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# ON THE EXCLUDED MIDDLE LAW AND THE CONTRADICTION LAW FOR FUZZY PROBABILITY

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ABSTRACT. We consider a probability measure $P$ on a probability space $(\Omega, \mathfrak{F}, P)$ and a fuzzy probability $\widetilde{P}$ on $2^{\Omega}$. We prove that the fuzzy probability becomes a probability measure and satisfies elementary properties. But, the excluded middle law and the contradiction law do not hold in fuzzy probability.

## 1. INTRODUCTION

Let $\Omega$ be a nonempty set. Let $\mathfrak{F}$ be a $\sigma$-field of subsets of $\Omega$, that is, a nonempty class of subsets of $\Omega$ which is closed under countable union and complementation.

Let $P$ be a measure defined on $\mathfrak{F}$ satisfying $P(\Omega) = 1$. Then the triple $(\Omega, \mathfrak{F}, P)$ is called a probability space, and $P$, a probability measure. The set $\Omega$ is the sure event, and elements of $\mathfrak{F}$ are called events.

We note that, if $A_n \in \mathfrak{F}, n = 1, 2, \cdots$, then $A_n^c, \cup_{n=1}^{\infty} A_n, \cap_{n=1}^{\infty} A_n, \liminf_{n\to\infty} A_n$, $\limsup_{n\to\infty} A_n$, and $\lim_{n\to\infty} A_n$ (if it exists) are events. Also, the probability measure $P$ is defined on $\mathfrak{F}$, and for all events $A, A_n$,

$$P(A) \geq 0, \quad P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)(A_n\text{'s disjoint}), \quad P(\Omega) = 1.$$

A fuzzy set $A$ on $\Omega$ is called a *fuzzy event*. Let $\mu_A(\cdot)$ be the membership function of the fuzzy event $A$. Then the fuzzy probability of a fuzzy event $A$ is defined by Zadeh([12]) as

$$\widetilde{P}(A) = \int_{\Omega} \mu_A(\omega)\, dP(\omega), \quad \mu_A(\omega) : \Omega \to [0, 1].$$

In this paper, we prove that the fuzzy probability of a fuzzy event becomes a probability measure on $2^{\Omega}$, i.e., satisfies the following (P.1), (P.2) and (P.3) and has some properties (1)~(10).

(P.1) For every fuzzy events $A \subset \Omega$, $0 \leq \widetilde{P}(A) \leq 1$.

(P.2) $\widetilde{P}(\Omega) = 1$.

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(P.3) For disjoint fuzzy events $A_i$ $(i = 1, 2, \cdots)$,

$$\widetilde{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \widetilde{P}(A_i).$$

(1) $\widetilde{P}(\emptyset) = 0$.

(2) For disjoint fuzzy events $A_i$ $(i = 1, 2, \cdots, n)$,

$$\widetilde{P}(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \widetilde{P}(A_i).$$

For any fuzzy events $A$ and $B$,

(3) If $A \subset B$, then $\widetilde{P}(A) \leq \widetilde{P}(B)$.

(4) $\widetilde{P}(A \cup B) = \widetilde{P}(A) + \widetilde{P}(B) - \widetilde{P}(A \cap B)$.

(5) $\widetilde{P}(A^c) = 1 - \widetilde{P}(A)$.

(6) $\widetilde{P}(A \hat{+} B) = \widetilde{P}(A) + \widetilde{P}(B) - \widetilde{P}(A \cdot B)$.

(7) $\widetilde{P}(A \cap (B \cup C)) = \widetilde{P}((A \cap B) \cup (A \cap C)) = \widetilde{P}(A \cap B) + \widetilde{P}(A \cap C) - \widetilde{P}(A \cap B \cap C)$.

(8) $\widetilde{P}(A \cup (A \cap B)) = \widetilde{P}(A) + \widetilde{P}(A \cap B) - \widetilde{P}(A \cap A \cap B) = \widetilde{P}(A)$.

(9) $\widetilde{P}(A \cup \emptyset) = \widetilde{P}(A) + \widetilde{P}(\emptyset) - \widetilde{P}(\emptyset) = \widetilde{P}(A)$.

(10) $\widetilde{P}(A \cup \Omega) = \widetilde{P}(\Omega)$, $\quad \widetilde{P}(A \cap \Omega) = \widetilde{P}(A)$.

In probability theory, the following excluded middle law and the contradiction law hold, i.e., for any events $A, B \in \mathfrak{F}$,

(1) $P(A \cup A^c) = P(\Omega) = 1$.

(2) $P(A \cap A^c) = P(\emptyset) = 0$.

(3) $A \subset B \;\Rightarrow\; P(B - A) = P(B) - P(A)$.

But, it does not hold in fuzzy probability, i.e., for any fuzzy events $A$ and $B$,

(1) $\widetilde{P}(A \cup A^c) \neq \widetilde{P}(\Omega)$.

(2) $\widetilde{P}(A \cap A^c) \neq \widetilde{P}(\emptyset)$.

(3) $A \subset B \;\not\Rightarrow\; \widetilde{P}(B - A) = \widetilde{P}(B) - \widetilde{P}(A)$.

## 2. Fuzzy set operator

When interval is defined on real number $\mathbb{R}$, this interval is said to be a subset of $\mathbb{R}$. For instance, if interval is denoted as $A = [a_1, a_3]$, $a_1, a_3 \in \mathbb{R}$, $a_1 < a_3$, we may regard this as one kind of sets. Expressing the interval as membership function is

$$\mu_A(x) = \begin{cases} 0, & x < a_1, \quad a_3 < x, \\ 1, & a_1 \leq x \leq a_3. \end{cases}$$

If $a_1 = a_3$, this interval indicate a point. That is $[a_1, a_1] = a_1$.

Let $X$ be a set of elements, called the universe, whose elements are denoted $x$. Membership in a classical subset $A$ of $X$ is often viewed as a characteristic function $\mu_A$ from $X$ to $[0, 1]$ such that $\mu_A(x) = 1$ iff $x \in A$, and $\mu_A(x) = 0$ iff $x \notin A$. $[0,1]$ is called a valuation set.

**Definition 2.1.** If the valuation set is allowed to be the real interval $[0,1]$, $A$ is called a fuzzy set. $\mu_A(x)$ is to 1, the more $x$ belongs to $A$.

Clearly, $A$ is a subset of $X$ that has no sharp boundary. $A$ is completely characterized by the set of pairs

$$A = \{(x, \mu_A(x)), x \in X\}.$$

When $X$ is a finite set $\{x_1, \cdots, x_n\}$, a fuzzy set $A$ on $X$ is expressed as

$$A = \mu_A(x_1)/x_1 + \cdots + \mu_A(x_n)/x_n = \sum_{i=1}^{n} \mu_A(x_i)/x_i.$$

When $X$ is not finite, we write

$$A = \int_X \mu_A(x)/x.$$

Two fuzzy sets $A$ and $B$ are said to be equal(denoted $A = B$) if and only if $\mu_A(x) = \mu_B(x), \forall x \in X$.

**Example 2.2.** $X = \{1, 2, 3, 4, 5, 6\}$. Membership function for $A = \{\text{three or so}\}$ is given as follows;

$$\mu_A(1) = 0.3, \ \ \mu_A(2) = 0.6, \ \ \mu_A(3) = 1, \ \ \mu_A(4) = 0.5, \ \ \mu_A(5) = 0.1, \ \ \mu_A(6) = 0,$$

i.e., $A = 0.3/1 + 0.6/2 + 1/3 + 0.5/4 + 0.1/5 + 0/6$.

**Example 2.3.** $X = \mathbb{R}$. Let $\mu_A(x) = \dfrac{1}{1 + (x - 7)^2}$, i.e.,

$$A = \int_{\mathbb{R}} \frac{1}{1 + (x - 7)^2} / x.$$

$A$ is a fuzzy set of real numbers clustered around 7.

**Definition 2.4.** Operations of fuzzy sets are defined as

(1) Union $A \cup B$ :

$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}, \quad \forall x \in X.$$

(2) Intersection $A \cap B$ :

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, \quad \forall x \in X.$$

(3) Complement $A^c$ :

$$\mu_{A^c}(x) = 1 - \mu_A(x), \quad \forall x \in X.$$

(4) Probabilistic sum $A\hat{+}B$ :

$$\mu_{A\hat{+}B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x), \quad \forall x \in X.$$

(5) Probabilistic product $A \cdot B$ :

$$\mu_{A\cdot B}(x) = \mu_A(x) \cdot \mu_B(x), \quad \forall x \in X.$$

(6) Bounded sum $A \oplus B$ :

$$\mu_{A\oplus B}(x) = \min\{1, \mu_A(x) + \mu_B(x)\}, \quad \forall x \in X.$$

(7) Bounded product $A \odot B$ :

$$\mu_{A\odot B}(x) = \max\{0, \mu_A(x) + \mu_B(x) - 1\}, \quad \forall x \in X.$$

(8) Drastic sum $A \sqcup B$ :

$$\mu_{A\sqcup B}(x) = \begin{cases} \mu_A(x), & \text{if } \mu_B(x) = 0, \\ \mu_B(x), & \text{if } \mu_A(x) = 0, \\ 1, & \text{otherwise.} \end{cases}$$

(9) Drastic product $A \sqcap B$ :

$$\mu_{A\sqcap B}(x) = \begin{cases} \mu_A(x), & \text{if } \mu_B(x) = 1, \\ \mu_B(x), & \text{if } \mu_A(x) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(10) Difference $A - B$ :

$$A - B = A \cap B^c.$$

**Definition 2.5.** For two fuzzy sets $(A, \mu_A)$ and $(B, \mu_B)$, $A$ is a subset of $B$ denoted $A \subset B$ if for all $x \in X, \mu_A(x) \le \mu_B(x)$.

**Theorem 2.6.** For two fuzzy sets $(A, \mu_A)$ and $(B, \mu_B)$,

(1) $A \cup B \subset A\hat{+}B \subset A \oplus B$.

(2) $A \odot B \subset A \cdot B \subset A \cap B$.

*Proof.*

(1) For every $x \in X$, since $0 \le \mu_A(x) \le 1$ and $0 \le \mu_B(x) \le 1$,

$$\begin{aligned}\mu_{A\cup B}(x) &= \max\{\mu_A(x), \mu_B(x)\} \\ &\leq \mu_A(x) + \mu_B(x) - \mu_A(x)\cdot\mu_B(x) \\ &= \mu_{A\hat{+}B}(x) \\ &\leq \min\{1, \mu_A(x) + \mu_B(x)\} \\ &= \mu_{A\oplus B}(x).\end{aligned}$$

(2) For every $x \in X$, since $0 \leq \mu_A(x) \leq 1$, $0 \leq \mu_B(x) \leq 1$ and

$$\mu_A(x)\cdot\mu_B(x) - (\mu_A(x) + \mu_B(x) - 1) = (\mu_A(x) - 1)(\mu_B(x) - 1) \geq 0,$$

we have

$$\begin{aligned}\mu_{A\odot B}(x) &= \max\{0, \mu_A(x) + \mu_B(x) - 1\} \\ &\leq \mu_A(x)\cdot\mu_B(x) \\ &= \mu_{A\cdot B}(x) \\ &\leq \min\{\mu_A(x), \mu_B(x)\} \\ &= \mu_{A\cap B}(x).\end{aligned}$$

□

**Example 2.7.** Let $A = \{(1, 0.5), (2, 0.9), (3, 1), (4, 0.9), (5, 0.5), (6, 0.3)\}$, $B = \{(2, 0.4), (3, 0.8), (4, 1), (5, 1), (6, 0.8), (7, 0.7)\}$ and $X = \{1, 2, \cdots, 10\}$.

$$\begin{aligned}
&A \cup B = \{(1, 0.5), (2, 0.9), (3, 1), (4, 1), (5, 1), (6, 0.8), (7, 0.7)\}. \\
&A \cap B = \{(2, 0.4), (3, 0.8), (4, 0.9), (5, 0.5), (6, 0.3)\}. \\
&A^c = \{(1, 0.5), (2, 0.1), (4, 0.1), (5, 0.5), (6, 0.7), (7, 1), (8, 1), (9, 1), (10, 1)\}. \\
&A\hat{+}B = \{(1, 0.5), (2, 0.94), (3, 1), (4, 1), (5, 1), (6, 0.86), (7, 0.7)\}. \\
&A \cdot B = \{(2, 0.36), (3, 0.8), (4, 0.9), (5, 0.5), (6, 0.24)\}. \\
&A \oplus B = \{(1, 0.5), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (7, 0.7)\}. \\
&A \odot B = \{(2, 0.3), (3, 0.8), (4, 0.9), (5, 0.5), (6, 0.1)\}. \\
&A \sqcup B = \{(1, 0.5), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (7, 0.7)\}. \\
&A \sqcap B = \{(3, 0.8), (4, 0.9), (5, 0.5)\}. \\
&A - B = \{(1, 0.5), (2, 0.6), (3, 0.2), (6, 0.2)\}.
\end{aligned}$$

**Theorem 2.8.** Fuzzy set operator have the following properties.

(1) Commutative law : $A \cup B = B \cup A, \quad A \cap B = B \cap A.$

(2) Associative law :

$$A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C,$$
$$A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C.$$

(3) Distributive law :

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

(4) Involution : $(A^c)^c = A.$

(5) Idempotency : $A \cup A = A, \quad A \cap A = A.$

(6) Absorption : $A \cup (A \cap B) = A, \quad A \cap (A \cup B) = A.$

(7) Identity : $A \cup \phi = A, \quad A \cap \phi = \phi.$

(8) Absorption by $\phi$ and $U$ : $A \cap \phi = \phi, \quad A \cup U = U.$

(9) De Morgan's law :

$$(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c.$$

## 3. Probability Theory

Let $\Omega$ be a nonempty set. Let $\mathfrak{F}$ be a $\sigma$-field of subsets of $\Omega$, that is, a nonempty class of subsets of $\Omega$ which is closed under countable union and complementation.

Let $P$ be a measure defined on $\mathfrak{F}$ satisfying $P(\Omega) = 1$. Then the triple $(\Omega, \mathfrak{F}, P)$ is called a probability space, and $P$, a probability measure. The set $\Omega$ is the sure event, and elements of $\mathfrak{F}$ are called events.

We note that, if $A_n \in \mathfrak{F}, n = 1, 2, \cdots$, then $A_n^c, \cup_{n=1}^{\infty} A_n, \cap_{n=1}^{\infty} A_n, \liminf_{n\to\infty} A_n$, $\limsup_{n\to\infty} A_n$, and $\lim_{n\to\infty} A_n$ (if it exists) are events. Also, the probability measure $P$ is defined on $\mathfrak{F}$, and for all events $A, A_n$,

$$P(A) \geq 0, \quad P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)(A_n\text{'s disjoint}), \quad P(\Omega) = 1.$$

It follows that

$$P(\emptyset) = 0, \quad P(A) \leq P(B) \ \text{ for } \ A \subset B, \quad P(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P(A_n).$$

Moreover,

$$P(\liminf_{n\to\infty} A_n) \leq \liminf_{n\to\infty} P(A_n) \leq \limsup_{n\to\infty} P(A_n) \leq P(\limsup_{n\to\infty} A_n),$$

and, if $\lim_{n\to\infty} A_n$ exists, then

$$P(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} P(A_n).$$

This is known as the continuity property of probability measures.

**Definition 3.1.** Let $(\Omega, \mathfrak{F}, P)$ be a probability space. A real-valued function $X$ defined on $\Omega$ is said to be a random variable if

$$X^{-1}(E) = \{\omega \in \Omega : X(\omega) \in E\} \in \mathfrak{F} \quad \text{for all} \quad E \in \mathcal{B},$$

where $\mathcal{B}$ is the $\sigma$-field of Borel sets in $\mathbb{R} = (-\infty, \infty)$ ; that is, a random variable $X$ is a measurable transformation of $(\Omega, \mathfrak{F}, P)$ into $(\mathbb{R}, \mathcal{B})$.

It suffices to require that $X^{-1}(I) \in \mathfrak{F}$ for all intervals $I = (-\infty, b]$, and so on.

We note that a random variable $X$ defined on $(\Omega, \mathfrak{F}, P)$ induces a measure $P_X$ on $\mathcal{B}$ defined by the relation

$$P_X(E) = P\{X^{-1}(E)\} \quad (E \in \mathcal{B}).$$

Clearly, $P_X$ is a probability measure on $\mathcal{B}$ and is called the probability distribution or the distribution of $X$. We note that $P_X$ is a Lebesgue-Stieltjes measure on $\mathcal{B}$.

**Definition 3.2.** For every $x \in \mathbb{R}$ set

$$F_X(x) = P_X(-\infty, x] = P\{\omega \in \Omega : X(\omega) \leq x\}.$$

We call $F_X = F$ the distribution of the random variable $X$.

**Theorem 3.3.** The distribution function $F$ of a random variable $X$ is a nondecreasing, right-continuous function on $\mathbb{R}$ which satisfies

$$F(-\infty) = \lim_{x\to-\infty} F(x) = 0$$

and

$$F(\infty) = \lim_{x\to\infty} F(x) = 1.$$

Let $(\Omega, \mathfrak{F}, P)$ be a probability space, and $X$ be a random variable defined on it. Let $g$ be a real-valued Borel-measurable function on $\mathbb{R}$. Then $g(X)$ is also a random variable.

**Definition 3.4.** We say that the mathematical expectation of $g(X)$ exists if $E[g(X)]$ of the random variable $g(X)$

$$E[g(X)] = \int_\Omega g(X(\omega))\, dP(\omega) = \int_\Omega g(X)\, dP$$

is finite.

We note that a random variable $X$ defined on $(\Omega, \mathfrak{F}, P)$ induces a measure $P_X$ on a Borel set $B \in \mathcal{B}$ defined by the relation $P_X(B) = P\{X^{-1}(B)\}$. Then $P_X$ becomes a probability measure on $\mathcal{B}$ and is called the probability distribution of $X$. It is known that if $E[g(X)]$ exists, then $g$ is also integrable over $\mathbb{R}$ with respect to $P_X$. Moreover, the relation

$$\int_\Omega g(X)\, dP = \int_{\mathbf{R}} g(t)\, dP_X(t) \tag{3.1}$$

holds. We note that the integral on the right-hand side of (3.1) is the Lebesgue-Stieltjes integral of $g$ with respect to $P_X$.

In particular, if $g$ is continuous on $\mathbb{R}$ and $E[g(X)]$ exists, we can rewrite (3.1) as follows

$$\int_\Omega g(X)\, dP = \int_{\mathbf{R}} g\, dP_X = \int_{-\infty}^{\infty} g(x)\, dF(x),$$

where $F$ is the distribution function corresponding to $P_X$, and the last integral is a Riemann-Stieltjes integral.

Let $F$ be absolutely continuous on $\mathbb{R}$ with probability density function $f(x) = F'(x)$. Then $E[g(X)]$ exists if and only if $\int_{-\infty}^{\infty} |g(x)| f(x) dx$ is finite and in that case we have

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx.$$

We note some elementary properties of random variables with finite expectations which follow as immediate consequences of the properties of integrable functions. Denote by $\mathfrak{F}_1 = \mathfrak{F}_1(\Omega, \mathfrak{F}, P)$ the set of all random variables with finite expectations. In the following we write a.s. to abbreviate "almost surely with respect to the probability distribution of $X$ on $(\mathbb{R}, \mathfrak{B})$".

(1) $X, Y \in \mathfrak{F}_1$ and $\alpha, \beta \in \mathbb{R} \Rightarrow \alpha X + \beta Y \in \mathfrak{F}_1$ and $E(\alpha X + \beta Y) = \alpha E[X] + \beta E[Y]$.

(2) $X \in \mathfrak{F}_1 \Rightarrow |E[X]| \le E[|X|]$.

(3) $X \in \mathfrak{F}_1, X \ge 0$ a.s. $\Rightarrow E[X] \ge 0$.

(4) Let $X \in \mathfrak{F}_1$. Then $E[|X|] = 0 \Leftrightarrow X = 0$ a.s..

(5) For $E \in \mathfrak{F}$, write $\chi_E$ for the indicator function of the set $E$, that is, $\chi_E = 1$ on $E$ and $\chi_E = 0$ otherwise. Then $X \in \mathfrak{F}_1 \Rightarrow X \cdot \chi_E \in \mathfrak{F}_1$, and we write

$$\int_E X dP = E[X \cdot \chi_E].$$

Also, $E[|X| \cdot \chi_E] = 0 \Leftrightarrow$ either $P(E) = 0$ or $X = 0$ a.s. on $E$.

(6) If $X \in \mathfrak{F}_1$, then $X = 0$ a.s. $\Leftrightarrow E[X \cdot \chi_E] = 0$ for all $E \in \mathfrak{F}$.

(7) Let $X \in \mathfrak{F}_1$ and $E \in \mathfrak{F}$. If $\alpha \leq X \leq \beta$ a.s. on $E$ for $\alpha, \beta \in \mathbb{R}$, then

$$\alpha P(E) \leq \int_E X dP \leq \beta P(E).$$

(8) Let $Y \in \mathfrak{F}_1$, and $X$ be a random variable such that $|X| \leq |Y|$ a.s.. Then $X \in \mathfrak{F}_1$ and $E[|X|] \leq E[|Y|]$. In particular, if $X$ is bounded a.s., then $X \in \mathfrak{F}_1$.

**Example 3.5.** Let the random variable $X$(denoted $X \sim N(m, \sigma^2)$) have the normal distribution given by the probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-m)^2}{2\sigma^2}}, \quad x \in \mathbb{R},$$

where $\sigma^2 > 0$ and $m \in \mathbb{R}$. Then $E[|X|^\gamma] < \infty$ for every $\gamma > 0$, and we have

$$E[X] = m \quad \text{and} \quad E[(X - m)^2] = \sigma^2.$$

The induced measure $P_X$ is called the normal distribution.

## 4. Main Results

A fuzzy set $A$ on $\Omega$ is called a *fuzzy event*. Let $\mu_A(\cdot)$ be the membership function of the fuzzy event $A$. Then the fuzzy probability of a fuzzy event $A$ is defined by Zadeh([12]) as

$$\widetilde{P}(A) = \int_\Omega \mu_A(\omega)\, dP(\omega), \quad \mu_A(\omega) : \Omega \to [0, 1].$$

**Theorem 4.1.** The fuzzy probability of a fuzzy event becomes a probability measure on $2^\Omega$, i.e., satisfies the following properties.

(P.1) For every fuzzy events $A \subset \Omega$, $0 \leq \widetilde{P}(A) \leq 1$.

(P.2) $\widetilde{P}(\Omega) = 1$.

(P.3) For disjoint fuzzy events $A_i$ $(i = 1, 2, \cdots)$,

$$\widetilde{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \widetilde{P}(A_i).$$

*Proof.*

(P.1) For every fuzzy events $A \subset \Omega$, $0 \le \mu_A(\omega) \le 1$. Thus

$$0 \le \widetilde{P}(A) \le 1.$$

(P.2) Since $\mu_\Omega(\omega) = 1$,

$$\widetilde{P}(\Omega) = \int_\Omega \mu_\Omega(\omega) dP(\omega) = \int_\Omega dP(\omega) = 1.$$

(P.3) Since $A_i'$s disjoint, $A_i \cap A_j = \emptyset$ if $i \neq j$. Thus $\cap_{i=1}^{\infty} A_i = \emptyset$. Therefore $\mu_{\cup_i A_i}(\omega) = \sum_{i=1} \mu_{A_i}(\omega)$, and thus

$$\begin{aligned}\widetilde{P}(\bigcup_{i=1}^{\infty} A_i) &= \int_\Omega \mu_{\cup_i^\infty A_i}(\omega) dP(\omega) \\ &= \int_\Omega \sum_{i=1}^{\infty} \mu_{A_i}(\omega) dP(\omega) \\ &= \sum_{i=1}^{\infty} \int_\Omega \mu_{A_i}(\omega) dP(\omega) \\ &= \sum_{i=1}^{\infty} \widetilde{P}(A_i).\end{aligned}$$

Therefore, the fuzzy probability of a fuzzy event becomes a probability measure on $2^\Omega$. □

**Theorem 4.2.** The fuzzy probability of a fuzzy event satisfies the following properties.

(1) $\widetilde{P}(\emptyset) = 0$.

(2) For disjoint fuzzy events $A_i$ $(i = 1, 2, \cdots, n)$,

$$\widetilde{P}(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \widetilde{P}(A_i).$$

For any fuzzy events $A$ and $B$,

(3) If $A \subset B$, then $\widetilde{P}(A) \le \widetilde{P}(B)$.

(4) $\widetilde{P}(A \cup B) = \widetilde{P}(A) + \widetilde{P}(B) - \widetilde{P}(A \cap B)$, equivalently,
$\widetilde{P}(A \cap B) = \widetilde{P}(A) + \widetilde{P}(B) - \widetilde{P}(A \cup B)$.

(5) $\widetilde{P}(A^c) = 1 - \widetilde{P}(A)$.

(6) $\widetilde{P}(A \hat{+} B) = \widetilde{P}(A) + \widetilde{P}(B) - \widetilde{P}(A \cdot B)$, equivalently,

$$\widetilde{P}(A \cdot B) = \widetilde{P}(A) + \widetilde{P}(B) - \widetilde{P}(A \widehat{+} B).$$

*Proof.*

(1) Let $A_i = \emptyset$ for all $i = 1, 2, \cdots$, then $A_i'$s disjoint and $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} \emptyset = \emptyset$. Thus by (P.3),

$$\begin{aligned}\widetilde{P}(\emptyset) &= \widetilde{P}(\cup_{i=1}^{\infty} A_i) \\ &= \sum_{i=1}^{\infty} \widetilde{P}(A_i) \\ &= \sum_{i=1}^{\infty} \widetilde{P}(\emptyset).\end{aligned}$$

By (P.1), $0 \leq \widetilde{P}(\emptyset) \leq 1$. Thus $\widetilde{P}(\emptyset) = 0$.

(2) Let $A_j = \emptyset$ for all $j > n \in \mathbb{N}$, then $A_i'$s disjoint and $\cup_{i=1}^{n} A_i = \cup_{i=1}^{\infty} A_i$ by Theorem 2.8. Thus by (P.3) and (1),

$$\begin{aligned}\widetilde{P}(\bigcup_{i=1}^{n} A_i) = \widetilde{P}(\bigcup_{i=1}^{\infty} A_i) &= \sum_{i=1}^{\infty} \widetilde{P}(A_i) \\ &= \sum_{i=1}^{n} \widetilde{P}(A_i).\end{aligned}$$

(3) Since $A \subset B$, $\mu_A(\omega) \leq \mu_B(\omega)$ for all $\omega \in \Omega$. Thus

$$\begin{aligned}\widetilde{P}(A) &= \int_{\Omega} \mu_A(\omega) dP(\omega) \\ &\leq \int_{\Omega} \mu_B(\omega) dP(\omega) \\ &= \widetilde{P}(B).\end{aligned}$$

(4) Since $\mu_{A\cup B}(\omega) = \mu_A(\omega) + \mu_B(\omega) - \mu_{A\cap B}(\omega)$,

$$\begin{aligned}\widetilde{P}(A \cup B) &= \int_{\Omega} \mu_{A\cup B}(\omega) dP(\omega) \\ &= \int_{\Omega} (\mu_A(\omega) + \mu_B(\omega) - \mu_{A\cap B}(\omega)) dP(\omega) \\ &= \int_{\Omega} \mu_A(\omega) dP(\omega) + \int_{\Omega} \mu_B(\omega) dP(\omega) - \int_{\Omega} \mu_{A\cap B}(\omega) dP(\omega) \\ &= \widetilde{P}(A) + \widetilde{P}(B) - \widetilde{P}(A \cap B).\end{aligned}$$

(5) Since $\mu_{A^c}(\omega) = 1 - \mu_A(\omega)$,

$$
\begin{aligned}
\widetilde{P}(A^c) &= \int_\Omega \mu_{A^c}(\omega)dP(\omega) \\
&= \int_\Omega (1-\mu_A(\omega))dP(\omega) \\
&= 1-\widetilde{P}(A).
\end{aligned}
$$

(6) Since $\mu_{A\hat{+}B}(\omega)=\mu_A(\omega)+\mu_B(\omega)-\mu_A(\omega)\cdot\mu_B(\omega)$,

$$
\begin{aligned}
\widetilde{P}(A\hat{+}B) &= \int_\Omega \mu_{A\hat{+}B}(\omega)dP(\omega) \\
&= \int_\Omega (\mu_A(\omega)+\mu_B(\omega)-\mu_A(\omega)\cdot\mu_B(\omega))dP(\omega) \\
&= \int_\Omega \mu_A(\omega)dP(\omega)+\int_\Omega \mu_B(\omega)dP(\omega)-\int_\Omega \mu_A(\omega)\cdot\mu_B(\omega)dP(\omega) \\
&= \widetilde{P}(A)+\widetilde{P}(B)-\widetilde{P}(A\cdot B).
\end{aligned}
$$

□

**Theorem 4.3.** We have the following properties.

(1) $\widetilde{P}(A\cap(B\cup C))=\widetilde{P}((A\cap B)\cup(A\cap C))=\widetilde{P}(A\cap B)+\widetilde{P}(A\cap C)-\widetilde{P}(A\cap B\cap C)$.

(2) $\widetilde{P}(A\cup(A\cap B))=\widetilde{P}(A)+\widetilde{P}(A\cap B)-\widetilde{P}(A\cap A\cap B)=\widetilde{P}(A)$.

(3) $\widetilde{P}(A\cup\emptyset)=\widetilde{P}(A)+\widetilde{P}(\emptyset)-\widetilde{P}(\emptyset)=\widetilde{P}(A)$.

(4) $\widetilde{P}(A\cup\Omega)=\widetilde{P}(\Omega)$, $\quad\widetilde{P}(A\cap\Omega)=\widetilde{P}(A)$.

*Proof.* It is clear by Theorem 2.8 and Theorem 4.2. □

By Theorem 4.2 and Theorem 4.3,

$$
\begin{aligned}
\widetilde{P}(A\cup B\cup C) = {} & \widetilde{P}(A)+\widetilde{P}(B)+\widetilde{P}(C)-\widetilde{P}(A\cap B)-\widetilde{P}(B\cap C)-\widetilde{P}(C\cap A) \\
& +\widetilde{P}(A\cap B\cap C).
\end{aligned}
$$

Thus we have the following theorem by induction.

**Theorem 4.4.** For any fuzzy events $A_i$, $i=1,2,\cdots,n$ and $J\subset\{1,2,\cdots,n\}$, put

$$S_k=\sum_{|J|=k}\widetilde{P}(\bigcap_{i\in J}A_i).$$

Then

$$\widetilde{P}(\bigcup_{k=1}^{n}A_k)=\sum_{k=1}^{n}(-1)^{k-1}S_k.$$

Let $A \subset B$. Since $B - A = B \cap A^c$,

$$\mu_{B-A}(\omega) = \min\{\mu_B(\omega), \mu_{A^c}(\omega)\} = \min\{\mu_B(\omega), 1 - \mu_A(\omega)\}.$$

Thus $\mu_{B-A}(\omega)$ can not be represented by $\mu_B(\omega), \mu_A(\omega)$ and $\min\{\mu_B(\omega), \mu_A(\omega)\}$. This implies that

$$A \subset B \quad \not\Rightarrow \quad \widetilde{P}(B - A) = \widetilde{P}(B) - \widetilde{P}(A).$$

Furthermore, the excluded middle law and the contradiction law do not hold in fuzzy events, i.e.,

$$A \cup A^c \neq \Omega \quad \text{and} \quad A \cap A^c \neq \emptyset.$$

Thus we have the following main theorem.

**Theorem 4.5.** For any fuzzy events $A$ and $B$,

(1) $\widetilde{P}(A \cup A^c) \neq \widetilde{P}(\Omega)$.

(2) $\widetilde{P}(A \cap A^c) \neq \widetilde{P}(\emptyset)$.

(3) $A \subset B \quad \not\Rightarrow \quad \widetilde{P}(B - A) = \widetilde{P}(B) - \widetilde{P}(A)$.

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