The Gauss Map of A Non-flat Complete Minimal Surface

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비편평 완비 극소 곡면의 **Gauss** 사상

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Summary

In this paper, we show that the Gauss map of a non-flat complete minimal surface possibly omits 1 to 4 points but can not omit seven points of the sphere.

Introduction

Throughout this paper, all surfaces are considered to be connected and orientable submanifolds of R^3 with the induced Reimannian metric. A surface is ***minimal*** if its mean curvature vanishes at all points, and is flat if its sectional curvature $k \equiv o$ on the surface.

A well-know theorem of Osserman states that the Gauss map of a complete minimal surface $M^2 \subset R^3$ cannot omit a set of positive logarithmic capacity unless the surface is a plane. In this paper we improve Osserman's theorem by showing that the Gauss map of M^2 cannot omit 7 points of the sphere (provided M^2 is not flat). It should be pointed out, however, that no example is known where the omitted set has 5 points. Therefore the problem of determing the exact size of the omitted set remains unsolved.

1. Some examples

1. Enneper's surface. This surface is given analytically by the equations:

$$x = \mathrm{Re}\ [w - \frac{1}{3} w^3]$$

$$y = \mathrm{Re}\ [i\ (w + \frac{1}{3} w^3)]$$

$$z = \mathrm{Re}\ [w^2]$$

where w ranges over the complex plane C . The Gauss map omits one point (0,0,1).

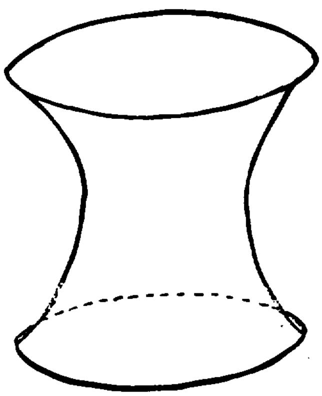
2. The Catenoid. This is generated by revolving the catenary z=cosh(x) about the x-axis in (x,y,z)-space. The Gauss map is 1-1 and omits 2 points (±1, 0, 0). This surface is given explicitly by the equation.

$$z^2 + y^2 = (\cosh x)^2 . \qquad \text{(Fig. 1)}$$

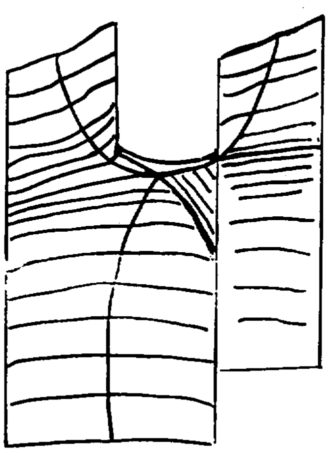
3. Scherk's surface. This is a complete, doubly periodic, minimal surface, which is invariant under the translations $(x,y,z) \rightarrow (x,y+2\pi,z)$ and $(x,y,z) \rightarrow$

$(x+2\pi, y, z)$. The interior of a fundamental domain of the surface can be expressed as the graph of the function $z=\log(\cos y/\cos x)$ in the square:

$|x|<\pi/2$ and $|y|<\pi/2$. This function goes to ∞ as $(x,y)\to(\pm\frac{\pi}{2}, y)$ for $|y|<\pi/2$ and goes to $-\infty$ as $(x,y)\to(x,\pm\pi/2)$ for $|x|<\pi/2$. The resulting surface assumes the four lines $|x|=|y|=\pi/2$ as boundary. The surface can now be continued indefinitely by reflection. The Gauss map omits 4 points $(\pm 1, 0, 0)$ and $(0, \pm 1, 0)$. (Fig. 2).



(Fig. 1)



(Fig. 2)

We have given examples of minimal surfaces whose Gauss map omits 1, 2 and 4 points. But, alternatively, using Weierstrass representation theorem of minimal surfaces, we get:

Let E be an arbifrary set of k points on S^2, where $2\leqslant k\leqslant 4$. Then there exists a complete regular minimal surface in R^3 whose Gauss map omits precisely the set E. The proof will be given in the later.

2. Main Theorems

Theorem 1. Let D be a domain in the complex w-plane $(w=x+iy)$, g an arbitrary meromorphic function in D and f an analytic function in D having the property that at each point where g has a pole of order m, f has a zero of order at least 2m. Put

(1.1) $\phi_1=\frac{1}{2}(1-g^2)f$,

$\phi_2=\frac{i}{2}(1+g^2)f$,

$\phi_3=gf$.

Then the function $\Psi=(\Psi_1, \Psi_2, \Psi_3): D\to R^3$, where

(1.2) $\Psi_k(w)=\mathrm{Re}\left(\int^w \phi_k(z)\,dz\right)$

will define a minimal surface M in R^3 whose metric is given by $ds^2=\lambda^2|dw|^2$, i.e. $g_{ij}=\lambda^2\delta_{ij}$ where $\lambda=\frac{|f|}{2}(1+|g|^2)$.

The equation (1.1) is called the Weierstrass representations of minimal surfaces in R^3. This representation makes it easy to write down an enormous number of complete minimal surfaces in R^3. For example, if we set $D=C$, $f\equiv 1$ and $g(z)=z$, we get Enneper's surface. If we set $D=C\sim[0]$, $f=(\frac{1}{z^2})$, $g(z)=z$ we get the catenoid.

For simply connected regular surfaces, we get the following result.

Theorem 2. Every simply connected minimal surface M in R^3 can be represented in the form (1.2), where the domain D is either the unit disk or the entire plane, g and f have properties stated in Theorem 1. The surface will be regular if and only if f satisfies the further property that it vanishes only at the poles of g, and the order of its zero at such a point is exactly twice the order of the pole of g.

The proof will be found in [6].

It will be convenient, for later use, to make some geometric observations about the Weierstrass representation. Let $\Psi : D \to M \subset R^3$ be the surface in Theorem 1. Observe that g can be thought of as a conformal map $g:D \to C \cup [\infty] = S^2$. In this sense, g is exactly the Gauss map of the surface. In particular, let N be the unit normal vector field on M and $\pi : S^2 \sim [(0, 0, 1)] \to R^2$ be the stereographic projection. We already know

(1.3) $N(q) = \dfrac{\Psi_x \times \Psi_y}{|\Psi_x \times \Psi_y|}(p) \in S^2 \subset R^3$, where $\Psi(p)=q$.

Then

(1.4) $g = \pi \cdot N \cdot \Psi$.

To see this, we note that $\dfrac{\partial \Psi}{\partial x} - i\dfrac{\partial \Psi}{\partial y} = (\phi_1, \phi_2, \phi_3)$, and theorefore

$$\frac{\partial \Psi}{\partial x} \times \frac{\partial \Psi}{\partial y} = \mathrm{Im}\,[(\phi_2\overline{\phi_3}, \phi_3\overline{\phi_1}, \phi_1\overline{\phi_2})]$$

$$= \frac{(1+|g|^2)|f|^2}{4}(\,2\,\mathrm{Re}\,g, 2\mathrm{Im}\,g, |g|^2-1).$$

Hence

$$N \circ \Psi = \frac{\Psi_x \times \Psi_y}{|\Psi_x \times \Psi_y|} = \frac{(2\mathrm{Re}g, 2\mathrm{Im}g, |g|^2-1)}{|g|^2+1} = \pi^{-1} \circ g.$$

Equation (1.4) means that the poles of g occur exactly at those points $q \in M$ where $N(q)=(0, 0, 1)$. Thus, if the Gauss map N omits at least one point of S^2 we may, by making a rotation of coordinates, assume that g has no poles on M (and, therefore, f also has no zeros.)

Let us return now to the general case of a minimal surface $\Psi : M \to R^n$, where Ψ is a minimal immersion and M is 2-dimensional orientable manifold, not necessarily submanifold of R^3. Receall that if in a local coordinate z on M the metric is expressed as $ds^2 = 2F|dz|^2$, the Gauss curvature K of the surface is given by

$$K = -\frac{1}{F}\frac{d}{dz}\frac{d}{d\bar{z}}\log F.$$

We then have that, in terms of the functions $\phi = \dfrac{\partial \Psi}{\partial z}$, K can be expressed as

$$K = -\frac{|\phi \wedge \phi'|^2}{|\phi|^6}$$

where $|\phi \times \phi'|^2 = |\phi|^2|\phi'|^2 - |\langle \phi, \phi' \rangle|^2$

$$= \sum_{i<j} |\phi_i\phi'_j - \phi_j\phi'_i|^2$$

We introduce on CP^{n-1} the Fubini–Study metric

$$ds^2 = \frac{|z \wedge dz|^2}{|z|^4}$$

We have renormalized the metric here (a factor of 2 instead of 4) so that the induced metric on the quadratic Q_1 is of constant curvature 1. The equivalence, $S^2 \approx Q_1$, is now an isometry. Each of the linear subspaces $CP^1 \subset CP^{n-1}$ has the form

$$d\sigma^2 = 2G|dz|^2$$

where

$$G = \frac{|\phi \wedge \phi'|^2}{|\phi|^4}$$

Hence, as a generalization of the classical case in R^3, we have

$$K = -\frac{d\sigma^2}{ds^2}$$

Letting $C(\Psi)$ denote the total curvature of M and $A(\Phi)$ the area induced by the Gauss map, we see that therefore

$$C(\Psi) = -A(\Phi).$$

Theorem 3. Let E be an arbitrary set of k points on S^2, where $2 \leqslant k \leqslant 4$. Then there exists a complete regular minimal surface in R^3 whose Gauss map omits precisely the set E.

Proof. By a rotation we may assume that E contains the north pole. Let the other points of E correspond to the points ω_i, $i=1,2, \ldots, k-1$, under the stereographic projection. If we set $g(w)=\frac{1}{\prod_{i=1}^{k-1}(w-w_i)}$, $f(w)=w$ and $D=C-[w_1, \ldots, w_{k-1}]$ in Theorem 1, we obtain a minimal surface

$$\Psi = (\Psi_1, \Psi_2, \Psi_3) : D \to R^3,$$
$$\Psi_k(w) = \mathrm{Re}\int^w \phi_k(z)dz, \ k=1, 2, 3.$$

Since $\pi^{-1} \circ g = N \circ \Psi$, $g(w) \neq w_1, \ldots, w_{k-1}$, the Gauss map must omit $\pi^{-1}(w_1), \ldots, \pi^{-1}(w_{k-1})$. Futhermore, there is no $w \in D$ such that $\pi^{-1}(g(w)) = (0, 0, 1)$, $N \circ \Psi(w) \neq (0, 0, 1)$ for $v_w \notin D$. Thus, $\Psi: D \to R^3$ is a minimal surface whose Gauss map omits precisely the points of E, and which is complete, because a divergent path γ must tend either to ∞ or to one of the points w_m, and in either case, we have

$$\int_\gamma \lambda|dw| = \frac{1}{2}\int_\gamma |f|(1+|g|^2)|dw| = \infty$$

The following theorem is the object of this thesis. For the sake of clarity we shall state our result more precisely.

Theorem 4. The complement of the image of the Gauss map of a non-flat complete minimal surface in R^3 contains at most 6 points of S^2

We need the following some results.

Let M be a connected Riemannian m-manifold. By the *Laplace-Beltrami operator* on M we mean a map $\Delta: C^\infty(M) \to C^\infty(M)$ defined in any of the following equivalent ways. Let $p \in M$ and $f \in C^\infty(M)$; then;

(a) If $\mathcal{E}_1, \ldots, \mathcal{E}_m \in \aleph_p$ are pointwise orthonormal, then

$$\Delta f = \sum_{k=1}^{m} [\mathcal{E}_k \mathcal{E}_k f - (\Delta_{\mathcal{E}_k} \mathcal{E}_k) f]$$

in a neighborhood of p.

(b) If $(x^1, \ldots, x^m)$ are local coordinates at p, then in the coordinate neighborhood

$$\Delta f = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij} \frac{\partial f}{\partial x^j}\right)$$

where the metric $ds^2 = \Sigma g_{ij} dx^i dx^j$, the matrix $((g^{ij})) = ((g_{kl}))^{-1}$ and $g = \det((g_{ij}))$

c) $\Delta f = -{*}d{*}df$.

Theorem([7], Theorem 1) Let M be a complete Riemannian manifold of infinite volume and μ a non-negative function satisfying $\Delta \log \mu = 0$ almost everywhere. Then $\int_M \mu^p = \infty$ for $p > 0$.

Let U be the unit disk in the complex plane. A function $f: U \to C$ is called normal if the family $[f(S(z))]$, where S is a conformal transformation of U into itself, is normal in Montel's sense, i.e. any sequence in the family contains a subsequence converging uniformly on compact subsets of U.

Lemma. Let f be a holomorphic function in the unit disk D and let $f\neq 0$, a. Let $\alpha=1-\frac{1}{k}$, $k\epsilon Z^+$

Then we have

$$\frac{|f'|}{|d|^{\alpha}+|f|^{2-\alpha}}\ \epsilon\ L^p(D)$$

for every p with $0<p<1$.

Proof. Since $f^{\frac{1}{k}}$ omits two values, it is normal (see [2], page 169). By Theorem 6.5 of [2], there is a constant C such that

$$\frac{|g'|}{1+|g|^2}\leqslant\frac{C}{1-|z|^2}$$

Applying this estimate on the spherical derivative to $f^{1/k}$, we have

$$\frac{|f'|}{k|f|^{1-1/k}(1+|f|)^{2/k}}\leqslant\frac{C}{1-|z|^2}$$

so that

$$\frac{|f'|}{|f|^{1-1/k}+|f|^{2(1-\frac{1}{k})}}\leqslant\frac{k}{1-|z|^2}$$

In particular, $|f'|/(|f|^{\alpha}+|f|^{2-\alpha})\ \epsilon\ L^p(D)$, $0<p<1$, because

$$\int_D\left(\frac{1}{1-|z|^2}\right)^p dz=\int_0^{2\pi}\int_0^1\frac{r}{(1-r^2)^p}drd\theta<\infty.$$

Proof of Theorem 4.

Suppose that M is a complete non-flat minimal surface whose Gauss map misses 7 points. By passing to the universal covering space, we may assume M is simply connected. By Theorem 2, M can be represented in the form (1.2), where the domain D is either U or C. Recall (1.3) . Since M is not flat, g is non-constant. If D=C, then g is entire (because g has no poles in C) and, hence, by Picard Theorem g takes all complex values with at most one exception. Thus the Gauss map N takes all values of S^2 with at most one exception Futhermore, the functions f and g are holomorphic in U and $|f|>0$. From (1.3) we also notice that the north pole is among the omitted points since g has no poles. In view of the above we are reduced to proving the following:

(*) Let f,g be holomorphic functions on U, $|f|>0$. Suppose that for six distinct numbers a_1, $a_2, \ldots, a_6$ the equation $g(z)=a_i$ has no solution $(i=1, 2, \ldots, 6)$. Then the metric $|f|^2(1+|g|^2)^2|dz|^2$ on U is not complete.

For the proof consider the function

$$h=f^{-\frac{2}{p}}g'\prod_{i=1}^{6}(g-a_i)^{-\alpha}$$

where $\frac{5}{6}<\alpha<1$ is as in the previous Lemma and $p=5/6\alpha$. Note that $f^{\cdot\frac{2}{p}}$ is well-defined because $|f|>0$. The Laplace-Beltrami operator Δ of the metric

$$\lambda|dz|^2\ (\lambda=|f|^2(1+|g|^2)^2)$$

is given by $(1/\lambda)\ (\partial/\partial z)\ (\partial/\partial\bar{z})$. Hence the function $\mu=|h|$ satisfies $\Delta\log\mu=0$ almost everywhere in U (there may be a discrete set where g' vanishes). We assert that $\mu\notin L^p(M)$. Indeed, if μ is a (necessarily non-zero) constant, this follows from the fact that complete simply-connected surfaces of nonpositive curvature have infinite area. If μ is not constant this follows from Yau's theorem ([7], Theorem 1). Since the area element is $\lambda dxdy$, the condition $\mu\notin L^p(D)$ can be written

$$\int_U\frac{|g'|^p\ (1+|g|^2)^2}{\prod_{i=1}^6|g-a_i|^{p\alpha}}dxdy=\infty.$$

The contradiction will be achieved by showing that this integral is actually finite. Let

$D_j = [z \epsilon U | \, |g(z) - a_j| \leq \ell]$,

where

$0 < \ell < (\frac{1}{4}) \min_{i \neq k\,;\, k=1, \ldots, 6} |a_i - a_k|$.

Also, let $U' = U \setminus \cup_{j=1}^{6} D_j$. Denoting by $H(z)$ the in-integrand of the last integral we have

$$\int_U H\, dxdy = \sum_{j=1}^{6} \int_{D_j} H\, dxdy + \int_{U'} H\, dxdy.$$

On each D_j we have an estimate $H \leq c(|g'|^p / |g-a_j|^{p\alpha})$. We may also assume $\ell < 1$, so that

$$\frac{|g'|^p}{|g-a_j|^{p\alpha}} \leq 2^p \frac{|g'|^p}{(|g-a_i|^{\alpha} + |g-a_i|^{2-\alpha})^p}$$

Hence $\int_{D_j} H\, dx\, dy < \infty$ by the lemma. The integral over U' can be handled in a similar way. We observe that

$$\frac{(1+|g|^2)^2}{\pi_{i=1}^{5} |g-a_j|^{p\alpha}} = \frac{(1+|g|^2)^2}{\pi_{j=1}^{5} |g-a_j|^{5/6}}$$

is bounded over = U'. Hence

$$\int_{U'} H dxdy \leqq c \int_{U'} \frac{|g'|^p}{|g-a_6|^{p\alpha}} dxdy < \infty,$$

as before. This completes the proof of (*).

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국 문 초 록

비편평 완비 극소 곡면의 Gauss 사상

비편평 완비 극소 곡면 위에서의 Gauss 사상이 취하지 못하는 단위구면 위의 점들의 갯수는 4개까지는 가능하나 7개 이상은 불가능함을 증명하였다.