Structural Analysis of Nonsquare Matrices Using Permanent Theory

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퍼머넨트 理論에 의한 行列의 構造 分析

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Summary

Many studies of permanents theory are related on doubly stochastic n-square matrices. In this paper, we define a partial doubly stochastic $m\times n$ matrices and analyze the structures of such matrices and their permanents. And we investigate the structures of fully indecomposable $m\times n$ matrices, partly decomposable $m\times n$ matrices and contraction matrices of partial doubly stochastic matrices.

1. Introduction and preliminaries

Many studies on permanents theory are related on doubly stochastic n-square matrices. In this paper, we define a partial doubly stochastic $m\times n$ matrices and investigate such matrices.

Let $A=(a_{ij})$ be an $m\times n$ matrix over any commutative ring, $m\leq n$. The *permanent* of A, written Per(A), is defined by

$$\text{Per}(A)=\sum_{\sigma} a_{1\sigma(1)}a_{2\sigma(2)}\cdots\cdots a_{m\sigma(m)}$$

where the summation extends over all one-to-one functions from $\{1, \cdots, m\}$ to $\{1, \cdots, n\}$. The sequence $(a_{1\sigma(1)}, \cdots, a_{m\sigma(m)})$ is called a *diagonal* of A.

Let $\Gamma_{r,n}$ denote the set of all n^r sequences $w=(w_1, \cdots, w_r)$ of integers, $1\leq w_i\leq n$, $i=1, \cdots, n$. Let $Q_{r,n}$ denote the subset of $\Gamma_{r,n}$ consisting of all increasing sequences,

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$$Q_{r,n}=\{(w_1, \cdots, w_r)\in \Gamma_{r,n} : 1\leq w_1<\cdots<w_r\leq n\}.$$

Let $A=(a_{ij})$ denote the $m\times n$ matrix with entries from real numbers and let $\alpha\in Q_{h,m}$ and $\beta\in Q_{k,n}$. Then $A[\alpha : \beta]$ denotes the $h\times k$ submatrix of A whose (i, j) entry is $a_{\alpha_i\beta_j}$ And $A(\alpha|\beta)$ denotes the $(m-h)\times(n-k)$ submatrix of A complementary to $A[\alpha|\beta]$ — that is, the submatrix obtained from A by deleting rows α and columns β. The other definitions are refered to [4] "*Permanents*".

In this paper, we analyze the structures of fully indecomposable $m\times n$ matrices and partly decomposable $m\times n$ matrices. In particular, we define doubly c(k)-stochastic $m\times n$ matrices and investigate such matrices.

we assume that $m\leq n$ for all $m\times n$ matrices in this paper.

2. The structure and permanents of doubly c(k)-stochastic m×n matrices.

DEFINITION 1. A nonnegative $m\times n$ matrix is called ***doubly c(k)-stochastic*** if all its row sums and k column sums are 1 but its remaining (n-k) column sums are $\frac{m-k}{n-k}$ for some $0\leq k<m\leq n$. If m=n, a doubly c(n)-stochastic matrix is called a ***doubly stochastic matrix***.

For examples, let

$$A=\begin{bmatrix}\frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\\ \frac{1}{2} & 0 & \frac{1}{2} & 0\end{bmatrix},\quad B=\begin{bmatrix}\frac{1}{8} & \frac{3}{8} & \frac{1}{8} & \frac{3}{8}\\ \frac{2}{8} & \frac{3}{8} & 0 & \frac{3}{8}\\ \frac{3}{8} & 0 & \frac{5}{8} & 0\end{bmatrix}$$

Then A is a doubly c(1)-stochastic 3×4 matrix and B is a doubly c(0)-stochastic 3×4 matrix.

DEFINITION 2. ([4]) A nonnegative $m\times n$ matrix A is called ***fully indecomposable*** if $Per(A(i|j))>0$ for $i=1, \cdots, m$, and $j=1, \cdots, n$. Otherwise, A is called ***partly decomposable***.

THEOREM 1. ([4]) Let A be a nonnegative $m\times n$ matrix. Then $Per(A)=0$ if and only if A contains an $s\times(n-s+1)$ zero submatrix. (Extended version of ***Frobenius*** and ***König*** Theorem)

LEMMA 2. An $m\times n$ matrix A is partly decomposable if and only if there exist permutation matrices P and Q of orders m and n respectively, such that

$$PAQ=\begin{bmatrix}B & C\\ 0 & D\end{bmatrix}_{m\times n} \qquad (1)$$

where 0 is an $s\times(n-s)$ submatrix $(s\geq 1)$.

Proof. If A is an $m\times n$ partly decomposable matrix, then there exist i and j such that $Per(A(i|j))=0$. By Theorem 1, $A(i|j)$ contains an $s\times((n-1)-s+1)$ zero submatrix. Hence A contains an $s\times(n-s)$ zero submatrix.

Now, assume that there exist permutation matrices P and Q of orders m and n respectively such that PAQ is of the form (1). Let (i, j) be a position in the submatrix C of PAQ. Then $(m-1)\times(n-1)$ matrix $PAQ(i|j)$ contains an $s\times((n-1)-s+1)$ zero submatrix. By Theorem 1, $Per(PAQ(i|j))=0$ and hence $Per(A(i|j))=0$. Therefore A is a partly decomposable $m\times n$ matrix.

THEOREM 3. If an $m\times n$ matrix A is partly decomposable doubly c(k)-stochastic $(k>0)$, then there exist permutation matrices P and Q of orders m and n respectively such that PAQ is a direct sum of an s-square doubly stochastic matrix and an $(m-s)\times(n-s)$ doubly $c(k-s)$-stochastic matrix.

Proof. Since A is partly decomposable, for some permutation matrices P and Q of orders m and n respectively, Lemma 2 implies that PAQ is of the form (1) in Lemma 2. Since PAQ is doubly $c(k)$-stochastic, the sum of entries in last s rows of PAQ is s and hence

$$\sigma(D)=s$$

Where $\sigma(X)$ denotes the sum of entries in the matrix X. Since D is s-square and nonnegative, the sum of every column in D is 1 and hence $s \leqq k$. Similarly, considering the entries in the first n-s columns of doubly $c(k)$-stochastic matrix PAQ, we can conclude that

$$\sigma(B) \geq \frac{m-k}{n-k}\times(n-k)+1\times(k-s)=m-s.$$

But

$$m=\sigma(PAQ)=\sigma(B)+\sigma(C)+\sigma(D)$$
$$\geq(m-s)+\sigma(C)+s=m+\sigma(C)$$

and therefore

$$\sigma(C)\leqq 0.$$

Since C is nonnegative, we must have

$$C=0$$

and thus

$$PAQ=B\oplus D$$

where D is an s-square doubly stochastic matrix and B is an $(m-s)\times(n-s)$ doubly $c(k-s)$-stochastic matrix.

THEOREM 4. A doubly $c(0)$-stochastic $m\times n$ matrix is fully indecomposable if $m<n$.

Proof. Let A be a doubly $c(0)$-stochastic $m\times n$ matrix with $m<n$. Assume that A is not fully indecomposable. Then there exist permutation matrices P and Q of orders m and n respectively such that PAQ is of the form (1). Since the sum of entries in the last s rows is s and the nonzero entries in them are all in the submatrix D, we have

$$\sigma(D)=s.$$

Since A is doubly $c(0)$-stochastic and nonneqative, the sum of entries in the last k columns is greater than or equals to the sum of entries in the submatrix D. Hence

$$\frac{m}{n}\times s\geq\sigma(D)=s.$$

Therefore $m \geq n$, which is impossible. Hence A is fully indecomposable if $m<n$.

THEOREM 5. The permanent of a doubly $c(k)$-stochastic $m\times n$ matrix is positive.

Proof. If $Per(A)=0$, then by Theorem 1, there exist permutation matrices P and Q such that PAQ is of the form (1), where the zero submatrix is $s\times(n-s+1)$ matrix. Since A is a doubly $c(k)$-stochastic $m\times n$ matrix, we have

$$m=\sigma(PAQ)\geq\sigma(B)+\sigma(D)$$

Now, all the nonzero entries in the last s rows are contained in D and thus

$$\sigma(D)=s.$$

This implies that $s\leqq k$. Similarly, all the nonzero entries in the first $(n-s+1)$ columns are contained in B and thus

$$\sigma(B) \geq \frac{m-k}{n-k} \times (n-k) + 1 \times ((n-s+1)-(n-k))$$
$$= (m-k) + (k-s+1) = m-s+1$$

But

$$m \geq \sigma(B) + \sigma(D) \geq (m-s+1) + s = m+1$$

which is impossible.

COROLLARY 6. Every doubly c(k)-stochastic matrix has a positive diagonal.

LEMMA 7. If A is a fully indecomposable $m \times n$ matrix and $c > 0$, then for every i and j,

$$Per(A + cE_{ij}) > Per(A)$$

where E_{ij} denotes the $m \times n$ matrix with 1 in the (i, j) position and zeros elsewhere.

Proof. Using the expansion theorem for permanents, we have

$$Per(A+cE_{ij}) = \sum_{j=1}^{n} a_{ij} Per((A+cE_{ij})(i|j))$$
$$= Per(A) + cPer(A(i|j)).$$

Since A is fully indecomposable, $Per(A(i|j)) > 0$ for all i and j. Hence we have the result.

THEOREM 8. If an $m \times n$ matrix A is a fully indecomposable (0, 1)-matrix, then

$$Per(A + \sum_{t=1}^{r} E_{i_t j_t}) \geq Per(A) + r.$$

Proof. Since A is a (0, 1)-matrix, definition 2 implies that $Per(A(i|j)) \geq 1$ for all i and j. Therefore,

$$Per(A+E_{i_1 j_1}) = Per(A) + Per(A(i_1|j_1))$$
$$\geq Per(A) + 1$$

by lemma 7. Clearly $A+E_{i_1 j_1}$ is fully indecomposable. The result now follows by induction on r.

THEOREM 9. Let

$$A = \begin{bmatrix} A_1 & B_1 & 0 & \cdots & 0 \\ 0 & A_2 & B_2 & & \vdots \\ \vdots & & \ddots & \ddots & \\ 0 & 0 & & A_{r-1} & B_{r-1} \\ B_r & 0 & \cdots & 0 & A_r \end{bmatrix}_{m \times n} \qquad (2)$$

be a nonnegative $m \times n$ matrix, where A_i is a fully indecomposable $m_i \times n_i$ matrix, $i=1, \cdots, r$, and $B_i \neq 0$, $i = 1, \cdots, r$. Then A is fully indecomposable.

Proof. Suppose that A is partly decomposable—i. e., that $A[\alpha|\beta]=0$ for some $\alpha \in Q_{s,m}$ and $\beta \in Q_{t,n}$, where $s+t=n$. Let s_j of rows α and t_j of columns β intersect the submatrix A_j, $j=1, \cdots, r$. Then $s_1+s_2+\cdots+s_r = s > 1$, so that at least one of the s_j must be positive. Similarly, at least one of the t_j is not zero. Now, since each A_j is fully indecomposable and it contains an $s_j \times t_j$ zero submatrix (unless either $s_j=0$ or $t_j=0$), we must have $s_j+t_j \leq n_j$, where equality can hold only if $s_j=0$ or $t_j=0$. But

$$n = s + t = \sum_{j=1}^{r} s_j + \sum_{j=1}^{r} t_j = \sum_{j=1}^{r} (s_j + t_j)$$
$$\leq \sum_{j=1}^{r} n_j = n$$

and thus $s_j+t_j=n_j$ for every j. It follows that either $s_j=0$ or $t_j=0$ for $j=1, \cdots, r$. But not all the s_j nor all the t_j can be zero, and therefore there must exist an integer k such that $s_k=n_k$ and $t_{k+1}=n_{k+1}$ (subscripts reduced modulo r). It follows that B_k is a submatrix of a zero submatrix, contradicting our hypothesis.

THEOREM 10. A fully indecomposable $m\times n$ matrix A has a row stochastic matrix which has the same zero pattern as A.

Proof. Since A is fully indecomposable, $Per(A(i|j))>0$ for all i, j, by definition 2. Let $S=(s_{ij})$ be the $m\times n$ matrix defined by

$$s_{ij}=a_{ij}Per(A(i|j))/Per(A)$$

$i=1, \cdots, m$ and $j=1, \cdots, n$. Clearly S is nonnegative, and it has the same zero pattern as A. Also for $i=1, \cdots, m$,

$$\sum_{j=1}^{n} s_{ij} = \frac{1}{Per(A)} \sum_{j=1}^{n} a_{ij} Per(A(i|j))$$

$$= \frac{1}{Per(A)} Per(A) = 1$$

Hence S is row stochastic.

DEFINITION 3 ([1, 3]). If column h of an $m\times n$ matrix A contains exactly two nonzero entries, say, in rows i and j, then the $(m-1)\times(n-1)$ matrix C(A) obtained from A by replacing row i with the sum of rows i and j and deleting row j and column h is called a *contraction* of A.

THEOREM 11. Let A be a nonnegative $m\times n$ matrix and let C(A) be a contraction of A on column h relative to rows i and j.

(i) If rows i and j each contain at least two positive entries, then A is fully indecomposable if and only if C(A) is fully indecomposable.

(ii) If A is a doubly c(k)-stochastic matrix such that $a_{ih}+a_{jh}=1$, $k\geq1$, then C(A) is a doubly c(k-1)-stochastic matrix.

Proof. It suffices to consider the case where C(A) is the contraction of A on column 1 relative to rows 1 and 2. Thus A and C(A) have the form

$$A=\begin{bmatrix} a_{11} & U \\ a_{21} & V \\ 0 & B \end{bmatrix}_{m\times n} \quad C(A)=\begin{bmatrix} U+V \\ \\ B \end{bmatrix}_{(m-1)\times(n-1)}$$

where $a_{11}\neq0\neq a_{21}$.

(i) Suppose C(A) is not fully indecomposable. Then there exists an $s\times t$ zero submatrix $0_{s\times t}$ of C(A) where $s+t=n-1$. If $0_{s\times t}$ is a submatrix of B, then clearly A has an $s\times(t+1)$ zero submatrix where $s+(t+1)=n$. Hence in this case A is not fully indecomposable. Suppose $0_{s\times t}$ is not a submatrix of B. Since a_{11} and a_{21} are positive while U and V are nonnegative, A has an $(s+1)\times t$ zero submatrix where $(s+1)+t=n$. Therefore A is not fully indecomposable.

Conversely, suppose A is not a fully indecomposable $m\times n$ matrix. Thus A contains an $s\times t$ zero submatrix $0_{s\times t}$ with $s+t=n$. If $0_{s\times t}$ is contained in the last m-2 rows of A, then B, and thus C(A), contains an $s+(t-1)$ zero submatrix with $s+(t-1)=n-1$. Let $0_{s\times t}$ not be contained in the last m-2 rows of A. Then, since a_{11} and a_{21} are positive, $0_{s\times t}$ is contained in the last n-1 columns of A. Since rows 1 and 2 of A each contain at least two positive entries by assumption, $0_{s\times t}$ is a submatrix of neither U nor V. Hence C(A) contains an $(s-1)\times t$ zero submatrix with $(s-1)+t=n-1$. Therefore C(A) is not fully indecomposable.

(ii) Since A is a doubly c(k)-stochastic matrix, the sum of entries in the first two rows of A is 2. If $a_{11}+a_{21}=1$ in A, then the sum of entries in the first row of C(A), that is $\sigma(U+V)$, is 1. therefore C(A) is row stochastic. Since C(A) is a contraction on the first column of A, C(A) has only (k-1) columns such that the sums of their columns each are 1. And the sums of the other (n-1)-(k-1) columns each are $\frac{m-k}{n-k}$, that is $\frac{(m-1)-(k-1)}{(n-1)-(k-1)}$. Hence C(A) is a doubly c(k-1)-stochastic $(m-1)\times(n-1)$ matrix.

Theorem 12. Let P and Q be m- and n-square (0, 1)-matrices respectively such that P has no zero rows and Q has no zero columns. Then

(1) if PAQ is partly decomposable for arbitrary $m\times n$ (0, 1)-matrix A having a zero row, then P is a permutation matrix.

(2) if PAQ is partly decomposable for arbitrary $m\times n$ (0, 1)-matrix A having a zero column, then Q is a permutation matrix.

Proof. (1) Suppose PAQ is partly decomposable for every A with a zero row. Let A_1 be the matrix all of whose entries equal 1 except those in the first row which equal 0. Since Q has no zero columns, it follows that $A_1Q\geqq A_1$. Let $P'=P[\cdot, \{2, \cdots, m\}]_{m\times(m-1)}$ and let $A_1'=A_1[\{2, \cdots, m\}, \cdot]_{(m-1)\times n}$ so that all entries of A_1' equal 1. Then

$$PA_1Q\geqq PA_1=P'A_1'$$

Since PA_1Q is partly decomposable, it now follows that P' has a zero row. Since P has no zero rows, we conclude that some row of P equals $(1, 0, \cdots, 0)$. By considering the matrix A_i all of whose entries equal 1 except those in row i which equal 0 $(i=1, \cdots, m)$, we conclude in a similar way, that for each $i=1, \cdots, m$, some row of P contains only o's except for a 1 in column i. Hence P is a permutation matrix.

(2) Suppose PAQ is partly decomposable for every A with a zero column. Let A_1 be the matrix all of whose entries equal 1 except those in the first column which equal 0. Since P^t has no zero columns, it follows that

$A_1^tP_1^t\geqq A_1^t$. Let $(Q^t)'=Q^t[\cdot, \{2, \cdots, n\}]_{n\times(n-1)}$ and let $(A_1^t)'=A_1^t[\{2, \cdots n\}, \cdot]_{(n-1)\times n}$ so that all entries of $(A_1^t)'$ equal 1. Then

$$Q^tA_1^tP^t\geqq Q^tA_1^t=(Q^t)'(A_1^t)'.$$

Since $Q^tA_1^tP^t$ is partly decomppsable, it now follows that $(Q^t)'$ has a zero row. Since Q^t has no zero rows, we conclude that some row of Q^t equals $(1, 0, \cdots, 0)$. By considering the matrix A_j all of whose entries equal 1 except in column j which equal 0 $(j=1, \cdots, n)$, we conclude in a similar way, that for each $j=1, \cdots, n$. some row of Q^t contains only 0's except for a 1 in a column j. Hence Q^t and Q are permutation matrices.

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摘　　要

퍼머넨트 理論에 관한 많은 研究들은 주로 n次의 正方行列에 관련되어 왔다. 本 論文에서는 이러한 正方行列에 관한 理論을 一般的인 m×n 行列로 확장시켰다. 곧 分解할 수 없는 m×n 行列과 分解가능한 m×n 行列의 構造에 관한 정리들과 縮約에 관한 정리들을 一般的으로 확장시켜 證明하였다.