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ON THE ARC LENGTH UNDER INVERSION

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Abstract

Two points P and P′ of the plane are said to be inverse with respect to a given circle $(O)_R$, if $OP \cdot OP' = R^2$ and also if both points are on the same side of O. Circle $(O)_R$ is called the circle of inversion and the transformation which sends point P into point P′ is known as an inversion.

In this paper we consider the curves in two dimensional Euclidean space R^2 and prove that the length of a regular new curve segment $\beta(t)$ of the inside curve $\alpha(t)$ under inversion is equal to the length of a regular curve segment $\alpha(t)$ by scalar multiple.

Introduction

In this paper, our study of curves will be restricted to the certain plane curves in two dimensional Euclidean space R^2.

In Section 1, we present the basic definitions and examples with respect to reparametrized curves and study some properies of the differential geometry, in particular, the arc length of curve segment $\alpha : [a, b] \to R^2$.

Next, in Section 2, we introduce the definition and some properties of in-

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verse curve under inversion. That is, the symbol $(O)_R$ is given by $OP \cdot OP' = R^2$ where its two points and O are collinear.

Finally, in Section 3, from the definition and the properties in Section 2, we prove the main theorem; the length of a regular new curve segment $\beta(t)$ of the inside curve $\alpha(t)$ under inversion is equal to the length of a regular curve segment $\alpha(t)$ by scalar multiple.

1. The arc length of a regular curve

Let α be an injective function from an interval into R^2 and $\alpha(t)$ denote the curve in the plane. Then we have the derivative $\frac{d\alpha}{dt}(t_0)$ of α evaluated at $t=t_0$ if $\alpha(t)$ is defferentiable in interval (a, b).

Definition 1.1 A curve $\alpha : (a, b) \to R^2$ is called a regular curve if $\alpha \in C^k$ for some $k \geq 1$ and if $\frac{d\alpha}{dt} \neq 0$ for all $t \in (a, b)$.

If t is time, then the velocity vector of a regular curve $\alpha(t)$ at $t=t_0$. is the derivative evaluated at $t=t_0$. The speed of $\alpha(t)$ at $t=t_0$ is the length of the velocity vector at $t=t_0$, $\left|\frac{d\alpha}{dt}(t_0)\right|$.

Let $g : (c, d) \to (a, b)$ be an one-to-one and onto function, and let g and its inverse $h : (a, b) \to (c, d)$ be of class C^k for some $k \geq 1$. Then g is called a reparametrization of a curve $\alpha : (a, b) \to R^2$.

Proposition 1.2 If $\alpha : (a, b) \to R^2$ is a regular curve then the new curve $\beta = \alpha \circ g$ is a regular curve, if $\frac{dg}{dr} \neq 0$.

Proof.

$$\frac{d\beta}{dr} = \frac{d}{dr}[\alpha \circ g(r)] = \frac{d\alpha}{dt} \cdot \frac{dg}{dr}, \tag{1.1}$$

that is, if $\frac{dg}{dr} \neq 0$ then $\frac{d\beta}{dr} \neq 0$.

Example 1.3 Let $g : (0,1) \to (1,2)$ be given by $g(r) = 1 + r^2$. Then g is a one-to-one and with inverse $h'(t) = \sqrt{t-1}$, $g \in C^k$, on $(0,1)$ and $h \in C^k$ on $(1,2)$ for some $k \geqq 1$. Thus g is a reparametrization of any regular curve on $(1,2)$.

A regualr curve segment is a function $\alpha : [a, b] \to R^2$ together with an open interval (c, d), with $c < a < b < d$, and a regular curve $r : (c, d) \to R^2$ such that $\alpha(t) = r(t)$ for all $t \in [a, b]$.

Definition 1.4 The legth of a regular curve segment $\alpha : [a, b] \to R^2$ is defined by

$$\int_a^b \left| \frac{d\alpha(t)}{dt} \right| dt. \tag{1.2}$$

Theorem 1.5. The length of a curve is a geometric property, that is, it does not depend on the choice of reparametrization.

Proof. Let $g : [c, d] \to [a, b]$ be a reparametrization of a curve segment $\alpha : [a, b] \to R^3$, and let the new curve $\beta = \alpha \circ g$. Then, for $r \in [c, d]$, since $g(r) = t$, $t \in [a, b]$, the length of β is

$$\begin{aligned} \int_c^d \left| \frac{d\beta}{dr} \right| dr &= \int_c^d \left| \frac{d}{dr}(\alpha \circ g) \right| dr \\ &= \int_c^d \left| \left(\frac{d\alpha}{dt} \right) \left(\frac{dg}{dr} \right) \right| dr \\ &= \int_c^d \left| \frac{d\alpha}{dt} \right| \left| \frac{dg}{dr} \right| dr. \end{aligned}$$

If $\frac{dg}{dr} > 0$, then $\left| \frac{dg}{dr} \right| = \frac{dg}{dr}$ and $g(c) = a$, $g(d) = b$.

Thus

$$\int_c^d \left|\frac{d\alpha}{dt}\right| \left|\frac{dg}{dr}\right| dr = \int_c^d \left|\frac{d\alpha}{dt}\right| \left(\frac{dg}{dr}\right) dr$$

$$= \int_a^b \left|\frac{d\alpha}{dt}\right| dt.$$

If $\frac{dg}{dr} < 0$, then $\left|\frac{dg}{dr}\right| = -\frac{dg}{dr}$ and

$$g(c) = b, \quad g(d) = a.$$

Hence

$$\int_c^d \left|\frac{d\alpha}{dt}\right| \left|\frac{dg}{dr}\right| dr = -\int_b^a \left|\frac{d\alpha}{dt}\right| \left(\frac{dg}{dr}\right) dr$$

$$= \int_a^b \left|\frac{d\alpha}{dt}\right| dt.$$

Example 1.6. Let $\alpha(t) = (r\cos t,\ r\sin t)$ with $r > 0$. Then $\frac{d\alpha}{dt} = (-r\sin t,\ r\cos t)$. Consider the arc length $s = s(t)$ of $\alpha(t)$.

Then

$$s = \int_c \left|\frac{d\alpha}{dt}\right| dt$$

$$= \int_c \sqrt{r^2 \sin^2 t + r^2 \cos^2 t}\ dt$$

$$= rt.$$

That is,

$$s = rt \text{ and } t = g(s) = \frac{s}{r}.$$

Hence,

$\beta(s) = (r\cos\frac{s}{r}, r\sin\frac{s}{r})$ is the unit speed parametrization of a circle of radius r.

2. The properties of inverse curve under inversion

In order to study the theorems is section 3, we will see the properties of inverse curve.

Let the symbol $(O)_R$ denote the circle with center O and radius R.

Definition 2.1. Two points P and P′ of the plane are said to be inverse with respect to a given circle $(O)_R$, if $OP \cdot OP' = R^2$ and if p, p′ are on the same side of O and the (O, P, P′) are collinear.

A circle $(O)_R$ is called the circle of inversion, and the transformation which sends point P into P′ is called an inversion. As point P moves on a curve C its inverse point P′ moves on a curve C′ which is the inverse curve of C. But the center O of the circle of inversion has no inverse point C, for when P is at point O, OP=0 and the relation $OP' = \frac{R^2}{OP}$ is meaningless.

Proposition 2.2 A line through O inverts into a line through O.
proof. It is evident from the fact that O and inverse points are collinear.

Proposition 2.3 A line not through O inverts into a circle through O. Conversely, a circle through O inverts into a line not through O.

Proof. Let l be a line not through O and Q be the foot of the perpendicular from O to l, and let P be any point an l (Fig. 2.1).
Then, there are the inverse point Q′ and P′ of Q and P, respectively.

That is,

(2.1.a) $$OQ \cdot OQ' = OP \cdot OP' = R^2$$

and

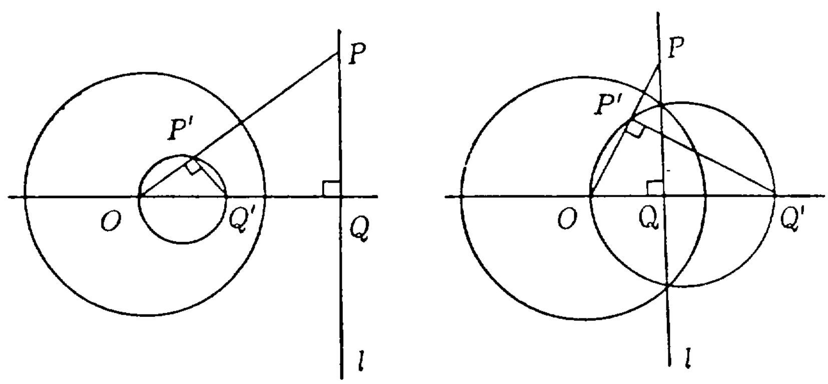
(2.1.b) $$\frac{OQ}{OP} = \frac{OP'}{OQ'}.$$

Therefore, △ OQP and △ OQ'P' have a common angle ∠POQ. By(2.1), △ OQP is similar to △ OP'Q'.

Thus
$$\angle OQP = \angle OP'Q' = 90°.$$

But the arc in which a 90° angle is inscribed is a semicircle. Thus the point P' lies on a circle whose diameter is OQ'.

A reversal of these arguments completes the proof of this theorem.



⟨Fig. 2.1⟩

Proposition 2.4 The angle between any two curves intersecting at a point which is different from the center O of the circle of inversion is unchanged under inversion.

Proof. Let the given curves C_1 and C_2(Fig. 2.2) intersect in a point P distinct from the center O of the circle of inversion and let any line l through O intersect these curves in the respective points A and B. Then the inverse curves to C_1 and C_2, namely C_1' and C_2', intersect at the inverse point P′ to P.

If curves C_1' and C_2' are met by line l in the inverse points A′ and B′ of A and B, respectively. Let θ be the angle between the tangents at P to curves C_1 and C_2 and let θ' be the angle between the the tangents at P′ to curves C_1' and C_2'. We must show that $\theta=\theta'$. Consider the triangles OPA and OP′A′. Then we have

(2.2) $$\frac{OA}{OP}=\frac{OP'}{OA'}.$$

Hence $\triangle OPA$ and $\triangle OP'A'$ are similar, so are $\triangle OPB$ and $\triangle OP'B'$. Therefore

(2.3) $$\angle\, OPA = \angle\, OA'P'$$

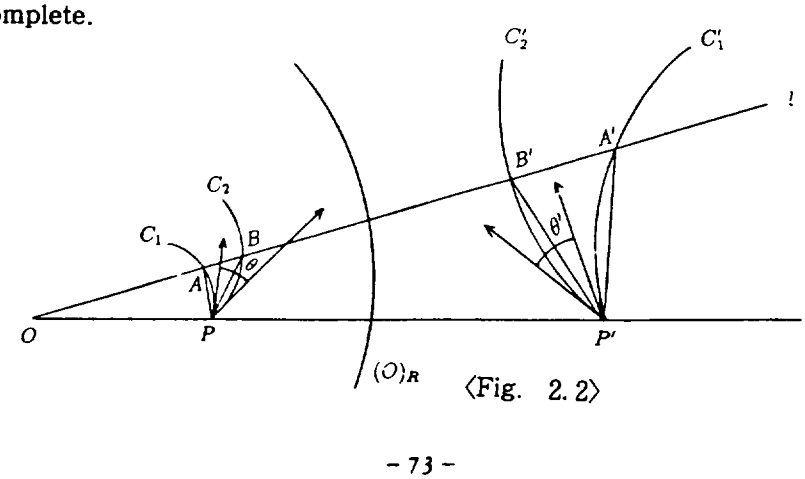
and

(2.4) $$\angle\, OPB = \angle\, OB'P'.$$

Subtraction (2.3) from (2.4) gives

$$\angle\, APB = \angle A'P'B'.$$

Therefore $\lim_{l\to OP} \angle\, APB = \theta$ and $\lim_{l\to O'P'} \angle A'P'B' = \theta'$. Hence the proof is complete.



⟨Fig. 2.2⟩

3. The arc length under inversion

Let α : (a, b) $\rightarrow R^2$ be the curve C_1 inside of inversion circle $(O)_R$.

Then, for all $t \in (a, b)$, $\alpha(t)$ the image of α is the points P_t on curve C_1. There exists a inverse curve $C_2=\beta(t)$ outside of $(O)_R$.

Let OP_t be a distance from O to point P_t on curve C_1. If a function $g : C_1 \rightarrow C_2$ is defined by

$$(3.1) \qquad g(P_t) = P_t' \text{ for } P_t \in C_1,$$

then we can take a new curve $\beta(t) = g \circ \alpha(t)$ and see that the following properties hold.

Theorem 3.1 If curve $C_1=\alpha(t)$ is a regular curve, then the inverse curve $C_2=\beta(t)$ is also a regular curve.

Proof. Let $\alpha(t)=P_t$, for each $t \in (a, b)$. Then $\frac{d\alpha(t)}{dt} \neq 0$ for all $t \in (a, b)$, since $\alpha(t)$ is regular on (a, b). Since $g(x, y) = \left(\frac{R^2 x}{x^2+y^2}, \frac{R^2 y}{x^2+y^2} \right)$, g is of class C^1 in $R^2 - \{(0, 0)\}$.

Now
$$\frac{d\beta(t)}{dt} = \begin{pmatrix} \frac{R^2(y^2-x^2)}{(x^2+y^2)^2} & \frac{-2R^2xy}{(x^2+y^2)^2} \\ \frac{-2R^2xy}{(x^2+y^2)^2} & \frac{R^2(x^2-y^2)}{(x^2+y^2)^2} \end{pmatrix} \frac{d\alpha(t)}{dt}.$$

since $\frac{R^4(y^2-x^2)(x^2-y^2)}{(x^2+y^2)^4} - \frac{4R^4x^2y^2}{(x^2+y^2)^4} \neq 0$ for all (x, y) except $(x, y) = (0, 0)$, $\frac{d\beta}{dt} \neq (0, 0)$, and hence β is regular in (a, b).

Let OP_t and OP_t' be distances from the center of inversion circle $(O)_R$ to point P_t and P_t' on curves C_1 and C_2, respectively. Consider the curve equation $OP_t = \alpha(t)$ with respect to the polar coordinate.

Then the equation of the new curve $\beta(t)$ is given by $OP'_t = \beta(t)$.

Theorem 3.2 The length of a regular curve segment of new curve $\beta(t)$ of the inside curve $\alpha(t)$ under inversion is given by

$$\int_{t_1}^{t_2} \left| \frac{d\beta(t)}{dt} \right| dt = R^2 \int_{t_1}^{t_2} \frac{\sqrt{\alpha^2(t) + [\alpha'(t)]^2}}{\alpha^2(t)} dt$$

where t is the between OP_t and horizontal line.

Proof. Let $OP^t = \alpha(t)$, $OP'_t = \beta(t)$ and let $t_1 < t_2$.

$$\text{Then } \int_{t_1}^{t_2} \left| \frac{d\beta(t)}{dt} \right| dt = \int_{t_1}^{t_2} \sqrt{\beta^2(t) + \left[\frac{d\beta(t)}{dt} \right]^2} dt.$$

From (2.1.a), we have

$$\int_{t_1}^{t_2} \sqrt{\beta^2(t) + \left[\frac{d\beta(t)}{dt} \right]^2} dt = \int_{t_1}^{t_2} \sqrt{\left[\frac{R^2}{\alpha(t)} \right]^2 + \left[\frac{d}{dt} \frac{R^2}{\alpha(t)} \right]^2} dt$$

$$= R^2 \int_{t_1}^{t_2} \sqrt{\left(\frac{1}{\alpha(t)} \right)^2 + \left[-\frac{1}{\alpha^2(t)} \frac{d\alpha(t)}{dt} \right]^2} dt$$

$$= R^2 \int_{t_1}^{t_2} \frac{\sqrt{\alpha^2(t) + [\alpha'(t)]^2}}{\alpha^2(t)} dt.$$

Thus we have the result.

Example 3.3 Let the circle through center of inversion circle $(O)_R$ be $\alpha(t) = \cos t$ and let $0 \le t \le \frac{\pi}{3}$.
Then we have

$$\int_0^{\frac{\pi}{3}} \left|\frac{d\beta(t)}{dt}\right| dt = R^2 \int_0^{\frac{\pi}{3}} \frac{\sqrt{\cos^2 t + \sin^2 t}}{\cos^2 t} dt$$

$$= R^2 \int_0^{\frac{\pi}{3}} \sec^2 t \, dt$$

$$= R^2 [\tan t]_0^{\frac{\pi}{3}}$$

$$= \sqrt{3} R^2.$$

On the other hand, in virtue of (2. 1. a), if $t = \frac{\pi}{3}$,

$$\beta(t) = \frac{R^2}{\alpha(t)} = \frac{R^2}{\cos t} = 2R^2.$$

Thus $PQ = \sqrt{3}R^2$ (Fig 2. 1).

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〈국 문 초 록〉

Inversion에 의한 곡선의 길이

중심이 O이고 반지름의 길이가 R인 원 $(O)_R$ 에서 두 점 P, P′이 중심 O의 같은 쪽에 있고, $OP \cdot OP' = R^2$을 만족할 때, 이 두 점, P, P′을 서로역(inverse)이라 하고, $(O)_R$를 전위 원(inversion circle)이라고 하며, 점 P에서 P′으로 보내어 주는 변환을 전위(inversion)라고 한다.

이 논문에서는 2차 Euclid 공간의 곡선으로 제한하여, 전위(inversion)에 의한 $(O)_R$의 내부의 곡선 $\alpha(t)$에 대응하는 새로운 곡선 $\beta(t)$의 길이는 곡선 $\alpha(t)$의 길이의 스칼라배로 나타낼 수 있음을 보였다.