J. of Basic Sciences, Cheju Nat. Univ.
9(1), 195~201, 1996

기초과학연구 제주대학교
9(1), 195~201, 1996

On Convertible Complex Matrices*

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ABSTRACT

A complex square matrix A is called *convertible* if there is a matrix B obtained by A from affixing $\pm$ signs to entries of A such that per A = det B. In this note it is proved that a complex matrix all of whose entrices are taken from a fixed sector of angle π/n is convertible if and only if its support is.

* This research was supported by TGRC-Kosef in 1992.

LINEAR ALGEBRA AND ITS APPLICATIONS 233:167–173 (1996)

0024-3795/96/$15.00
SSDI 0024-3795(94)00065-L

1. INTRODUCTION

For a field F of characteristic 0, let $F^{n\times n}$ denote the vector space of all $n \times n$ matrices over F. For $A = [a_{ij}] \in F^{n\times n}$, the permanent of A, per A, is defined by

$$\text{per } A = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)},$$

where S_n stands for the symmetric group on $\{1, 2, \ldots, n\}$.

Conversion of the permanent into the determinant is a classical problem. In 1913, Pólya [7] posed a problem of determining whether or not there exists a method of uniformly affixing $\pm$ signs to entries of matrices in $F^{n\times n}$ so that the permanent is converted into the determinant. Pólya's problem was solved by Szegö [9]. Generalizing Pólya and Szegö's result, Marcus and Minc [6] proved that there is no linear transformation $T : F^{n\times n} \to F^{n\times n}$ such that per $A = \det T(A)$ for all $A \in F^{n\times n}$.

However, there are matrices A such that per $A = \det B$ for some matrix B obtained from A by affixing $\pm$ signs to entries of A, i.e., such that per $A = \det(H \circ A)$ for some $(1, -1)$ matrix H of the same size as A, where $H \circ A$ denotes the Hadamard (entrywise) product of H and A. If that is the case, the matrix A is called *convertible* and the matrix H is called a *converter* of A [4].

For matrices A, B of the same size, A is said to be permutation-equivalent to B if there exist permutation matrices P, Q such that $PAQ = B$. If both A and B are real, we denote by $A \leqslant B$ that every entry of A is less than or equal to the corresponding entry of B.

Let $T_n = [t_{ij}]$ be the $n \times n$ (0, 1) matrix defined by $t_{ij} = 0$ if and only if $j > i + 1$. Gibson proved that every $n \times n$ real matrix A such that $A \leqslant T_n$ is convertible [3] and also that the number of 1's of an $n \times n$ convertible (0, 1) matrix B is less than or equal to $(n^2 + 3n - 2)/2$, with equality if and only if B is permutation-equivalent to T_n [2]. A graph theoretical characterization of convertible (0, 1) matrices was obtained by Little [5]. An $n \times n$ real matrix S is called *sign-nonsingular* if every $n \times n$ real matrix with the same sign pattern as S is nonsingular. It is noted in [1] that an $n \times n$ (0, 1) matrix is convertible if and only if there exists an $n \times n$ $(1, -1)$ matrix H such that $H \circ A$ is sign-nonsingular. The convertibility of complex matrices does not seem to be easily linked to something like sign-nonsingularity.

In this paper we study the convertibility of complex matrices in connection with that of their supports.

2. MAIN RESULTS

Let $\boldsymbol{R}$ and $\boldsymbol{C}$ denote the real field and the complex field respectively. For $A = [a_{ij}] \in \boldsymbol{C}^{n\times n}$, the $n \times n$ matrix supp $A = [s_{ij}]$ defined by

$$s_{ij} = \begin{cases} 1 & \text{if} \quad a_{ij} \neq 0, \\ 0 & \text{if} \quad a_{ij} = 0 \end{cases}$$

is called the *support* of A. It is proved in [4] that an $n \times n$ real matrix A is convertible if there is an $n \times n$ convertible (0, 1) matrix B such that supp $A \leqslant B$. However, given a complex (or even real) matrix A, it is not easy to decide whether A is convertible or not by checking only the support of A. This is possible for some special classes of complex matrices. In the following we prove a theorem which may be used as a convertibility test for certain class of complex matrices. From now on in the sequel, for any real number α, $0 \leqslant \alpha < 2\pi$, let $R_{\alpha, n}$ denote the subset of $\boldsymbol{C}$ defined by

$$R_{\alpha, n} = \left\{ z \in \boldsymbol{C} \,|\, z \neq 0,\ \alpha - \frac{\pi}{2n} < \arg z < \alpha + \frac{\pi}{2n} \right\} \cup \{0\}.$$

We call such a set $R_{\alpha, n}$ an n-sector.

THEOREM 1. *For $A = [a_{ij}] \in \boldsymbol{C}^{n\times n}$, the following holds:*

(1) *If* supp A *is convertible, then so is A.*

(2) *If there is an α, $0 \leqslant \alpha < 2\pi$, such that $a_{ij} \in R_{\alpha, n}$ for all $i, j = 1, \dots, n$, then the converse of* (1) *holds.*

Proof. (1): Suppose that supp $A = [s_{ij}]$ is convertible with converter $H = [h_{ij}]$. Then

$$\sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} s_{i\sigma(i)} \right) \left(1 - (\operatorname{sgn} \sigma) \prod_{i=1}^{n} h_{i\sigma(i)} \right) = 0.$$

Since $s_{1\sigma(1)}, \dots, s_{n\sigma(n)} \geqslant 0$ and $(\operatorname{sgn} \sigma) h_{1\sigma(1)} \cdots h_{n\sigma(n)} \geqslant 0$ for all $\sigma \in S_n$, it follows that $(\operatorname{sgn} \sigma) h_{1\sigma(1)} \cdots h_{n\sigma(n)} = 1$ for all $\sigma \in S_n$ such that $s_{1\sigma(1)} \cdots s_{n\sigma(n)} = 1$. So, since $s_{1\sigma(1)} \cdots s_{n\sigma(n)} = 1$ whenever $a_{1\sigma(1)} \cdots a_{n\sigma(n)} \neq 0$, it follows that

$$\sum_{\sigma\in S_n}\left(\prod_{i=1}^{n} a_{i\sigma(i)}\right)\left(1-(\operatorname{sgn}\sigma)\prod_{i=1}^{n} h_{i\sigma(i)}\right)=0$$

and hence that per $A = \det(H \circ A)$.

(2): Conversely, suppose that $A = [a_{ij}]$ is convertible and also that there is an α, $0 \leqslant \alpha < 2\pi$, such that $a_{ij} \in R_{\alpha, n}$ for all $i, j = 1, 2, \ldots, n$. We prove the convertibility of supp A for the case $\alpha = 0$ first and then do the general case.

Case (i): $\alpha = 0$. For each $(i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$, we can choose $r_{ij} \geqslant 0$ and θ_{ij}, $-\pi/2n < \theta_{ij} < \pi/2n$, such that $a_{ij} = r_{ij}\exp(\sqrt{-1}\,\theta_{ij})$. Let $H = [h_{ij}]$ be a converter of A. Then

$$\begin{aligned} 0 &= \operatorname{per} A - \det(H \circ A) \\ &= \sum_{\sigma\in S_n}\left(1-(\operatorname{sgn}\sigma)\prod_{i=1}^{n} h_{i\sigma(i)}\right)\left(\prod_{i=1}^{n} a_{i\sigma(i)}\right) \\ &= \sum_{\sigma\in S_n}\left(1-(\operatorname{sgn}\sigma)\prod_{i=1}^{n} h_{i\sigma(i)}\right)\left(\prod_{i=1}^{n} r_{i\sigma(i)}\right)\exp\left(\sqrt{-1}\sum_{i=1}^{n}\theta_{i\sigma(i)}\right), \end{aligned}$$

from which it follows that

$$\sum_{\sigma\in S_n}\left(1-(\operatorname{sgn}\sigma)\prod_{i=1}^{n} h_{i\sigma(i)}\right)\left(\prod_{i=1}^{n} r_{i\sigma(i)}\right)\left(\cos\sum_{i=1}^{n}\theta_{i\sigma(i)}\right)=0.$$

Since $-\pi/2n < \theta_{i\sigma(i)} < \pi/2n$ for all $i = 1, 2, \ldots, n$ and all $\sigma \in S_n$, it follows that

$$-\frac{\pi}{2} < \sum_{i=1}^{n}\theta_{i\sigma(i)} < \frac{\pi}{2}$$

for all $\sigma \in S_n$ and hence that

$$\cos\sum_{i=1}^{n}\theta_{i\sigma(i)} > 0$$

for all $\sigma \in S_n$. Thus $(\operatorname{sgn} \sigma) h_{1\sigma(1)} \cdots h_{n\sigma(n)} = 1$ for all $\sigma \in S_n$ such that $r_{1\sigma(1)} \cdots r_{n\sigma(n)} \neq 0$. Therefore we see that the support of the matrix $\mathcal{B} = [r_{ij}]$ is convertible and hence that supp A is convertible because supp A = supp $\mathcal{B}$.

Case (ii): General case. Let $\beta = \exp(-\sqrt{-1}\,\alpha)$, and let $X = \beta A$. Then X is also convertible, and all the entries of X are in the n-sector $R_{0,n}$. So, by case (i), supp X, which equals supp A, is convertible, and the proof is complete. ∎

COROLLARY. *A real nonnegative square matrix is convertible if and only if its support is.*

The converse of assertion (1) of Theorem 1 does not, in general, hold for complex (or even real) matrices. In the following we give examples of convertible matrices with nonconvertible supports. Let

$$A = \begin{bmatrix} 1 & \omega & \omega^2 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix},$$

where $\omega = \exp(2\pi\sqrt{-1}/3)$, and let

$$H = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then per $A = 2 = \det(H \circ A)$, so that A is convertible, while supp A is the 3×3 matrix of 1's, which is well known to be nonconvertible [8]. Let

$$B = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then B is a real matrix with nonconvertible support: however, it is convertible, since per $B = 4 = \det(K \circ B)$, where

$$K = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The condition in (2) of Theorem 1 which enables a convertible complex matrix to have a convertible support can be weakened a bit, as we see in the following

THEOREM 2. *Let* $A \in \mathbf{C}^{n\times n}$ *be such that all the components of each column vector (or row vector) come from an n-vector. If A is convertible, then* supp A *is convertible.*

Proof. Let $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$, and let $\alpha_1, \ldots, \alpha_n \in [0, 2\pi)$ be such that all the components of $\mathbf{a}_j$ are in $R_{\alpha_j, n}$ for all $j = 1, \ldots, n$. Let $D = \operatorname{diag}(\exp(-\sqrt{-1}\,\alpha_1), \ldots, \exp(-\sqrt{-1}\,\alpha_n))$, and let $B = AD$. Then since $\operatorname{per} B = \exp(\sqrt{-1}\,\beta) \operatorname{per} A$ and $\det B = \exp(\sqrt{-1}\,\beta) \det A$ where $\beta = -\alpha_1 - \cdots - \alpha_n$, we see that B is also convertible. It is clear that supp A = supp B. Now since each entry of B lies in the sector $R_{0, n}$, we see that supp B is convertible by Theorem 1. ■

As we mentioned earlier, our Theorem 2 can be used to test the nonconvertibility of complex matrices of a certain type from that of their supports. For example, since the 3×3 matrix of 1's is not convertible, no matrices $A = [a_{ij}] \in \mathbf{C}^{3\times 3}$ such that $a_{ij} \neq 0$ for all $i, j = 1, 2, 3$ and such that

$$\max_{1 \leqslant j < l \leqslant 3} |\arg a_{ij} - \arg a_{il}| < \frac{\pi}{3}$$

for each $i = 1, 2, 3$ are convertible.

The authors wish to thank the referee for pointing out a technical error in the original manuscript and for some valuable comments.

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Received 10 December 1993; final manuscript accepted 1 March 1994