F}$, every transversal Killing spinor is an eigenspinor of D_b.*

Proof. Let Ψ be the transversal Killing spinor. From (4.6), if $\mathcal{F}$ is minimal, then

$$D_b\Psi = fq\Psi.$$

From Theorem 4.3, f is constant. Hence Ψ is an eigenspinor. □

Now we recall the generalized Myers' theorem.

Theorem 4.7 ([8]) *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a Riemannian foliation $\mathcal{F}$ and complete bundle-like metric g_M. If there is a positive lower bound of the transversal Ricci curvature, then the leaf spact $M/\mathcal{F}$ of $\mathcal{F}$ is compact, and the basic cohomology $H^1(M/\mathcal{F}) = 0$.*

Summing up Theorem 4.3 and Theorem 4.7, we have the following theorem.

Theorem 4.8 *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and complete bundle-like metric g_M. If M admits a transversal Killing spinor, then the leaf space $M/\mathcal{F}$ of $\mathcal{F}$ is compact and $H^1(M/\mathcal{F}) = 0$.*

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엽층구조를 가지는 리만다양체상에서의 횡단적 Killing 스피너의 성질

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요 약

본 논문에서는 횡단적 스핀구조를 가진 엽층적 리만다양체상에서의 횡단적 Killing 스피너들의 특성을 연구하였다.