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SOME VANISHING THEOREMS ON KÄHLER FOLIATIONS

SEOUNG DAL JUNG AND JIN SUK PAK

ABSTRACT. We shall prove some vanishing theorems for the transversal Dirac operators on Kähler foliations

1. Introduction

J. Brüning and F. W. Kamber ([1]) studied the transversal Dirac operators on compact foliated Riemannian manifolds and proved some vanishing theorems for the transversal Dirac operators. Also, J.S.Pak and S.D.Jung ([8]) extended the above results to the complete cases. In this paper, we shall prove some vanishing theorems on compact Kähler foliations. Throughout this paper, we shall be in c^∞-class. Manifolds are assumed to be connected, orientable, paracompact and hausdorff spaces. We also adopt the following ranges of indices :

$$1 \leqslant i, j, \cdots \leqslant p; \quad 1 \leqslant a, b, \cdots \leqslant n,$$

$$1 \leqslant \alpha, \beta, \cdots \leqslant q(= 2n), \quad 1 \leqslant A, B, \cdots \leqslant p + q.$$

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2. Preliminaries

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with an oriented foliation $\mathcal{F}$ of codimension $q(=2n)$ and a bundle-like metric g_M with respect to $\mathcal{F}$. Then there exists an exact sequence of vector bundles

$$O \to L \to TM \to Q \to O,$$

where L is the tangent bundle and Q is the normal bundle of $\mathcal{F}$ with respect to g_M ([9]). The foliation is assumed to be transversally Kähler. By a *Kähler foliation* $\mathcal{F}$ we mean a foliation satisfying the following conditions; (i) $\mathcal{F}$ is Riemannian, with a bundle-like metric g_M on M inducing the holonomy invariant metric g_Q on $Q \cong L^\perp$, (ii) there is a holonomy invariant almost complex structure $J : Q \to Q$, where $dimQ = q(=2n)$ (real dimension), with respect to which g_Q is Hermitian, i.e., $g_Q(JX, JY) = g_Q(X, Y)$ for X,$Y \in \Gamma(Q)$, and (iii) if ∇ denotes the unique metric and torsion free connection in Q, then ∇ is almost complex, i.e., $\nabla J = 0$. Note that $\Phi(X, Y) = g_Q(X, JY)$ defines a basic 2-form Φ, which is closed as a consequence of $\nabla g_Q = 0$ and $\nabla J = 0$. Let R_∇ be the curvature associated to the unique metric and torsion free connection ∇ in the normal bundle $\Gamma(Q)$ of the Riemannian foliation $\mathcal{F}$. Let similarily S_∇ be the Ricci curvature. For a Kähler foliation we have then the following properties :

(2.1) $$R_\nabla(X, Y)J = JR_\nabla(X, Y),$$

(2.2) $$R_\nabla(JX, JY) = R_\nabla(X, Y),$$

(2.3) $$S_\nabla(JX, JY) = S_\nabla(X, Y),$$

(2.4) $$R_\nabla(X, Y)Z + R_\nabla(Y, Z)X + R_\nabla(Z, X)Y = 0,$$

where X,Y and Z are elements of $\Gamma(Q)$. In the sequal it will be convinient to use the following orthonormal frame on M. For $x \in M$, let $\{e_A\}$ be an oriented orthonormal basis of T_xM with e_i in L_x and e_α in $L_x^\perp$ ($\mathcal{F}$ is of codimension $q = 2n$ on M^{p+2n}). The transversal Kähler property of $\mathcal{F}$ allows then to extend e_a, Je_a to local vector fields E_a,$JE_a \in \Gamma L^\perp$ such that

(2.5)

$$(\nabla_{E_a} E_b)_x = 0, \quad (\nabla_{E_a} JE_b)_x = 0, \quad (\nabla_{JE_a} E_b)_x = 0, \quad (\nabla_{JE_a} JE_b)_x = 0.$$

As a consequence of torsion freeness

$$[E_a, E_b]_x, \quad [E_a, JE_b]_x, \quad [JE_a, JE_b]_x \in L_x. \tag{2.6}$$

The E_a,JE_a can be chosen as (local) infinitesimal automorphisms of $\mathcal{F}$, so that

$$\nabla_X E_a = \pi[X, E_a] = 0 \quad \text{for} \quad X \in \Gamma L. \tag{2.7}$$

We can complete E_a,JE_a by the Gram-Schmidt process to a moving local frame by adding $E_i \in \Gamma L$ with $(E_i)_x = e_i$. In terms of such a moving frame the transversal Ricci operator and the scalar curvature are given by

$$\rho_\nabla = \sum JR_\nabla(E_a, JE_a) \quad \text{and} \tag{2.8}$$

$$\sigma_\nabla = \sum g_Q(\rho_\nabla(E_\alpha), E_\alpha) \tag{2.9}$$

respectively. Let $\Omega^r_B(\mathcal{F})$ be the space of all basic forms of degree r. The exterior differential d restrcts to $d_B : \Omega^r_B \to \Omega^{r+1}_B$ and let δ_B be the formal adjoint of d_B with respect to the induced scalar product $<, >_B$ on Ω_B ([8]). Now, assume that the mean curvature form k of the foliation $\mathcal{F}$ is isoparametric, i.e., $k \in \Omega^1_B(\mathcal{F})$. It is well known that if $k \in \Omega^1_B(\mathcal{F})$, then $dk = 0$ ([9]).

3. Vanishing Theorems on Kähler foliations

Let $Cl(Q)$ be the transversally Clifford algebra of Q and $\mathbf{C}l(Q) = Cl(Q)\otimes_{\mathrm{R}} \mathbf{C}$ the complexification of $Cl(Q)$. Set

$$\epsilon_a = \frac{1}{2}(E_a - iJE_a), \quad \bar{\epsilon_a} = \frac{1}{2}(E_a + iJE_a), \tag{3.1}$$

where $\{E_a, JE_a\}$ is an orthonormal basis of Q. Then $\{\epsilon_a, \bar{\epsilon_a}\}$ forms a basis of $Q\otimes\mathbf{C}$, complexification of Q and $\mathbf{C}l(Q)$ is generated by $\{\epsilon_a, \bar{\epsilon_a}\}$ which satisfies the relations

$$\epsilon_a\bar{\epsilon_b} + \bar{\epsilon_b}\epsilon_a = -\delta_{ab}, \quad \epsilon_a\epsilon_b = -\epsilon_b\epsilon_a, \quad \bar{\epsilon_a}\bar{\epsilon_b} = -\bar{\epsilon_b}\bar{\epsilon_a}. \tag{3.2}$$

Here we omitted the Clifford multiplication " $\cdot$ ". Let $E \to \dot{M}$ be the holomophic foliated bundle of left modules over $Cl(Q)$, i.e., the fiber E_x is a left module over $Cl(Q)_x$ for each $x \in M$, and the multiplication map is smooth. We assume that E carries a hermitian metric (,) such that ;

(i) Module multiplication by unit tangent vectors is unitary, i.e.,

$$(\varphi s, t) + (s, \bar{\varphi} t) = 0 \tag{3.3}$$

for all $\varphi \in Cl(Q)$ and for all $s, t \in \Gamma(E)$.

(ii) With respect to the cannonical hermitian connection, covariant differentiation is a derivation of module multiplication, i.e., for all $\varphi \in \Gamma(Cl(Q))$ and all $s \in \Gamma(E)$, we have

$$\nabla(\varphi, s) = (\nabla \varphi)s + \varphi(\nabla s). \tag{3.4}$$

We now introduce two differential operators $\mathcal{D}$, $\bar{\mathcal{D}} : \Gamma(E) \to \Gamma(E)$ by formulas

$$\begin{aligned} \mathcal{D} &= \sum \epsilon_a \nabla_{\bar{\epsilon_a}} - \frac{1}{4} H, \\ \bar{\mathcal{D}} &= \sum \bar{\epsilon_a} \nabla_{\epsilon_a} - \frac{1}{4} \bar{H}, \end{aligned} \tag{3.5}$$

where $H = \frac{1}{2}\{k - iJk\}, k$ is a mean curvature form of $\mathcal{F}$.

THEOREM 3.1. *The operators $\mathcal{D}$ and $\bar{\mathcal{D}}$ are formal adjoints of one another and transversally elliptic.*

PROOF. Fix $x \in M$ and choose a local frame $\epsilon_1, \cdots, \epsilon_n, \bar{\epsilon_1}, \cdots, \bar{\epsilon_n}$ as above such that $(\nabla \epsilon_a)_x = (\nabla \bar{\epsilon_a})_x = 0$. Then for all s, $t \in \Gamma(E)$, we have at the point x that

$$\begin{aligned} (\mathcal{D}s, t)_x &= \sum (\epsilon_a \nabla_{\bar{\epsilon_a}} s - \frac{1}{4} H s, t)_x \\ &= -\sum (\nabla_{\bar{\epsilon_a}} s, \bar{\epsilon_a} t)_x + \frac{1}{4}(s, \bar{H} t)_x \\ &= -\sum \bar{\epsilon_a}(s, \bar{\epsilon_a} t)_x \sum + (s, \bar{\epsilon_a} \nabla_{\epsilon_a} t)_x + \frac{1}{4}(s, \bar{H} t)_x \\ &= (divU)_x + \sum (s, \bar{\epsilon_a} \nabla_{\epsilon_a} t)_x + \frac{1}{4}(s, \bar{H} t)_x, \end{aligned}$$

where U is the complex vector field in $Q\otimes\mathbf{C}$ defined by the condition that $g_Q(V,U)=\frac{1}{4}((V-iJV)s,t)$ for all real vectors $V\in\Gamma(Q)$. Then

$$\begin{aligned}(divU)_x &= \sum g_Q(\nabla_{E_a}U,E_a)_x+\sum g_Q(\nabla_{JE_a}U,JE_a)_x\\ &= \sum E_a g_Q(U,e_a)_x+\sum JE_a g_Q(U,JE_a)_x\\ &= \frac{1}{4}\sum\{E_a((E_a-iJE_a)s,t)_x+JE_a((JE_a+ie_a)s,t)_x\}\\ &= \frac{1}{2}\sum\{E_a(\epsilon_a s,t)_x+iJE_a(\epsilon_a s,t)_x\}=\sum\bar{\epsilon_a}(\epsilon_a s,t)_x\\ &= -\sum\bar{\epsilon_a}(s,\bar{\epsilon_a}t)_x.\end{aligned}$$

By the Green's theorem ([10]),

$$\begin{aligned}\int_M divU &= \ll U,k\gg=\frac{1}{4}\int_M((k-iJk)s,t)\\ &= \frac{1}{2}\int_M(Hs,t)=-\frac{1}{2}\int_M(s,\bar{H}t),\end{aligned}$$

where $\ll U,V\gg=\int_M g_Q(U,V)$. It follows that

$$\int_M(\mathcal{D}s,t)=\int_M(s,\bar{\mathcal{D}}t)$$

for all $s,t\in\Gamma(E)$. Hence we have $\mathcal{D}^*=\bar{\mathcal{D}}$. Moreover, by straightforward calculation, $\sigma_{\mathcal{D}}(x,\xi_0)=\xi$ and $\sigma_{\bar{\mathcal{D}}}(x,\xi_0)=\bar{\xi}$ for $\xi_0\in\Gamma(Q^*)\equiv\Gamma(Q)$, where $\xi=\frac{1}{2}(\xi_0-iJ\xi_0)$. □

Now, we define the subspace $\Gamma_B(E)$ of basic or holonomy invariant section of E by

$$\Gamma_B(E)=\{s\in\Gamma(E)|\quad\nabla_X s=0,\quad X\in\Gamma(L)\}. \tag{3.6}$$

If we consider the vector bundle $E=\Lambda Q^*\otimes\mathbf{C}$, then we have

$$\Gamma_B(E)=\Omega_B^*(\mathcal{F})\otimes\mathbf{C}. \tag{3.7}$$

From (3.5), we see that $\mathcal{D}$ and $\bar{\mathcal{D}}$ leaves $\Gamma_B(E)$ invariant if and only if the foliation $\mathcal{F}$ is isoparametric. Put

$$\mathcal{D}_b \equiv \mathcal{D}_{|\Gamma_B(E)} \quad \text{and} \quad \bar{\mathcal{D}}_b \equiv \bar{\mathcal{D}}_{|\Gamma_B(E)}.$$

Now, let $\Omega_B^{r,s}(\mathcal{F})$ be the standard Dolbealt decomposition of $\Omega_B^*(\mathcal{F})\otimes\mathbf{C}$. Then there are operators

$$\partial : \Omega_B^{r,s}(\mathcal{F}) \otimes \mathbf{C} \to \Omega_B^{r+1,s}(\mathcal{F}) \otimes \mathbf{C},$$
$$\bar{\partial} : \Omega_B^{r,s}(\mathcal{F}) \otimes \mathbf{C} \to \Omega_B^{r,s+1}(\mathcal{F}) \otimes \mathbf{C}$$

are given by the followings :

(3.9)
$$\partial = \bar{\epsilon}_a \wedge \nabla_{\epsilon_a}, \quad \bar{\partial} = \epsilon_a \wedge \nabla_{\bar{\epsilon}_a},$$

where ∇ is the Kähler connection on $Q\otimes\mathbf{C}$ and their formal adjoints of ∂ and $\bar{\partial}$ are

(3.10)
$$\partial^* = -i(\epsilon_a)\nabla_{\bar{\epsilon}_a} + \frac{1}{2}i(H),$$
$$\bar{\partial}^* = -i(\bar{\epsilon}_a)\nabla_{\epsilon_a} + \frac{1}{2}i(\bar{H}).$$

These follows from d_B and δ_B by breaking up the formulas of d_B and δ_B into (1,0) and (0,1) components and using $d_B = 2(\partial + \bar{\partial}), \delta_B = 2(\partial^* + \bar{\partial}^*)$. Moreover, if $\mathcal{F}$ is harmonic kähler, by the well known facts; $\partial\bar{\partial}^* + \bar{\partial}^*\partial = 0,\ \mathcal{D}_b = \bar{\partial} + \partial^*$ and $\bar{\mathcal{D}}_b = \partial + \bar{\partial}^*$, we have

(3.11)
$$\mathcal{D}_b\bar{\mathcal{D}}_b + \bar{\mathcal{D}}_b\mathcal{D}_b = \frac{1}{4}\Delta_B,$$

where $\Delta_B = d_B\delta_B + \delta_B d_B$ is the basic Laplacian. Also, we define invariant operators on $\Gamma(E)$ by

(3.12)
$$\nabla_{tr}^*\nabla_{tr}s = -\nabla_{\epsilon_a}\nabla_{\bar{\epsilon}_a}s + \frac{1}{2}\nabla_{\bar{H}}s,$$
$$\bar{\nabla}_{tr}^*\nabla_{tr}s = -\nabla_{\bar{\epsilon}_a}\nabla_{\epsilon_a}s + \frac{1}{2}\nabla_H s,$$
$$\mathcal{R} = \sum \epsilon_a\bar{\epsilon}_b R^E(\bar{\epsilon}_a, \epsilon_b),$$
$$\bar{\mathcal{R}} = \sum \bar{\epsilon}_a\epsilon_b R^E(\epsilon_a, \bar{\epsilon}_b),$$

where R^E is the curvature tensor field on $\Gamma(E)$. Then we have

PROPOSITION 3.2. *The operators $\nabla_{tr}^*\nabla_{tr}$ and $\bar{\nabla}_{tr}^*\bar{\nabla}_{tr}$ are nonnegative, transversally elliptic, formally self-adjoint differential operators.*

PROOF. Fix $x \in M$. If we choose a local frame $\{\epsilon_a, \bar{\epsilon_a}\}$ such that $(\nabla\epsilon_a)_x = (\nabla\bar{\epsilon_a})_x = 0$, then for $s, t \in \Gamma(E)$, we have

$$\begin{aligned}
(\nabla_{tr}^*\nabla_{tr}s, t)_x &= -\sum(\nabla_{\epsilon_a}\nabla_{\bar{\epsilon_a}}s, t)_x + \frac{1}{2}(\nabla_H s, t)_x \\
&= -\sum \epsilon_a(\nabla_{\bar{\epsilon_a}}s, t)_x + \sum(\nabla_{\bar{\epsilon_a}}s, \nabla_{\bar{\epsilon_a}}t)_x + \frac{1}{2}(\nabla_{\bar{H}}s, t)_x \\
&= -(divU)_x + \sum(\nabla_{\bar{\epsilon}_a}s, \nabla_{\bar{\epsilon_a}}t)_x + \frac{1}{2}(\nabla_{\bar{H}}s, t)_x \\
&= -(divU)_x + (divW)_x - \sum(s, \nabla_{\epsilon_a}\nabla_{\bar{\epsilon_a}}t) + \frac{1}{2}(\nabla_{\bar{H}}s, t)_x.
\end{aligned}$$

Here U is the complex vector field in $Q\otimes\mathbf{C}$ defined by the relation : $g_Q(V, U) = \frac{1}{4}(\nabla_{V+iJV}s, t)$ for all real vectors $V \in \Gamma(Q)$. Also, W is defined as

$$g_Q(V, W) = \frac{1}{4}(s, \nabla_{V+iJV}t).$$

Note that at the point $x \in M$,

$$\begin{aligned}
(divU)_x &= \sum\{g_Q(\nabla_{E_a}U, E_a)_x + g_Q(\nabla_{JE_a}U, JE_a)_x\} \\
&= \sum\{E_a g_Q(U, E_a)_x + JE_a g_Q(U, JE_a)_x\} \\
&= \frac{1}{4}\sum\{E_a(\nabla_{E_a+iJE_a}s, t)_x + JE_a(\nabla_{JE_a-iE_a}s, t)_x\} \\
&= \sum \epsilon_a(\nabla_{\bar{\epsilon_a}}s, t)_x.
\end{aligned}$$

By the Green's theorem ([10]),

$$\begin{aligned}
\int_M divU = \ll U, k \gg &= \frac{1}{4}\int_M(\nabla_{k+iJk}s, t) \\
&= \frac{1}{2}\int_M(\nabla_H s, t)
\end{aligned}$$

and similarly

$$(divW)_x = \sum \bar{\epsilon_a}(s, \nabla_{\bar{\epsilon_a}} t)_x.$$

Hence

$$\int_M divW = \frac{1}{2}\int_M (s, \nabla_{\bar{H}} t)$$

Therefore, by integrating

$$\int_M (\nabla_{tr}^* \nabla_{tr} s, t) = \int_M (\nabla_{tr} s, \nabla_{tr} t) = \int_M (s, \nabla_{tr}^* \nabla_{tr} t).$$

where $(\nabla_{tr} s, \nabla_{tr} t) = \sum(\nabla_{\epsilon_a} s, \nabla_{\bar{\epsilon_a}} t)$. Hence $\nabla_{tr}^* \nabla_{tr}$ is nonnegative, formally self adjoint operator. Also, by simple calculation, $\nabla_{tr}^* \nabla_{tr}$ and $\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D}$ have the same principal symbols. So $\nabla_{tr}^* \nabla_{tr}$ is transversally elliptic. Arguments for $\nabla_{tr}^* \nabla_{tr}$ are similar. □

THEOREM 3.3. *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with an isoparametric Kähler foliation $\mathcal{F}$ and a bundle-like metric g_M. Then on $\Gamma(E)$, we have the following identity*

$$2(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})s = \frac{1}{2}\nabla_T^* \nabla_T s + \mathcal{R}^E(s) + \mathcal{K}s,$$

where

$$\nabla_T^* \nabla_T s = 2(\nabla_{tr}^* \nabla_{tr} s + \bar{\nabla}_{tr}^* \bar{\nabla}_{tr} s) = -\sum(\nabla^2_{e_a, e_a} + \nabla^2_{Je_a, Je_a})s,$$

$$\mathcal{R}^E = \frac{1}{4}\sum E_\alpha E_\beta R^E(E_\alpha, E_\beta) \quad \text{and}$$

$$\mathcal{K} = -\frac{1}{2}\{(\partial^* \bar{H} + \bar{\partial}^* H) - \frac{1}{2}|H|^2\}.$$

PROOF. If we choose a local frame $\{\epsilon_a, \bar{\epsilon_a}\}$ such that $(\nabla \epsilon_a)_x = (\nabla \bar{\epsilon_a})_x = 0$, then for any $s \in \Gamma(E)$, using (3.2) and (3.4) we have

$$\begin{aligned}(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})s = &-\sum \nabla_{\epsilon_a}\nabla_{\bar{\epsilon_a}} s + \mathcal{R}(s)\\ &-\frac{1}{4}\sum\{(\epsilon_a \bar{H} + \bar{H}\epsilon_a)\nabla_{\bar{\epsilon_a}} s + (H\bar{\epsilon_a} + \bar{\epsilon_a}H)\nabla_{\epsilon_a} s\}\\ &-\frac{1}{4}\sum\{\epsilon_a \nabla_{\bar{\epsilon_a}} \bar{H} s + \bar{\epsilon_a}\nabla_{\epsilon_a} H s\} - \frac{1}{8}|H|^2 s.\end{aligned}$$

Since $H\bar{\epsilon_a} + \bar{\epsilon_a}H = -2g_Q(H, \bar{\epsilon_a})$, we have $\sum(\bar{\epsilon_a}H + H\bar{\epsilon_a})\nabla_{\epsilon_a}s = -\nabla_H s$. Similarily, $\sum(\epsilon_a\bar{H} + \bar{H}\epsilon_a)\nabla_{\bar{\epsilon_a}}s = -\nabla_{\bar{H}}s$. From (3.9) and (3.10), we have

$$(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})s = -\sum \nabla_{\epsilon_a}\nabla_{\bar{\epsilon_a}}s + \mathcal{R}(s) + \frac{1}{4}(\nabla_H s + \nabla_{\bar{H}}s) - \frac{1}{4}\{(\bar{\partial} + \partial^*)\bar{H} + (\partial + \bar{\partial}^*)H\}s + \frac{1}{8}|H|^2 s$$

because of $i(H)\bar{H} = g_Q(H, \bar{H}) = |H|^2$. and by another calculation, we obtain

$$(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})s = -\sum \nabla_{\bar{\epsilon_a}}\nabla_{\epsilon_a}s + \bar{\mathcal{R}}(s) + \frac{1}{4}(\nabla_H s + \nabla_{\bar{H}}s) - \frac{1}{4}\{(\bar{\partial} + \partial^*)\bar{H} + (\partial + \bar{\partial}^*)H\}s + \frac{1}{8}|H|^2 s.$$

Summing up the above two equations, we have

$$2(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})s = (\nabla_{tr}^*\nabla_{tr} + \bar{\nabla}_{tr}^*\nabla_{tr})s + (\mathcal{R} + \bar{\mathcal{R}})s - \frac{1}{4}\{(\bar{\partial} + \partial^*)\bar{H} + (\partial + \bar{\partial}^*)H\}s + \frac{1}{4}|H|^2 s.$$

Since $dk = 0$ and $dJk = 0$. we have $\partial H = \bar{\partial}\bar{H} = 0$. Moreover, by straight calculation, we have

$$\mathcal{R} + \bar{\mathcal{R}} = \frac{1}{4}\sum E_\alpha E_\beta R^E(E_\alpha, E_\beta).$$

Hence this proof is completed. □

Corollary 3.4. *Let $(M, g_M, \mathcal{F})$ be as in Theorem3.3. Then on $\Gamma_B(E)$, we have*

$$2(\mathcal{D}_b\bar{\mathcal{D}}_b + \bar{\mathcal{D}}_b\mathcal{D}_b) = \Delta|_{\Gamma_B}(E),$$

where $\Delta = \frac{1}{2}\nabla^*\nabla + \mathcal{R}^E + \mathcal{K}$ is a Laplace type operator.

Corollary 3.4 may be used to prove vanishing theorems for Ker $\mathcal{D}_b$ provided one is able to control the divergence term $\delta\mathcal{K}$ in the above expression for $\mathcal{K}$. In fact, we assume that $\delta k = 0$. Then we have $\mathcal{K} =$

$\frac{1}{4}|H|^2$ and the resulting equation is given as following from Corollary 3.4 :

$$g_E((\mathcal{D}_b\bar{\mathcal{D}}_b + \bar{\mathcal{D}}_b\mathcal{D}_b)s, s) = \frac{1}{2}\|\nabla s\|^2 + \frac{1}{2}g_E(\mathcal{R}^E(s), s) + \frac{3}{8}|H|^2\|s\|^2$$

for any $s \in \Gamma_B(E)$, where g_E is a pointwise inner product on E. Thus we have

THEOREM 3.5. *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with an isoparametric Kähler foliation $\mathcal{F}$ and a bundle-like metric g_M. Suppose that the mean curvature form k of $\mathcal{F}$ satisfies $\delta k = 0$. If $\mathcal{R}^E$ is non-negative, then every $s \in Ker\ \mathcal{D}_b \cap Ker\ \overline{\mathcal{D}}_6$ is parallel. Moreover, If $\mathcal{R}^E$ is non-negative and positive at some point of M, then every $s \in Ker\ \mathcal{D}_b \cap Ker\ \bar{\mathcal{D}}_b$ vanishes.*

Moreover, since $\mathbf{C}l(Q)$ is a left module of itself, we can calculate $\mathcal{R}^E$ on $\mathbf{C}l(Q)$ as following : for any $s \in \Gamma(Q)$

(3.13)

$$\begin{aligned}
\mathcal{R}^E(s) &= \frac{1}{4}\sum E_\alpha E_\beta R_\nabla(E_\alpha, E_\beta)s \\
&= \frac{1}{4}\sum E_\alpha E_\beta g_Q(R_\nabla(E_\alpha, E_\beta)s, E_\gamma)E_\gamma \\
&= \frac{1}{4}\sum\{E_a E_b g_Q(R_\nabla(E_a, E_b)s, E_c)E_c \\
&\quad + E_a E_b g_Q(R_\nabla(E_a, E_b)s, JE_c)JE_c \\
&\quad + JE_a JE_b g_Q(R_\nabla(JE_a, JE_b)s, E_c)E_c \\
&\quad + JE_a JE_b g_Q(R_\nabla(JE_a, JE_b)s, JE_c)JE_c \\
&\quad + JE_a E_b g_Q(R_\nabla(JE_a, E_b)s, E_c)E_c \\
&\quad + JE_a E_b g_Q(R_\nabla(JE_a, E_b)s, JE_c)JE_c \\
&\quad + E_a JE_b g_Q(R_\nabla(E_a, JE_b)s, E_c)E_c \\
&\quad + E_a JE_b g_Q(R_\nabla(E_a JE_b)s, JE_c)JE_c\}.
\end{aligned}$$

By using the first Bianchi identity, we have

(3.14)

$$\begin{aligned}
&\sum E_aE_bg_Q(R_\nabla(E_a,E_b)s,E_c)E_c\\
&=-\sum g_Q(R_\nabla(E_a,E_b)E_c,s)E_aE_bE_c\\
&=-\frac{1}{3}\sum_{a\neq b\neq c\neq a} g_Q(R_\nabla(E_a,E_b)E_c\\
&\quad+R_\nabla(E_b,E_c)E_a+R_\nabla(E_c,E_a)E_b,s)E_aE_bE_c\\
&\quad+\sum(E_a,E_b)E_b,s)E_a-\sum g_Q(R_\nabla(E_a,E_b)E_a,s)E_b\\
&=-2\sum g_Q(R_\nabla(E_a,s)E_a,E_b)E_b.
\end{aligned}$$

Similarily, we have

$$\begin{aligned}
&\sum JE_aJE_bg_Q(R_\nabla(JE_a,JE_b)s,JE_c)JE_c\\
&=\;-2\sum g_Q(R_\nabla(JE_a,s)JE_a,JE_b)JE_b.
\end{aligned}\tag{3.15}$$

Also, by straight calculation, we have

$$\begin{aligned}
&\sum\{E_aE_bg_Q(R_\nabla(E_a,E_b)s,JE_c)JE_c\\
&+JE_aE_bg_Q(R_\nabla(JE_a,E_b)s,E_c)E_c\\
&+E_aJE_bg_Q(R_\nabla(E_a,JE_b)s,E_c)E_c\}\\
&=\;-2\sum g_Q(R_\nabla(E_a,JE_b)E_a,s)JE_b\\
&=\;-2\sum g_Q(R_\nabla(E_a,s)E_a,JE_b)JE_b,
\end{aligned}\tag{3.16}$$

$$\begin{aligned}
&\sum\{JE_aJE_bg_Q(R_\nabla(JE_a,JE_b)s,E_c)E_c\\
&+JE_aE_bg_Q(R_\nabla(JE_a,E_b)s,JE_c)JE_c\\
&+E_aJE_bg_Q(R_\nabla(E_a,JE_b)s,JE_c)JE_c\}\\
&=\;-2\sum g_Q(R_\nabla(JE_a,s)JE_a,E_b)E_b.
\end{aligned}\tag{3.17}$$

Substituting (3.14),(3.15),(3.16) and (3.17) into (3.13), we have

$$\begin{aligned}
\mathcal{R}^E(s)&=\frac{1}{2}\sum\{R_\nabla(E_a,s)E_a+R_\nabla(JE_a,s)JE_a\}\\
&=\frac{1}{2}\rho_\nabla(s).
\end{aligned}$$

Thus we have

THEOREM 3.6. *Let $(M, g_M, \mathcal{F})$ be as in Theorem 3.5 . Then on $\Gamma(Q)$ we have*

$$2(\mathcal{D}_b\bar{\mathcal{D}}_b + \bar{\mathcal{D}}_b\mathcal{D}_b) = \frac{1}{2}\nabla_\Gamma^*\nabla_\Gamma + \frac{1}{2}\rho_\nabla + \mathcal{K},$$

where ρ_∇ is the transversal Ricci operator on $\Gamma(Q)$.

THEOREM 3.7. *Let $(M, g_M, \mathcal{F})$ be an isoparametric Kähler foliation with bundle-like metric g_M. Suppose that the mean curvature form k satisfies $\delta k = 0$, then*

a) *If ρ_∇ is nonnegative and positive at some point of M , then every normal section $s \in Ker\ \mathcal{D}_b \cap Ker\ \bar{\mathcal{D}}_b$ vanishes, and*
b) *If ρ_∇ is non-negative , then every $s \in Ker\ \mathcal{D}_b \cap Ker\ \bar{\mathcal{D}}_b$ is parallel.*

By means of (3.11) and Theorem 3.7, we get

COROLLARY 3.8. *Let $(M, g_M, \mathcal{F})$ be a harmonic kähler foliation with bundle like metric g_M. Then if the transversal Ricci curvature is non-negative and positive at some point of M, then there are no nontrivial basic harmonic 1-forms.*

4. Vanishing theorems on Kähler spin foliations

Let $(M, g_M, \mathcal{F})$ be an isoparametric Kähler spin foliation. In this case there exists a principal *Spin* $(2n)$-bundle, $P_{Spin}(Q) \to M$, with a *Spin* $(2n)$-equivalent map, $\xi : P_{Spin}(Q) \to P_{So}(Q)$, to the bundle of (oriented) transversal orthonormal frames on M. The *foliated spinor bundle*, S, is then defined to be the vector bundle associated to the unitary representation τ of *Spin (2n)* given by the unique inreducible complex representation of $Cl(2n)$, i.e., $S = P_{Spin}(Q) \otimes_\tau \mathbf{C}^{2^n}$. This bundle is naturally a bundle of modules over $\mathbf{C}l(Q)$ and carries a cannonical connection induced from the lift of the Riemannian connection on $P_{So}(Q)$ ([4]). Since $\mathcal{F}$ is Kähler foliation, this bundle S is naturally holomorphic and its connection is hermitian. To compute the term $\mathcal{R}$ and $\bar{\mathcal{R}}$ in (3.12) we need to know the curvature tensor R^S of S. This is given in terms of the Riemannian curvature tensor on Q by the formula ([5])

$$R^S(X,Y)s = \frac{1}{4}\sum_{\alpha,\beta} g_Q(R_\nabla(X,Y)E_\alpha, E_\beta)E_\alpha E_\beta s \tag{4.1}$$

for all $X, Y \in \Gamma(Q)$ and all $s \in S$, where $\{E_\alpha\}$ is any real orthonormal basis of $\Gamma(Q)$. Choosing a basis $\{E_a, JE_a\}$, we can reexpress (4.1) as

(4.2)
$$\begin{aligned} R^S(X,Y) &= \sum\{g_Q(R_\nabla(X,Y)\epsilon_a\bar{\epsilon_b})\bar{\epsilon_a}\epsilon_b + g_Q(R_\nabla(X,Y)\bar{\epsilon_a},\epsilon_b)\epsilon_a\bar{\epsilon_b}\} \\ &= 2\sum g_Q(R_\nabla(X,Y)\epsilon_a,\bar{\epsilon_b})\bar{\epsilon_a}\epsilon_b + \sum g_Q(R_\nabla(X,Y)\epsilon_a,\bar{\epsilon_a}), \end{aligned}$$

where we have used the fact : $\epsilon_a\bar{\epsilon_b} + \bar{\epsilon_b}\epsilon_a = -\delta_{ab}$. It follows that from (3.12) and (4.2),

(4.3)
$$\begin{aligned} \mathcal{R} &= \sum \epsilon_a\bar{\epsilon_b}R^S(\bar{\epsilon_a},\epsilon_b) \\ &= \sum g_Q(R_\nabla(\bar{\epsilon_a},\epsilon_b)\epsilon_c,\bar{\epsilon_c})\epsilon_a\bar{\epsilon_b}. \end{aligned}$$

Here we have used the Bianchi identity and the curvature properties on Kähler foliation. Similarily, we have

(4.4)
$$\bar{\mathcal{R}} = \sum g_Q(R_\nabla(\epsilon_a,\bar{\epsilon_b})\bar{\epsilon_c},\epsilon_c)\bar{\epsilon_a}\epsilon_b.$$

Therefore we have

(4.5)
$$\begin{aligned} \mathcal{R}^E &= \mathcal{R} + \bar{\mathcal{R}} \\ &= \sum g_Q(R_\nabla(\epsilon_a,\bar{\epsilon_b})\epsilon_b,\bar{\epsilon_a}) \\ &= \frac{1}{8}\sigma_\nabla, \end{aligned}$$

where σ_∇ is the scalar curvature on Q. Thus we have

THEROEM 4.1. *Let $(M, g_M, \mathcal{F})$ be an isoparametric Kähler spin foliation. Then on the foliated spinor bundle S, we have*

$$2(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D}) = \frac{1}{2}\nabla_T^*\nabla_T + \frac{1}{8}\sigma_\nabla + \mathcal{K}.$$

By means of Theorem 3.5 and Theorem 4.1, we have

THEOREM 4.2. *Let $(M, g_M, \mathcal{F})$ be an isoparametric Kähler spin foliation. Suppose that the mean curvature k satisfies $\delta k = 0$. If $\sigma_\nabla \geqslant 0$ and > 0 at some point, then every $s \in Ker\ \mathcal{D}_b \cap Ker\ \bar{\mathcal{D}}_b$ vanishes, and if $\sigma_\nabla \geqslant 0$, then every $s \in Ker\ \mathcal{D}_b \cap ker\ \bar{\mathcal{D}}_b$ is parallel.*

REMARK. *To understand $\mathcal{D}$ and $\bar{\mathcal{D}}$ in (3.5), we now introduce the transversal Dirac operator $\mathcal{D}_{tr}$ on $\Gamma(E)$:*

$$\mathcal{D}_{tr} = \sum\{E_a \nabla_{E_a} + (JE_a)\nabla_{JE_a}\} - \frac{1}{2}k.$$

Then they are related as follows ([6]) ;

$$\mathcal{D}_{tr} = 2(\mathcal{D} + \bar{\mathcal{D}}).$$

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Seoung Dal Jung
Department of Mathematics
Cheju University
Cheju 690-756, Korea

Jin Suk Pak
Department of Mathematics Education
Kyungpook University
Daegu 702-701, Korea