J. of Basic Sciences, Cheju Nat'l Univ.
17(1), 47~58, 2004

기초과학연구 제주대학교
17(1), 47~58, 2004

Linear preservers of roots of matrix polynomials

Seok-Zun Song*

Department of Mathematics, Cheju National University, Jeju 690-756, Korea

Abstract. In this paper we classify linear operators on matrices with coefficients from fields or semirings that preserve different relations that can be defined by multivariable matrix polynomials.

1 Introduction

In the last decades much attention has been paid to Linear Preserver Problems over semirings. A lot of results have their parallel solutions for matrices over semirings. Among the investigated relations there were also several polynomial conditions, see [2, 4, 5, 6, 7].

Let $\mathbf{Z}_+$ be the set of nonnegative integers and $\mathcal{M}_n(\mathbf{Z}_+)$ denote the set of $n \times n$ matrices with coefficients from $\mathbf{Z}_+$.

Many authors have studied the problem of determining the linear maps of $n \times n$ matrix algebra $\mathcal{M}_n(\mathcal{F})$ over a field $\mathcal{F}$ that leave certain matrix relations, subsets, or properties invariant. Among the relations that were considered the central role was played by relations that can be determined in terms of matrix polynomials, for example, commutativity, nilpotence, idempotence, etc., see [1, 3, 8, 9, 10, 11, 12].

The aim of the present paper is to consider linear preservers of roots of matrix polynomials in a systematic way.

*Basic Science Research Institute, Cheju National University

2 Definitions and Notations

By the Amitsur-Levitsky Theorem $n \times n$ matrix algebras over fields satisfy standard polynomial identities of degree $2n$ and not polynomial identities of lower degrees. Moreover, the set of noncommutative polynomials that are identities in matrix algebra is of a very special type. Therefore, together with each noncommutative polynomial equation evaluated in a matrix algebra we can consider the set of matrices that satisfy this polynomial equation. The structure of this variety is unknown. The usual way to generate elements in such variety is to find a concrete n-tuple of matrices in this variety and act on it by various linear transformations that preserves roots of polynomial under consideration. Thus the problem of classification of linear transformations preserving roots of matrix polynomials arises.

Below we start with some polynomials that arise naturally in the study of associative algebras and their matrix representations.

Commutative algebras: Let us denote by $[x, y]$ the additive commutator, i.e., $[x, y] = xy - yx$. The class of commutative algebras is defined by the identity $[x, y] = 0$. In the other words commutativity condition on matrices can be expressed in a following way: Consider the non-commutative matrix polynomial $P_2 = [x, y] = xy - yx$. Thus the set of pairs of commuting matrices is the set of its matrix roots, namely the set

$$\mathcal{V}(P_2) = \{(X, Y) \in \mathcal{M}_n(\mathbf{Z}_+)^2 | XY = YX\}.$$

The infinite dimensional exterior Grassman algebra is

$$\langle 1, e_1, e_2, \ldots | e_i^2 = 0,\ e_i e_j = -e_j e_i \rangle.$$

It is easy to show that the center of the Grassman algebra is the set of words on $e_1, \ldots, e_n, \ldots$ with even length which satisfy the triple commutator polynomial identity, i.e., $P_3(x, y, z) = 0$, where $P_3(x, y, z) = [[x, y], z] = xyz + zyx - yxz - zxy$. Let

$$\mathcal{V}(P_3) = \{(X, Y, Z) \in (\mathcal{M}_n(\mathbf{Z}_+))^3 | XYZ + ZYX = YXZ + ZXY\}.$$

We will also consider a long commutator of arbitrary length:

$$P_m(x_1, \ldots, x_m) = [[\ldots[[x_1, x_2], x_3], \ldots], x_m]$$

and denote

$$\begin{aligned}\mathcal{V}(P_4) &= \{(X,Y,Z,U) \in (\mathcal{M}_n(\mathbf{Z}_+))^4 | \\ &\quad XYZU + ZYXU + UZXY + UYXZ = \\ &\quad ZXYU + YXZU + UXYZ + UZYX\}.\end{aligned}$$

More generally, let $f(x_1,\dots,x_m) = P_m(x_1,\dots,x_m) = [\dots[[x_1,x_2],x_3],\dots x_m]$ be a polynmial with real coefficients. Denote by $f_+(x_1,\dots,x_m)$ the polynomial which consists of all terms of f that have positive coefficients, and by $f_-(x_1,\dots,x_m)$ the polynomial which consists of all terms of f that have negative coefficients. We denote by $\mathcal{V}(P_m)$ the set $\mathcal{V}(P_m) =$

$$\{(X_1X_2,\cdots,X_m) \in (\mathcal{M}_n(\mathbf{Z}_+))^m | f_+(X_1X_2,\cdots,X_m) = f_-(X_1X_2,\cdots,X_m)\}.$$

A 2×2 upper triangular matrix algebra satisfies the polynomial identity which is the product of two commutators: $P_{2,2}(x_1,x_2,x_3,x_4) = [x_1,x_2][x_3,x_4]$. Denote the corresponding variety in $(\mathcal{M}_n(\mathbf{Z}_+))^4$ by

$$\mathcal{V}(P_{2,2}) = \{(X,Y,Z,U) \in (\mathcal{M}_n(\mathbf{Z}_+))^4 | XYZU+YXUZ = XYUZ+YXZU\}.$$

Polynomial identities for $k \times k$ upper triangular matrix algebras have the form

$$P_{2,k}(x_1,\dots,x_{2k}) = [x_1,x_2]\cdots[x_{2k-1,2k}].$$

In the case of antinegative semiring for the simplicity of notations we write $[X,Y] = 0$ for matrices X,Y satisfying $XY = YX$, $[[X,Y],Z] = 0$ for triples $(X,Y,Z) \in \mathcal{V}(P_3)$, $[[[X,Y],Z],U] = 0$ for 4-tuples $(X,Y,Z,U) \in \mathcal{V}(P_4)$, and $[X,Y][Z,U] = 0$ for 4-tuples of matrices $(X,Y,Z,U) \in \mathcal{V}(P_{2,2})$. More generally, we define the set $\mathcal{V}(P_{2,k})$ to be the set of $2k$-tuples of matrices $X_1,\dots,X_{2k}$ satisfying $f_+(X_1,\dots,X_{2k}) = f_-(X_1,\dots,X_{2k})$.

3 Linear Preservers of $\mathcal{V}(P_m)$ for $m \geq 4$.

In what follows we obtain a characterization of bijective linear transformations of matrices over semirings that leave the sets $\mathcal{V}(P_m)$ invariant for $m \geq 3$.

Lemma 3.1 *If $T : \mathcal{M}_n(\mathbf{Z}_+) \to \mathcal{M}_n(\mathbf{Z}_+)$ is a bijective linear transformation, then there exists a permutation σ of the set $\{(i,j)|1 \leq i \leq m, 1 \leq j \leq n\}$ such that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all (i,j).*

Proof. By the fact that T is surjective, there is some X such that , $T(X) = E_{i,j}$ for each pair (i,j).

We say that $A \geq B$ or $B \leq A$ if and only if $b_{i,j} \neq 0$ implies that $a_{i,j} \neq 0$.

If $E_{r,s} \leq X$, let $Y = (y_{i,j})$ be the matrix such that $y_{i,j} = x_{i,j}$ if $(i,j) \neq (r,s)$, and $y_{r,s} = 0$. Then

$$T(x_{r,s}E_{r,s}) \leq T(x_{r,s}E_{r,s}) + T(Y) = T(x_{r,s}E_{r,s} + Y) = T(X) = E_{i,j}$$

since $\mathbf{Z}_+$ is antinegative and T is additive. Hence, $T(x_{r,s}E_{r,s}) \leq E_{i,j}$. Thus $T(x_{r,s}E_{r,s}) = \alpha E_{i,j}$ for a certain $\alpha \in \mathbf{Z}_+$.

Let us check that T induces a permutation on the set of cells. Assume to the contrary that the inverse images of the cells under the action of T do not partition the set of cells. Then there exists (i,j) such that

$$T(E_{i,j}) = a_1 E_{i_1,j_1} + \cdots + a_k E_{i_k,j_k}$$

Since the preimage of a certain multiple of a cell is a cell we have that there exists corresponding indices $s_1, \ldots, s_k, r_1, \ldots, r_k$ and constants $b_1, \ldots, b_k$, and $c_1, \ldots, c_k$ such that

$$T(b_1 E_{s_1,r_1}) = c_1 E_{i_1,j_1}, \quad \ldots, \quad T(b_k E_{s_k,r_k}) = c_k E_{i_k,j_k}$$

Thus one has that

$$T(b_1 a_1 c_2 \cdots c_k E_{s_1,r_1}) = a_1 c_1 c_2 \cdots c_k E_{i_1,j_1},$$

$$T(b_2 a_2 c_1 c_3 \cdots c_k E_{s_2,r_2}) = a_2 c_1 \cdots c_k E_{i_2,j_2},$$

$$\cdots\cdots\cdots$$

$$T(b_k a_k c_1 \cdots c_{k-1} E_{s_k,r_k}) = a_k c_1 \cdots c_k E_{i_k,j_k},$$

Hence,

$$T(c_1 \cdots c_k E_{i,j}) = a_1 c_1 \cdots c_k E_{i_1,j_1} + \cdots + a_k c_1 \cdots c_k E_{i_k,j_k} =$$

$$= T(b_1 a_1 c_2 \cdots c_k E_{s_1,r_1}) + \cdots + T(b_k a_k c_1 \cdots c_{k-1} E_{s_k,r_k})$$

By linearity and bijectivity it follows that

$$c_1 \cdots c_k E_{i,j} = b_1 a_1 c_2 \cdots c_k E_{s_1,r_1} + \ldots + b_k a_k c_1 \cdots c_{k-1} E_{s_k,r_k}$$

Thus since $\mathbf{Z}_+$ is antinegative and without root divisors, one has that for any $l = 1, \ldots, k$ either $b_l = 0$ or $s_l = i$ and $r_l = j$. Let us assume that there exists more than one index l such that $b_l \neq 0$. After reindexing we may assume that $b_1 \neq 0$. Thus $r_1 = i, s_1 = j$ and

$$b_1(a_1 E_{i_1,j_1} + \cdots + a_k E_{i_k,j_k}) = T(b_1 E_{i,j}) = c_1 E_{i_1,j_1}$$

It follows that $a_2 = \cdots = a_k = 0$.

Therefore, the image of a cell is a cell, and preimages of cells partition the set of cells. By bijectivity and the fact that the unique invertible element in $\mathbf{Z}_+$ is 1, one has that for any i, j, $1 \leq i, j \leq n$, there exists some $k = k(i), l = l(j)$ such that $T(E_{i,j}) = E_{k,l}$.

■

Recall that $\mathcal{V}(P_m) =$

$\{(X_1 X_2, \cdots, X_m) \in (\mathcal{M}_n(\mathbf{Z}_+))^m | f_+(X_1 X_2, \cdots, X_m) = f_-(X_1 X_2, \cdots, X_m)\}$.

In what follows we obtain a characterization of bijective linear transformations of matrices over semirings that leave the set $\mathcal{V}(P_m)$ invariant for all $m \geq 4$.

In the case of antinegative semirings, for simplicity of notation, we write $[X, Y] = 0$ for matrices X, Y satisfying $XY = YX$ and

$$[\cdots[[X_1, X_2], X_3], \cdots X_m] = O$$

for m-tuples $(X_1, X_2, \cdots X_m) \in \mathcal{V}(P_m)$.

Theorem 3.2 *Let $T : \mathcal{M}_n(\mathbf{Z}_+) \to \mathcal{M}_n(\mathbf{Z}_+)$ be a bijective linear transformation, $n \geq 3$. The transformation T preserves the set $\mathcal{V}(P_m)$ if and only if there exists a permutation matrix $P \in \mathcal{M}_n(\mathbf{Z}_+)$ such that $T(X) = PXP^{-1}$ for all $X \in \mathcal{M}_n(\mathbf{Z}_+)$ or $T(X) = PX^tP^{-1}$ for all $X \in \mathcal{M}_n(\mathbf{Z}_+)$.*

Proof. It is easily seen that if $T(X) = PXP^{-1}$ for all $X \in \mathcal{M}_n(\mathbf{Z}_+)$ or $T(X) = PX^tP^{-1}$ for all $X \in \mathcal{M}_n(\mathbf{Z}_+)$ then T preserves $\mathcal{V}(P_m)$.

Suppose that T preserves $\mathcal{V}(P_m)$. Since T is bijective, by Lemma 3.1, we have that there is a permutation σ such that $T(E_{i,j}) = E_{\sigma(i,j)}$.

Let us check that T maps lines to lines. Assume that there are i, j, $1 \leq i \neq j \leq n$, such that $T^{-1}(E_{i,i}), T^{-1}(E_{i,j})$ do not lie in one row and one column. Let

$$E_{i,i} = T(E_{r,s}),\ E_{i,j} = T(E_{p,q}),\ E_{j,j} = T(E_{u,v}),$$

$1 \leq p \neq r,\ q \neq s,\ u,\ v \leq n$. Thus

$$E_{r,s}E_{p,q} = 0 = E_{p,q}E_{r,s}$$

and, hence,

$$(E_{r,s}, E_{p,q}, E_{u,v}, E_{u,v}, \cdots, E_{u,v}) \in \mathcal{V}(P_m),$$

since each term of both

$$f_+(E_{r,s}, E_{p,q}, E_{u,v}, E_{u,v}, \cdots, E_{u,v}) \text{ and } f_-(E_{r,s}, E_{p,q}, E_{u,v}, E_{u,v}, \cdots, E_{u,v})$$

contain either $E_{r,s}E_{p,q}$ or $E_{p,q}E_{r,s}$. It is easy to see that

$$(T(E_{r,s}), T(E_{p,q}), T(E_{u,v}), T(E_{u,v}), \cdots, T(E_{u,v})) =$$

$$(E_{i,i}, E_{i,j}, E_{j,j}, E_{j,j}, \cdots, E_{j,j}) \notin \mathcal{V}(P_m)$$

since only one term of $f_+(E_{i,i}, E_{i,j}, E_{j,j}, E_{j,j}, \cdots, E_{j,j})$ is nonzero,

$$E_{i,i}E_{i,j}E_{j,j}E_{j,j} \cdots E_{j,j} = E_{i,j}$$

and all terms of $f_-(E_{i,i}, E_{i,j}, E_{j,j}, E_{j,j}, \cdots, E_{j,j})$ are zero.

This contradicts the assumption that T preserves the set $\mathcal{V}(P_m)$. Thus for any i, j, the preimages of $E_{i,i}$ and $E_{i,j}$ lie in one row or one column.

Similar arguments show that $T^{-1}(E_{k,i})$ lies in r^{th} row or s^{th} column for each k, $1 \leq k \leq n$. Thus, since T permutes the cells, from the coincidence of cardinalities it follows that r^{th} row and s^{th} column are mapped into i^{th} row and i^{th} column.

Let us now show that $r = s$. Suppose to the contrary that $r \neq s$. Then

$$(E_{r,s}, E_{p,s}, E_{u_3,v_3}, \cdots, E_{u_m,v_m}) \in \mathcal{V}(P_m)$$

for all $p \neq s$ and for all u_i, v_i since $E_{r,s}$ commutes with $E_{p,s}$. On the other hand, for $T(E_{p,q}) = E_{i,j}$ choose u_i, v_i such that $(u_i, v_i) = (u_3, v_3)$, $i = 4, \cdots m$ and $T(E_{u_3,v_3}) = E_{j,j}$. Then

$$(E_{i,i}, E_{i,j}, E_{j,j}, \cdots, E_{j,j}) \notin \mathcal{V}(P_m)$$

for $j \neq i$. This contradicts the assumption that T preserves $\mathcal{V}(P_m)$ since there exists u_3, v_3 such that $T(E_{u_3,v_3}) = E_{j,j}$.

Thus one has that for any s, $s = 1, \dots, n$ $T(E_{s,s}) = E_{i,i}$, s^{th} row and column are mapped to i^{th} row and column.

For any $p \neq s, q \neq s$ one has that $[E_{p,s}, E_{q,s}] = 0$, i.e., $(E_{p,s}, E_{q,s}, E_{u,v}) \in \mathcal{V}(P_3)$ for all $1 \leq p \neq s, q \neq s \leq n$ and for all $1 \leq u, v \leq n$. Let us assume that $T(E_{p,s})$ and $T(E_{q,s})$ do not lie in one row or one column. Without loss of generality one may assume that $T(E_{p,s}) = E_{i,k}$, $T(E_{q,s}) = E_{l,i}$ for certain k, l, $1 \leq k, l \leq n$. Note that $i \neq k, l$ since $s \neq p, q$. Let us denote $T(E_{u_t,v_t}) = E_{a_t,b_t}$, $t = 3, \cdots, m$. Thus $(E_{i,k}, E_{l,i}, E_{a_3,b_3}, \cdots, E_{a_m,b_m}) \in \mathcal{V}(P_m)$ for all $1 \leq a_t, b_t \leq n$, $t = 3, \cdots, m$, since $(E_{p,s}, E_{q,s}, E_{u_3,v_3}, \cdots, E_{u_m,v_m}) \in \mathcal{V}(P_m)$. That is ,

$$f_+ = E_{i,k}E_{l,i}E_{a_3,b_3} \cdots E_{a_m,b_m} + E_{a_3,b_3}E_{l,i}E_{i,k}E_{a_4,b_4} \cdots E_{a_m,b_m} + \cdots$$

$$= E_{a_3,b_3}E_{i,k}E_{l,i} \cdots E_{a_m,b_m} + \cdots = f_-.$$

Let us consider the case that $b_3 = l$, $a_4 = k$, and $b_t = a_{t+1} = b_4$ for $t = 4, \cdots, m$ but $a_3 \neq i, k$, $b_4 \neq l, a_3$. This is possible since $n \geq 3$. Thus left-hand side, f_+, of the last equality is nonzero while the right-hand side, f_-, is zero.

This contradiction shows that the s^{th} column is transformed to the i^{th} row or to the i^{th} column. Similarly, the s^{th} row is transformed to the i^{th} row or to the i^{th} column.

Thus T maps rows into rows or columns. Since T is bijective on the set of cells and two rows consist of $2n$ cells while a row and column consists of $2n - 1$ cells, one can conclude that T maps all rows to rows and all columns to columns, or all rows to columns and all columns to rows.

Thus, since the image of each cell is a cell, the transformation T only permutes rows and columns of any matrix and, possibly, transpose it. This means that there exist permutation matrices $P, Q \in \mathcal{M}_n(\mathbf{Z}_+)$ such that $T(X) = PXQ$ for all $X \in \mathcal{M}_n(\mathbf{Z}_+)$ or $T(X) = PX^tQ$ for all $X \in \mathcal{M}_n(\mathbf{Z}_+)$.

We now show that $Q = P^{-1}$. Since P and Q are permutation matrices one has that $T(E_{i,j}) = E_{\tau(i),\pi(j)}$ for certain permutations $\tau, \pi \in S_n$. Let us assume that $\tau \neq \pi$. Thus there exists i, $1 \leq i \leq n$ such that $\tau(i) \neq \pi(i)$. Hence there exists $k \neq i$ such that $\tau(i) = \pi(k)$. For any $l \neq i, j$, one has that $[E_{l,k}, E_{i,j}] = 0$. Thus

$$(E_{l,k}, E_{i,j}, E_{a,b}, \cdots, E_{c,d}) \in \mathcal{V}(P_m)$$

for all $1 \leq a, b, c, \cdots, d \leq n$. Then

$$(T(E_{l,k}), T(E_{i,j}), T(E_{a,b}), \cdots, T(E_{c,d})) =$$

$$= (E_{\tau(l),\pi(k)}, E_{\tau(i),\pi(j)}, E_{\tau(a),\pi(b)}, \cdots, E_{\tau(c),\pi(d)}) \in \mathcal{V}(P_m).$$

By the choice of i, k one has that $[E_{\tau(l),\pi(k)}, E_{\tau(i),\pi(j)}] \neq 0$ since

$$E_{\tau(l),\pi(k)} E_{\tau(i),\pi(j)} = E_{\tau(l),\pi(j)} \neq E_{\tau(i),\pi(j)} E_{\tau(l),\pi(k)}$$

because $l \neq i$. By bijectivity of π and τ there exists a, $1 \leq a \leq n$ such that $\pi(j) = \tau(a)$. Thus for $b \neq j$ and appropriate c, d one has that $[\cdots[[E_{\tau(l),\pi(k)}, E_{\tau(i),\pi(j)}], E_{\tau(a),\pi(b)}], \cdots, E_{\tau(c),\pi(d)}] \neq 0$, i.e.,

$$(E_{\tau(l),\pi(k)}, E_{\tau(i),\pi(j)}, E_{\tau(a),\pi(b)}, \cdots E_{\tau(c),\pi(d)}) \notin \mathcal{V}(P_m).$$

This concludes the proof. ∎

A special case of the above theorem is:

Corollary 3.3 *Let $T : \mathcal{M}_n(\mathbf{Z}_+) \to \mathcal{M}_n(\mathbf{Z}_+)$ be a bijective linear transformation, $n \geq 3$. Transformation T preserves the set $\mathcal{V}(P_3)$ if and only if there exists a permutation matrix $P \in \mathcal{M}_n(\mathbf{Z}_+)$ such that $T(X) = PXP^{-1}$ for all $X \in \mathcal{M}_n(\mathbf{Z}_+)$ or $T(X) = PX^tP^{-1}$ for all $X \in \mathcal{M}_n(\mathbf{Z}_+)$.*

4 Linear Preservers of the Commutator Product

In this section we obtain a characterization of bijective linear transformations of matrices over $\mathbf{Z}$ that leave the set $\mathcal{V}(P_{2,2})$ invariant.

Theorem 4.1 *Let $T : \mathcal{M}_n(\mathbf{Z}_+) \to \mathcal{M}_n(\mathbf{Z}_+)$ be a bijective linear transformation. Then T preserves the set $\mathcal{V}(P_{2,k})$ if and only if there exists a permutation matrix $P \in \mathcal{M}_n(\mathbf{Z}_+)$ such that $T(X) = PXP^{-1}$ for all $X \in \mathcal{M}_n(\mathbf{Z}_+)$ or $T(X) = PX^tP^{-1}$ for all $X \in \mathcal{M}_n(\mathbf{Z}_+)$.*

Proof. By Lemma 3.1, since T is bijective, we have that there is a permutation σ such that $T(E_{i,j}) = E_{\sigma(i,j)}$. As in the previous theorem we will show that T^{-1} maps lines to lines.

Suppose that the image under T^{-1} of some line is not a line. Then, there are i, j with $1 \leq i, j \leq n$, such that $T^{-1}(E_{i,i})$ and $T^{-1}(E_{i,j})$ do not lie in one row or one column. Let $E_{i,i} = T(E_{p,q})$ and $E_{i,j} = T(E_{r,s})$ where, $1 \leq p, q, r, s \leq n$, $p \neq r$ and $q \neq s$. Then, $E_{p,q}E_{r,s} = O = E_{r,s}E_{p,q}$ while $E_{i,}E_{i,j} = E_{i,j} \neq O = E_{i,j}E_{i,i}$. Without loss of generality we may assume that $i = 1$ and $j = 2$. Now, let $B_{2l+1} = \begin{bmatrix} O & J_{n-1,n-1} \\ O & O \end{bmatrix}$ and $B_{2l+2} = \begin{bmatrix} O & O \\ O & J_{1,n-1} \end{bmatrix}$ for $l = 1, \cdots, k-1$. Let $A_1 = E_{p,q}$, $A_2 = E_{r,s}$, $A_{2l+1} = T^{-1}(B_{2l+1})$, and $A_{2l+2} = T^{-1}(B_{2l+2})$. Here $f_+(A_1, A_2, \cdots, A_{2k}) = f_-(A_1, A_2, \cdots, A_{2k}) = 0$ so that $(A_1, A_2, \cdots, A_{2k}) \in \mathcal{V}(P_{2,k})$, but $f_+(T(A_1), T(A_2), \cdots, T(A_{2k})) = f_+(E_{1,1}, E_{1,2}, B_3, B_4, \cdots, B_{2k}) =$

$$\begin{bmatrix} 0 & a & a & \cdots & a \\ \vec{0} & \vec{0} & \vec{0} & \cdots & \vec{0} \end{bmatrix},$$

where $a \neq 0$, in fact $a = (n-2)^{k-2}$. But, $f_-(T(A_1), T(A_2), \cdots, T(A_{2k})) = f_-(E_{1,1}, E_{1,2}, B_3, B_4, \cdots, B_{2k}) = O$ since each tern has a factor that is either $E_{1,2}E_{1,1}$ or $B_{2k+2}B_{2k+1}$ for some l, each of which is O. Thus,

$$(T(A_1), T(A_2), \cdots, T(A_{2k})) \notin \mathcal{V}(P_{2,k}),$$

a contradiction. Thus the inverse image of a line is a line, and thus, since T is bijective, the image of any line under T is also a line.

Since T is bijective we must have that either $T(X) = QXP$ for all $X \in \mathcal{M}_n(\mathbf{Z}_+)$ or $T(X) = QX^tP$ for all $X \in \mathcal{M}_n(\mathbf{Z}_+)$, for some permutation matrices P and Q. Since every transformation of the form $T(X) = PXP^{-1}$ or $T(X) = PX^tP^{-1}$ preserves $\mathcal{V}(P_{2,k})$, we may assume that $T(X) = QX$ We will now show that $Q = I$.

Suppose, without loss of generality, that T maps row 2 to row 1. Let $X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 1 \\ \vec{0} & \vec{0} & \vec{0} & \cdots & \vec{0} \end{bmatrix}$ then $Y = T(X) = \begin{bmatrix} 0 & 0 & 1 & \cdots & 1 \\ \vec{0} & \vec{0} & \vec{0} & \cdots & \vec{0} \end{bmatrix}$. Now, $(E_{2,1}, X, E_{2,1}, X, \cdots, E_{2,1}, X) \in \mathcal{V}(P_{2,k})$ but

$$(T(E_{2,1}), T(X), T(E_{2,1}), T(X), \cdots, T(E_{2,1}), T(X)) \notin \mathcal{V}(P_{2,k}),$$

which contradicts that T preserves $\mathcal{V}(P_{2,k})$. Thus $Q = I$. ■

A special case of the above is:

Corollary 4.2 *Let $T : \mathcal{M}_n(\mathbf{Z}_+) \to \mathcal{M}_n(\mathbf{Z}_+)$ be a bijective linear transformation, $n \geq 4$. Then T preserves the set $\mathcal{V}(P_{2,2})$ if and only if there exists a permutation matrix $P \in \mathcal{M}_n(\mathbf{Z}_+)$ such that $T(X) = PXP^{-1}$ for all $X \in \mathcal{M}_n(\mathbf{Z}_+)$ or $T(X) = PX^tP^{-1}$ for all $X \in \mathcal{M}_n(\mathbf{Z}_+)$.*

References

[1] L.B. Beasley, Linear transformations on matrices: the invariance of commuting pairs of matrices, *Linear and Multilinear Algebra*, **6**, 1978, P. 179-183.

[2] L.B. Beasley and S.-G. Lee, Linear operators strongly preserving r-potent matrices over semirings, *Linear Algebra Appl.*, **162**, 1992, P. 589-599.

[3] L.B. Beasley and N.J. Pullman, Linear operators preserving idempotent matrices over fields. *Linear Algebra Appl.*, **146**, 1990, P. 7-20.

[4] L.B. Beasley and N.J. Pullman, Linear operators strongly preserving idempotent matrices over semirings, *Linear Algebra Appl.*, **160**, 1992, P. 217-229.

[5] L.B. Beasley and N.J. Pullman, Linear operators that strongly preserve commuting pairs of Boolean matrices, *Linear Algebra Appl.*, **132**, 1990, P. 137-143.

[6] L.B. Beasley and N.J. Pullman, Linear operators that strongly preserve commuting pairs of fuzzy matrices, *Fuzzy Sets and Systems*, **41**, 1991, P. 167-173.

[7] L.B. Beasley and S.-Z. Song, Linear operators that preserve commuting pairs of nonnegative real matrices, Preprint.

[8] E.P. Botta, S. Pierce, and W. Watkins, Linear transformations that preserve the nilpotent matrices, *Pacific J. Math.*, **104**, 1983, P. 39-46.

[9] G.-H. Chan and M.-H. Lim, Linear preservers on powers of matrices, *Linear Algebra Appl.*, **162**, 1992, P. 615-626.

[10] R. Howard, Linear maps that preserve matrices annihilated by a polynomial, *Linear Algebra Appl.*, **30**, 1980, P. 167-176.

[11] S. Pierce et al, A Survey of Linear Preserver Problems, *Linear and Multilinear Algebra*, **33**, 1992, P. 1-119.

[12] W. Watkins, Linear maps that preserve commuting pairs of matrices, *Linear Algebra Appl.*, **14**, 1976, P. 29-35.

행렬다항식의 근을 보존하는 선형연산자

송 석 준
제주대학교 자연과학대학 정보수학과

요 약

본 논문은 체나 반환에서 원소들을 갖는 행렬들 상에서 선형연산자의 형태를 연구하였다. 이 선형연산자들은 다변수 행렬 다항식들에 의하여 정의될 수 있는 여러 가지 관계들을 보존하는 선형연산자들로서 선형보존자 문제에 관한 연구의 일환으로서, 최근에 매우 다양한 연구가 진행되는 중요한 과제임을 참고문헌들을 통하여 알 수 있다. 본 논문에서는 이 행렬다항식의 관계를 보존하는 선형연산자의 형태를 완전히 분류하여 정리들을 얻었다.