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Ranking fuzzy numbers by a statistical comparison

Chul Soo Kim
Department of Computer Science and Statistics,
Cheju National University, Cheju 690-756, Korea

Abstract

We consider a technique for ranking fuzzy numbers. The triangular and exponential fuzzy numbers are introduced and their membership functions are converted into probability density functions. Using the converted density function, the moment type transform is then introduced to obtain the mean and variance of a fuzzy numbers.

1. Introduction

Among the various types of fuzzy sets, there are special fuzzy sets that are defined on the set of real number. Under certain conditions, these fuzzy sets are viewed as fuzzy numbers.

Applications on fuzzy sets require higher level arithmetics that include the product of several fuzzy numbers and some power of fuzzy numbers. The fuzzy operations of this type may require computational complexity. It is well known that multiplication of two triangular fuzzy numbers does not necessarily result in a triangular fuzzy numbers.

Ranking fuzzy numbers is very important in fuzzy applications. The task of comparing do not always yield a totally ordered set.

In this paper we will present a statistical comparison technique for fuzzy numbers based on the moment type transform. The first step of this method is to derive a corresponding probability density function from a membership function. Then the mean and variance of fuzzy numbers are obtained in terms of the moment type transform. In section 3, we will rank two fuzzy numbers using the means of corresponding density functions.

2. Fuzzy numbers and converted probability density functions

Among the various types of fuzzy sets, of special significance are fuzzy sets that are defined on the set R of real numbers. Membership functions of these sets clearly have a quantitative meaning and may, under certain conditions, be viewed as fuzzy numbers.

Fuzzy numbers are a special kind of fuzzy sets which are normal and convex.

There is an infinite set of fuzzy numbers, but here we will define a special class of fuzzy numbers called triangular fuzzy numbers(TFN) and exponential fuzzy number(EFN). This study considers these two fuzzy numbers: triangular fuzzy number (TFN) and exponential fuzzy numbers (EFN).

Def 2.1 A triangular fuzzy number A denoted by TFN (l, m, μ) is defined as

$$\mu_A(x) = \begin{cases} \dfrac{x-l}{m-l}, & l \le x \le m \\ \dfrac{u-x}{\mu-m}, & m \le x \le \mu \\ 0 & \text{otherwise} \end{cases}$$

where $\mu_A(x)$ is the membership function of A

Def 2.2 A fuzzy number A is called exponential fuzzy number, $A=$ EFN(a,b), if the corresponding membership functions satisfies for all $x \in R$

$$\mu_A(x) = \exp\left(-\left(\frac{(x-a)^2}{b}\right)\right), \text{ where } -\infty < a < \infty \text{ and } b > 0.$$

In order to add fuzzy and random data, we will use a converted probability density function from a membership function by way of a simple transformation.

The conversion of a membership function into a probability density function can be made by a linear transformation, so-called the proportional probability distribution.

The occurrence probability of fuzzy event A should be proportional to the value of membership function $\mu_A(x)$. The converted density function $f(x)$ is of the form

$$f(x) = c \cdot \mu_A(x)$$

where c_1 is a proportional constant to satisfy the condition that the area under the continuous probability density function is equal to one. The converted density function retains the domain of variable X, but has lost the original shape of the membership function to some degree.

Consider a converted probability density function from a triangular fuzzy number.

For a triangular fuzzy number $A=$ TFN (l,m,u), we obtain a triangular density function $f(x)$ which have the form

$$f(x)=\begin{cases} 2(x-l)\,/\,(u-l)(m-l), & l\le x\le m \\ 2(u-x)\,/\,(u-l)(u-m), & m\le x\le u \end{cases} \tag{2.1}$$

For an exponential fuzzy number $A=$ EFN(a,b), the converted density function $f(x)$ is given by

$$f(x)=\frac{1}{\sqrt{\pi}b}\exp\left(-\frac{1}{2}\left(\frac{x-a}{b/\sqrt{2}}\right)^2\right) \tag{2.2}$$

where $-\infty < a < \infty$ and $b>0$.

3. The moment type transform

Let X be a continuous random variable with the probability density function $f(x)$. The moment type transform, $M_x(s)$ of the continuous probability density function $f(x)$ is defined as

$$M_X(s)=\int_{-\infty}^{\infty}x^{s-1}f(x)\,dx \tag{3.1}$$

For multivariate case, the moment type transform $M_x(s)$ is extended by

$$M_{X_1+\cdots+X_n}(s) = \int\cdots\int(x_1^{s-1}+\cdots+x_n^{s-1})f(x_1,\cdots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n \tag{3.2}$$

and

$$M_{X_1\cdots X_n}(s) = \int\cdots\int(x_1^{s-1}\cdots x_n^{s-1})f(x_1,\cdots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n \tag{3.3}$$

where $f(x_1,\cdots,x_n)$ is the jointly probability density function of the random variables $X_1,X_2,\cdots,X_n$.

The moment type transform $M_X(s)$ can be stated in terms of the expected value of X.
From (3.1), we obtain the relation

$$M_X(s) = E[X^{s-1}] \tag{3.4}$$

Thus, the first two statistical moments of the random variable X can be represented as follows:

$$\mu = E(x) = M_X(2)$$
$$\sigma^2 = Var(X) = E(X^2) - (E(X))^2 = M_x(3) - (M_x(2))^2$$

From the close relationship of the moment type transform and expected value, it is quite simple to establish some important properties of the moment type transform involving products, quotients and powers of random variables.

First, we will find the transform for the distribution kth power of X.

Let $Y = X^k$. Then

$$M_Y(s) = E(Y^{s-1}) = E((X^k)^{s-1})$$
$$= E(X^{ks-k-1}) = M_X(ks-k+1)$$

Proposition 1. If the random variables X and Y are independent and continuously distributed, then the moment type transform of the random variable $aX+bY$ is given by

$$M_{aX+bY}(s) = a^{s-1}M_X(s) + b^{s-1}M_Y(s)$$

Proof. From (3.2), it is clear.

Proposition 2. Let X and Y be continuously distributed, independent random variables with probability density functions $g(x)$ and $h(y)$, respectively and let $Z = XY$.

Then

$$M_Z(s) = M_X(s)M_Y(s)$$

Proof. From (3.3), it follows that $M_Z(s) = M_X(s)M_Y(s)$.

Proposition 3. Let X and Y be continuous random variables with joint

probability density function $f(x,y)$ and let $Z=X/Y$.
Then

$$M_Z(s)=M_X(s)M_Y(2-s)$$

Proof. From (3.3). The proof of Proposition 3 is simple.

We have seen that the moment type transform allows easy calculation of the statistical moments of any order of products and quotients without going through the inversion process. It is not even necessary to take derivatives.

In section 2, we have the conversion of triangular and exponential fuzzy membership functions into corresponding probability density functions.

For a triangular fuzzy number $A=$ TFN (l,m,u), the converted density function $f(x)$ is defined as in (2.1).
That is

$$f(x)=\begin{cases} f_1(x)=2(x-l)\ /\ (u-l)(m-l), & l\le x\le m \\ f_2(x)=2(u-x)\ /\ (u-l)(u-m), & m\le x\le u \end{cases}$$

Let X be a random variable corresponding a triangular fuzzy number $A=$ TFN (l,m,u).

Then the moment type transform $M_X(s)$ is obtained by

$$\begin{aligned} M_X(s) &= \int_{-\infty}^{\infty} x^{s-1}f(x)\,dx = \int_l^m x^{s-1}f_1(x)\,dx+\int_m^u x^{s-1}f_2(x)\,dx \\ &= \frac{2}{(u-l)s(s+1)}\left[\frac{u(u^s-m^s)}{(u-m)}-\frac{l(m^s-l^s)}{(m-l)}\right] \end{aligned}$$

We present one example here to show the applicability of the moment type transform in comparing fuzzy numbers.

Example

Let a fuzzy number X be the multiplication of two fuzzy numbers A and B.
That is $X=A(\cdot)B$
where $A=$ TFN(1,4,6) and $B=$ EFN(3,2).

Similarly, let $Y=C(\cdot)D$
where $C=$ TFN(2,3,4) and $D=$ EFN(3,4).

Now, compare two resulting fuzzy numbers X and Y by way of the moment type transform.

From the Proposition 2, the moment type transform of X would be $M_X(s)=M_A(s)M_B(s)$.

Since A is a TFN(1,4,6).

$$M_A(s)=\frac{2}{5s(s+1)}\left[\frac{6(6^s-4^s)}{2}-\frac{(4^s-1^s)}{3}\right]$$

Also, since B is EFN(3,2), the converted density is given by

$$f(x)=\frac{1}{2\sqrt{\pi}}\exp\left(-\left(\frac{x-3}{2}\right)^2\right)$$

Hence the moment type transform of B would be

$$M_B(s)=\int_{-\infty}^{\infty}x^{s-1}\frac{1}{2\sqrt{\pi}}\exp\left(-\left(\frac{x-3}{2}\right)^2\right)dx$$

The mean and variance of X are determined as

$$\mu_X=M_X(2)=M_A(2)\,M_B(2)=3.67\times3=11.01$$

$$\sigma_x^2=M_X(3)-\mu_X^2=M_A(3)M_B(3)-11.01^2$$

$$=14.5\times11-11.01^2=38.3$$

Similarly

$$M_C(s)=\frac{2}{2s(s+1)}[4(4^s-2^s)-2(3^s-2^s)],$$

$$M_D(s)=\int_{-\infty}^{\infty}x^{s-1}\frac{1}{4\sqrt{\pi}}\exp\left(-\left(\frac{x-3}{4}\right)^2\right)dx$$

So, the mean and variance of Y are determined as

$$\mu_Y=M_Y(2)=M_C(2)\,M_D(2)=3\times3=9$$

$$\sigma_Y^2=74.72$$

Hence we conclude that X is greater than Y because the mean of X is larger than that of Y.

4. Concluding remarks

The topic of ranking of fuzzy numbers is important and has many practical implications. One can proceed to rank the fuzzy numbers in various ways by providing various weightings to the different features of the fuzzy numbers. These methods presume the availability of exact membership function of fuzzy numbers. But the fuzzy numbers after arithmetic operations are highly complicated and their exact membership functions may not be available.

In this paper a statistical method was chosen. Membership functions of fuzzy numbers are converted into probability density functions, then the moment type transform is used to compute the mean and variance of triangular and exponential fuzzy numbers. The fuzzy number with the higher mean is then ranked higher than the fuzzy number with a lower mean.

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