On Compact Operator in Hilbert Sequence Space

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Hilbert 列空間에서 Compact 作用素에 關하여

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Introduction

DEFINITION 1.

Let H be a vector space over K(K=R or c)

A mapping $S: H\times H \to K$ is called a sesquilinear form on H if for all $f,g,h\in H$ and $a,b\in K$

1) $S(f,\ ag+bh)=aS(f,g)+bS(f,h)$

2) $S(af+bg,\ h)=a^*S(f,h)+b^*S(g,h)$ where * be the complex conjugated.

EXAMPLE 2.

For each positive integer m, let C^m be the complex vector space of the m-tuples $f=(f_1, f_2,\cdots,f_m)$, $g=(g_1,g_2,\cdots,g_m)$, …of complex numbers with the addtion

$f+g=(f_1+g_1,\cdots,f_m+g_m)$ and multiplication by a $\in C$

$af=(af_1,\cdots,af_m)$

If $(a_{jk})_{j,k=1,2,\cdots,m}$ is a complex $m\times m$ matrix, then $S(f,g)=\sum_{j,k=1}^{m} a_{jk}\cdot f_j^*\cdot g_k$ for $f,g\in C^m$ defines a sesquilinear form on C^m.

DEFINITION 3.

H be a Hilbert space.

A sesquilinear form S on H is said to be bounded if there is a $c\geq 0$ such that $|S(f,g)|\leq c\|f\|\,\|g\|$ for all $f,g\in H$.

(*) If $T\in B(H)$, then $t(f,g)=\langle Tf,g\rangle$ defines a bounded sesquilinear form on H where B(H) be a set of bounded operator H into H.

THEOREM 4.

Let H be a Hilbert space. If t is a bounded sesquilinear form on H, then there exists exactly one $T\in B(H)$ such that $t(f,g)=\langle Tf,g\rangle$ for all $f,g\in H$ and $\|T\|=\|t\|$.

(Proof)

For every $f\in H$ since $|t(f,g)|\leq\|t\|\,\|f\|\,\|g\|$ the function $g\to t(f,g)$ is a continuous linear functional on H.

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Therefore for each $f \in H$ there exists exactly one $\tilde{f} \in H$

such that $t(f,g) = \langle \tilde{f}, g \rangle$

The mapping $f \to \tilde{f}$ is linear. Let define T by $Tf = \tilde{f}$ for all $f \in H$.

The operator T is bounded with norm $\| T \| = \sup\{|\langle Tf, g\rangle| : f, g \in H, \ \|f\| = \|g\| = 1\} = \sup\{|t(f,g)| : f, g \in H, \ \|f\| = \|g\| = 1\} = \|t\|$

If $T_1 \in B(H)$ and $T_2 \in B(H)$, $\langle T_1 f, g\rangle = t(f,g) = \langle T_2 f, g\rangle$ for all $f, g \in H$, then $T_1 = T_2$.

DEFINITION 5.

A vector space T is said to be the direct sum of two subspaces T_1 and T_2 if each $f \in T$ has a unique representation $f = g + h$, $g \in T_1$, $h \in T_2$ and denoted by $T = T_1 \oplus T_2$

THEOREM 6.

Let H be a Hilbert space and let T be a closed subspace of H.

Then $H = T \oplus T^{\perp}$.

(Proof)

Since H is complete and T is closed, T is complete. Since T is convex, for every $f \in H$ there is a $g \in T$ such that $f = g + h$ — (*), $h \in T^{\perp}$.

Now we show uniqueness.

Assume that $f = g + h = g' + h'$ where $g, g' \in T$, $h, h' \in T^{\perp}$.

Then $g - g' = h - h'$

Since $g - g' \in T$ and $h - h' \in T^{\perp}$, $g - g' \in T \cap T^{\perp} = \{0\}$

Therefore $g = g'$ and $h = h'$

(In (*), g is called the orthogonal projection of f on T)

PROPOSITION 7.

Let H be a pre-Hilbert space and let T_1 and T_2 be orthogonal subspaces. If $T_1 \oplus T_2$ is closed, then T_1 and T_2 are closed.

THEOREM 8.

Let H be a pre-Hilbert space and let T_1 and T_2 be subspaces of H such that $T_1 \perp T_2$.

Then $\overline{T_1} \oplus \overline{T_2} \subset \overline{T_1 \oplus T_2}$

Particular, if H is a Hilbert space, then $\overline{T_1 \oplus T_2} = \overline{T_1} \oplus \overline{T_2}$

(Proof)

Let $f + g \in \overline{T_1} \oplus \overline{T_2}$, where $f \in \overline{T_1}$, $g \in \overline{T_2}$

Then there exist sequences $(f_n) \in T_1$, $(g_n) \in T_2$ such that $f_n \to f$ and $g_n \to g$, $f_n + g_n \in T_1 \oplus T_2$

Therefore $f + g = \lim (f_n + g_n) \in T_1 \oplus T_2$. Hence $f + g \in \overline{T_1 \oplus T_2}$

Next we will show that $\overline{T_1 \oplus T_2} \subset \overline{T_1} \oplus \overline{T_2}$.

Let $f + g \in \overline{T_1 \oplus T_2}$

Then there exist sequences $(f_n) \in T_1$, $(g_n) \in T_2$ such that $f_n \to f$, $g_n \to g$ and $T_1 \oplus T_2 \in f_n + g_n \to f + g$.

Therefore $f \in \overline{T_1}$, $g \in \overline{T_2}$ and $f + g \in \overline{T_1} \oplus \overline{T_2}$.

Compact operator in Hilbert space

DEFINITION 9.

Let H_1 and H_2 be Hilbert spaces. An operator $T : H_1 \to H_2$ is called a compact if T is linear and for every bounded subset B of H_1, the closure $\overline{T(B)}$ is compact.

THEOREM 10.

Let H_1 and H_2 be Hilbert spaces and let $T : H_1 \to H_2$ a linear operator.

Then T is compact if and only if it maps every bounded sequence (f_n) in H_1 onto a sequence (Tf_n) in H_2 which has a convergent subsequence.

(Proof)

If T is compact and (f_n) is bounded, then the closure of (Tf_n) in H_1 is compact and (Tf_n) contains a convergent subsequence.

Conversely, assume that every bounded sequence (f_n) contains a subsequence (f_{nk}) such that (Tf_{nk}) converges in H_2.

Consider any bounded subset $B \subset H_1$ and let (g_n) be any sequence in $T(B)$.

Then $g_n = Tf_n$ for some $f_n \in B$ and since B is bounded, so (f_n) is bounded. By assumption, (Tf_n) contains a convergent subsequence.

Hence $\overline{T(B)}$ is compact. Therefore T is compact.

THEOREM 11.

Let H_1, H_2 be Hilbert spaces. If (T_n) is a sequence of compat operator from $B(H_1, H_2)$ and $\| T_n - T \| \to 0$ for some $T \in B(H_1, H_2)$, then T is compact, where $B(H_1, H_2)$ be a set of bounded operator H_1 into H_2.

(Proof)

Let (f_n) be a weak null-sequence from H_1, then the sequence (f_n) is bounded, say $\| f_n \| \le c$ for all $n \in N$. Let $\varepsilon > 0$ be given. Since $\| T_n - T \| \to 0$, there exists an $m_0 \in N$ such that $\| T m_0 - T \| \le \varepsilon c^{-1}/2$. Since Tm_0 is compact, there exists an $n_0 \in N$ such that $\| T m_0 f_n \| \le \varepsilon/2$ for all $n \ge n_0$.

It follow from this that for all $n \ge n_0$ $\| Tf_n \| \le \| (T - Tm_0) f_n \| + \| T m_0 f_n \| \le \varepsilon$.

Therefore $Tf_n \to 0$ and T is compact.

PROPOSITION 12.

If $S \in B(H_2, H_3)$ and $T \in B(H_1, H_2)$ and if one of these operators is compact, then $ST \in B(H_1, H_3)$ is compact.

PROPOSITION 13.

Let $T : H_1 \to H_2$ be a linear operator and let T is bounded and $\dim T(H_1) < \infty$ Then T is compact.

(Proof)

Let (f_n) be any bounded sequence in H_1. Then $\| Tf_n \| \le \| T \| \| f_n \|$ and then (Tf_n) is bounded.

Since $\dim T(H_1) < \infty$, (Tf_n) is relatively compact.

It follows that (Tf_n) has a convergent subsequence.

Since (f_n) was an arbitrary bounded sequence in H_1. T is compact.

THEOREM 14.

Let ℓ^2 be a Hilbert sequence space and an operator $T : \ell^2 \to \ell^2$ be defineded by $T_x = y = (n_j)$, where $n_j = \varepsilon_j / 2^j$ for $j = 1, 2, \cdots$.

Then T is compact.

(Proof)

T is linear. If $x = (\varepsilon_j/2) \in \ell^2$, then $y = (n_j) \in \ell^2$. Let $T_n : \ell^2 \to \ell^2$ be defined by $T_n x = (\varepsilon_1/2, \varepsilon_2/2^2, \cdots, \varepsilon_n/2^n, 0, 0, \cdots)$

Then T_n is linear and bounded, by proposition 13 T_n is compact.

By theorem 11, we shall show that $T_n \to T$

$$\| (T - T_n) x \|^2 = \sum_{j=n+1}^{\infty} |\varepsilon_j / 2^j|^2 =$$

$$\sum_{j=n+1}^{\infty} 1/2^{2(j-1)} |\varepsilon_j/2|^2$$

$$\le 1/2^{2n} \sum_{j=n+1}^{\infty} |\varepsilon_j/2|^2$$

$$\le \| x \|^2 / 2^{2n}$$

Choose the supremum over all x of norm 1, then we have $\| T - T_n \| \le 1/2^n$.

Hence $T_n \to T$ and T is compact.

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Hilbert 列空間에서 Compact 作用素에 關하여

本 論文에서는 sesquilinear form과 Direct sum의 定義와 定理를 소개하면서 T_1과 T_2가 pre-Hilbert space의 subspace이면서 orthogonal이면 $\overline{T_1}\oplus\overline{T_2}\subset\overline{T_1\oplus T_2}$이고, T_1과 T_2가 Hilbert space의 subspace이면 $\overline{T_1\oplus T_2}=\overline{T_1}\oplus\overline{T_2}$임을 보였으며 ℓ^2가 Hilbert sequence space일 때 operator $T : \ell^2\rightarrow\ell^2$가 $T_x=y=(n_j)$, $n_j=\varepsilon_j/2^j$, $j=1.2.3,\cdots$로 定義하면 T가 compact 임을 보였다.