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Characterization of the foliation by the first eigenvalues of the basic Dirac operator

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Abstract. On a foliated Riemannian manifold with a transverse spin structure, any eigenvalue λ of the basic Dirac operator satisfies $\lambda^2 \geq \frac{q}{4(q-1)}(\mu_1 + \inf|\kappa|^2)$, where μ_1 is the first eigenvalue of the basic Yamabe operator and κ is a mean curvature form. Using the real spinor representation of *spin* and the generalized Lichnerowicz-Obata theorem, we prove that for the codimension of $\mathcal{F}$ is $q = 3, 4, 7$ and 8, the foliation $\mathcal{F}$ with the eigenvalue $\lambda^2 = \frac{q}{4(q-1)}(\mu_1 + \inf|\kappa|^2)$ is transversally isometric to the action of discrete subgroup of $O(q)$ acting on the q-sphere.

1 Introduction

The first estimate for the eigenvalues λ of the basic Dirac operator D_b restricted to the space of basic sections of a foliated spinor bundle on the foliated Riemannian manifold $(M, g_M, \mathcal{F})$ with a transverse spin structure was obtained by Jung[8]. Namely, by using a modified connection $\overset{f}{\nabla}$ defined by

$$\overset{f}{\nabla}_X \Psi = \nabla_X \Psi + f\pi(X)\cdot\Psi, \tag{1.1}$$

one proved that the following inequality

$$\lambda^2 \geq \frac{q}{4(q-1)} \inf(\sigma^\nabla + |\kappa|^2) \tag{1.2}$$

holds, where $q = codim\mathcal{F}$, σ^∇ is the transversal scalar curvature and κ is the mean curvature form of $\mathcal{F}$. Recently, Jung.et al.[9] proved that on the transverse

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spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and bundle-like metric g_M, any eigenvalue λ of the basic Dirac operator D_b satisfies

$$\lambda^2 \geq \frac{q}{4(q-1)}(\mu_1 + \inf|\kappa|^2), \tag{1.3}$$

where μ_1 is the smallest eigenvalue of the basic Yamabe operator Y_b, which is defined by

$$Y_b = 4\frac{q-1}{q-2}\Delta_B + \sigma^\nabla. \tag{1.4}$$

This paper is organized as follows. In section 2, we review the known facts on the foliated Riemannian manifold. In section 3, we recall the generalized Lichnerowicz and Obata theorem [13]. In section 4, we study the basic Yamabe operator and eigenvalues of the basic Dirac operator. In section 5, we study the limiting case. Moreover, by the generalized Lichnerowicz and Obata theorem for foliations, we prove that in case of $q = 3, 4, 7$ and 8, $\mathcal{F}$ is transversally isometric to the space of orbits a discrete subgroup of $O(q)$ acting on the standard q-sphere (See [7] for ordinary manifold).

Throughout this paper, we consider the bundle-like metric g_M for $(M, \mathcal{F})$ such that the mean curvature form κ is basic and harmonic. The existence of the bundle-like metric g_M for $(M, \mathcal{F})$ such that κ is basic, i.e., $\kappa \in \Omega_B^1(\mathcal{F})$, is proved in [3]. In [15,16], for any bundle-like metric g_M with $\kappa \in \Omega_B^1(\mathcal{F})$, it is proved that there exists another bundle-like metric $\tilde{g}_M$ for which the mean curvature form $\tilde{\kappa}$ is basic-harmonic.

2 Preliminaries and known facts

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and bundle-like metric g_M with respect to $\mathcal{F}$.

We recall the exact sequence

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0$$

determined by the tangent bundle L and the normal bundle $Q = TM/L$ of $\mathcal{F}$. The assumption of g_M to be a bundle-like metric means that the induced metric g_Q on the normal bundle $Q \cong L^\perp$ satisfies the holonomy invariance condition $\overset{\circ}{\nabla} g_Q = 0$, where $\overset{\circ}{\nabla}$ is the Bott connection in Q.

For a distinguished chart $\mathcal{U} \subset M$ the leaves of $\mathcal{F}$ in $\mathcal{U}$ are given as the fibers of a Riemannian submersion $f : \mathcal{U} \to \mathcal{V} \subset N$ onto an open subset $\mathcal{V}$ of a model Riemannian manifold N.

For overlapping charts $U_\alpha \cap U_\beta$, the corresponding local transition functions $\gamma_{\alpha\beta} = f_\alpha \circ f_\beta^{-1}$ on N are isometries. Further, we denote by ∇ the canonical connection of the normal bundle Q of $\mathcal{F}$. It is defined by

$$\begin{cases} \nabla_X s = \pi([X, Y_s]) & \text{for } X \in \Gamma L, \\ \nabla_X s = \pi(\nabla^M_X Y_s) & \text{for } X \in \Gamma L^\perp, \end{cases} \tag{2.1}$$

where $s \in \Gamma Q$ and $Y_s \in \Gamma L^\perp$ corresponding to s under the canonical isomorphism $Q \cong L^\perp$. The connection ∇ is metric and torsion free. It corresponds to the Riemannian connection of the model space N([10]). The curvature R^∇ of ∇ is defined by

$$R^\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \quad \text{for } X,\ Y \in TM.$$

Since $i(X)R^\nabla = 0$ for any $X \in \Gamma L$([10]), we can define the (transversal) Ricci curvature $\rho^\nabla : \Gamma Q \to \Gamma Q$ and the (transversal) scalar curvature σ^∇ of $\mathcal{F}$ by

$$\rho^\nabla(s) = \sum_a R^\nabla(s, E_a)E_a, \quad \sigma^\nabla = \sum_a g_Q(\rho^\nabla(E_a), E_a),$$

where $\{E_a\}_{a=1,\cdots,q}$ is an orthonormal basis of Q. The foliation $\mathcal{F}$ is said to be (transversally) *Einsteinian* if the model space N is Einsteinian, that is,

$$\rho^\nabla = \frac{1}{q}\sigma^\nabla \cdot id. \tag{2.2}$$

Let $\Omega^r_B(\mathcal{F})$ be the space of all *basic r-forms*, i.e.,

$$\Omega^r_B(\mathcal{F}) = \{\phi \in \Omega^r(M) |\ i(X)\phi = 0,\ \theta(X)\phi = 0, \text{ for } X \in \Gamma L\}.$$

The foliation $\mathcal{F}$ is said to be *isoparametric* if $\kappa \in \Omega^1_B(\mathcal{F})$. We already know that κ is closed, i.e., $d\kappa = 0$ if $\mathcal{F}$ is isoparametric ([18]). Since the exterior derivative preserves the basic forms (that is, $\theta(X)d\phi = 0$ and $i(X)d\phi = 0$ for $\phi \in \Omega^r_B(\mathcal{F})$), the restriction $d_B = d|_{\Omega^*_B(\mathcal{F})}$ is well defined. Let δ_B the adjoint operator of d_B. Then it is well-known([1,8]) that

$$d_B = \sum_a \theta_a \wedge \nabla_{E_a}, \quad \delta_B = -\sum_a i(E_a)\nabla_{E_a} + i(\kappa_B), \tag{2.3}$$

where κ_B is the basic component of κ, $\{E_a\}$ is a local orthonormal basic frame in Q and $\{\theta_a\}$ its g_Q-dual 1-form.

The *basic Laplacian* acting on $\Omega_B^*(\mathcal{F})$ is defined by

$$\Delta_B = d_B\delta_B + \delta_B d_B. \tag{2.4}$$

If $\mathcal{F}$ is the foliation by points of M, the basic Laplacian is the ordinary Laplacian. In the more general case, the basic Laplacian and its spectrum provide information about the transverse geometry of $(M, \mathcal{F})$([17]).

3 Generalized Lichnerowicz and Obata theorems

Now, we recall the generalized Lichnerowicz-Obata Theorem by J. Lee and K.Richardson for foliations([13]).

Definition 3.1 *Let G be a discrete group. Then $\mathcal{F}$ is* transversally isometric *to the isometric action of G on a Riemannian manifold N if there exitsts a smooth, surjective map $\phi : M \to N$ such that*

1. *The function ϕ induces a homeomorphism between the leaf spact $M/\mathcal{F}$ and the orbit space N/G.*

2. *For each $x \in M$, the push forward ϕ_* restricts to an isometry $\phi_* : Q_x \to T_{\phi(x)}N$, where Q is the normal bundle of the foliation and TN is the tangent bundle of N.*

Theorem 3.2 ([13])*(Generalized Lichnerowicz Theorem) Let $\mathcal{F}$ be a codimension q Riemannian foliation on a closed, connected Riemannian manifold M. Suppose that there exists a positive constant c such that the transversal Ricci curvature satisfies $\rho^\nabla(X) \geq c(q-1)X$ for every $X \in Q$. Then the smallest nonzero eigenvalue λ_B of the basic Laplacian Δ_B satisfies*

$$\lambda_B \geq cq.$$

Theorem 3.3 ([13]) *(Generalized Obata Theorem) The equality holds in Theorem 5.6 if and only if*

1. *$\mathcal{F}$ is transversally isometric to the action of a discrete subgroup of $O(q)$ acting on the q sphere of constant curvature c. Thus, there are at least two closed leaves(the poles).*

2. *If we choose the metric on M so that the mean curvature form is basic, then the mean curvature of the foliation is zero.*

3. *Each level set of the λ_B eigenfunction is the set of leaves corresponding to a latitude of the q sphere, and the volume $V(r)$ of this level set is the volume of the maximum leaf L times the volume of the latitude.*

For the classification of real Clifford algebra $Cl(n)$ of $\mathbb{R}^n$, we have the following proposition.

Proposition 3.4 ([12]) *For $1 \leq n \leq 8$, the Clifford algebra $Cl(n)$ and the dimension d_n of an irreducible $\mathbb{R}$-module for $Cl(n)$ are given by the followings:*

$$Cl(1) \cong \mathbb{C}, \quad Cl(2) \cong \mathbb{H}, \quad Cl(3) \cong \mathbb{H} \oplus \mathbb{H}, \quad Cl(4) \cong \mathbb{H}(2)$$
$$Cl(5) \cong \mathbb{C}(4), \quad Cl(6) \cong \mathbb{R}(8), \quad Cl(7) \cong \mathbb{R}(8) \oplus \mathbb{R}(8), \quad Cl(8) \cong \mathbb{R}(16)$$
$$d_1 = 2, \quad d_2 = 4, \quad d_3 = 4, \quad d_4 = 8, \quad d_5 = 8, \quad d_6 = 8, \quad d_7 = 8, \quad d_8 = 16,$$

where $K(n)$ denote the algebra of $n \times n$-matries with entries in $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

For $n > 8$, i.e., $n = m + 8k$ $(m, k \geq 1)$, $d_{m+8k} = 2^{4k} d_m$.

4 Basic Yamabe operator

Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$. Then the transversal Dirac operator D_{tr} is locally defined ([2,5]) by

$$D_{tr}\Psi = \sum_a E_a \cdot \nabla_{E_a}\Psi - \frac{1}{2}\kappa \cdot \Psi \quad \text{for } \Psi \in \Gamma S(\mathcal{F}), \tag{4.1}$$

where $\{E_a\}$ is a local orthonormal basic frame of Q. We define the subspace $\Gamma_B(S(\mathcal{F}))$ of *basic* or *holonomy invariant* sections of $S(\mathcal{F})$ by

$$\Gamma_B(S(\mathcal{F})) = \{\Psi \in \Gamma S(\mathcal{F}) | \nabla_X \Psi = 0 \quad \text{for } X \in \Gamma L\}.$$

Trivially, we see that D_{tr} leaves $\Gamma_B(S(\mathcal{F}))$ invariant if and only if the foliation $\mathcal{F}$ is isoparametric, i.e., $\kappa \in \Omega_B^1(\mathcal{F})$. Let $D_b = D_{tr}|_{\Gamma_B(S(\mathcal{F}))} : \Gamma_B(S(\mathcal{F})) \to \Gamma_B(S(\mathcal{F}))$. This operator D_b is called the *basic Dirac operator* on (smooth) basic sections. On an isoparametric transverse spin foliation $\mathcal{F}$ with $\delta\kappa = 0$, it is well-known([2,5,8]) that

$$D_{tr}^2\Psi = \nabla_{tr}^*\nabla_{tr}\Psi + \frac{1}{4}K_\sigma^\nabla\Psi, \tag{4.2}$$

where $K_\sigma^\nabla = \sigma^\nabla + |\kappa|^2$ and

$$\nabla_{tr}^*\nabla_{tr}\Psi = -\sum_a \nabla^2_{E_a,E_a}\Psi + \nabla_\kappa\Psi. \tag{4.3}$$

Now, we consider, for any real basic function u on M, the transversally conformal metric $\bar g_Q = e^{2u}g_Q$. Let $\bar P_{so}(\mathcal{F})$ be the principal bundle of $\bar g_Q$-orthogonal frames. Locally, the section $\bar s$ of $\bar P_{so}(\mathcal{F})$ corresponding a section $s = (E_1, \cdots, E_q)$ of $P_{so}(\mathcal{F})$ is $\bar s = (\bar E_1, \cdots, \bar E_q)$, where $\bar E_a = e^{-u}E_a$ $(a = 1, \cdots, q)$. This isometry will be denoted by I_u. Thanks to the isomorphism I_u one can define a transverse spin structure $\bar P_{spin}(\mathcal{F})$ on $\mathcal{F}$ in such a way that the diagram

$$\begin{array}{ccc} P_{spin}(\mathcal{F}) & \xrightarrow{\tilde I_u} & \bar P_{spin}(\mathcal{F}) \\ \downarrow & & \downarrow \\ P_{so}(\mathcal{F}) & \xrightarrow{I_u} & \bar P_{so}(\mathcal{F}) \end{array}$$

commutes.

The transversal Ricci curvature $\rho^{\bar\nabla}$ of $\bar g_Q = e^{2u}g_Q$ and the transversal scalar curvature $\sigma^{\bar\nabla}$ of $\bar g_Q$ are related to the transversal Ricci curvature ρ^∇ of g_Q and the transversal scalar curvature σ^∇ of g_Q by the following lemma.

Lemma 4.1 ([9]) *On a Riemannian foliation $\mathcal{F}$, we have that for any $X \in Q$,*

$$\begin{aligned} e^{2u}\rho^{\bar\nabla}(X) =& \rho^\nabla(X) + (2-q)\nabla_X grad_\nabla(u) + (2-q)|grad_\nabla(u)|^2 X \\ &+ (q-2)X(u)grad_\nabla(u) + \{\Delta_B u - \kappa(u)\}X. \end{aligned} \tag{4.4}$$

$$e^{2u}\sigma^{\bar\nabla} = \sigma^\nabla + (q-1)(2-q)|grad_\nabla(u)|^2 + 2(q-1)\{\Delta_B u - \kappa(u)\}. \tag{4.5}$$

From (4.5), we have

$$e^{2u}K_\sigma^{\bar\nabla} = \sigma^\nabla + |\kappa|^2 + 2(q-1)\Delta_B u + (q-1)(2-q)|grad_\nabla(u)|^2 - 2\kappa(u). \tag{4.6}$$

On the other hand, for $q \geq 3$, if we choose the positive function h by $u = \frac{2}{q-2}\ln h$, then we have

$$\Delta_B u = \frac{2}{q-2}\{h^{-2}|grad_\nabla(h)|^2 + h^{-1}\Delta_B h\}, \tag{4.7}$$

$$|grad_\nabla(u)|^2 = (\frac{2}{q-2})^2 h^{-2}|grad_\nabla(h)|^2. \tag{4.8}$$

Hence we have

$$e^{2u}K_\sigma^{\bar\nabla} = h^{\frac{4}{q-2}}K_\sigma^{\bar\nabla} = h^{-1}Y_b h + |\kappa|^2 - \frac{4}{q-2}h^{-1}\kappa(h), \tag{4.9}$$

where

$$Y_b = 4\frac{q-1}{q-2}\Delta_B + \sigma^\nabla, \tag{4.10}$$

which is called a *basic Yamabe operator* of $\mathcal{F}$.

Now we put $\mathcal{K}_u = \{u \in \Omega_B^0(\mathcal{F}) | \kappa(u) = 0\}$. If we choose $u \in \mathcal{K}_u$, then $\kappa(h) = 0 = \kappa(u)$. From (4.6) and (4.9), we have

$$e^{2u}K_\sigma^{\bar\nabla} = K_\sigma^\nabla + 2(q-1)\Delta_B u - (q-1)(q-2)|grad_\nabla(u)|^2 = h^{-1}Y_b h + |\kappa|^2, \tag{4.11}$$

where $K_\sigma^\nabla = \sigma^\nabla + |\kappa|^2$. Assume that the transversal scalar curvature σ^∇ is non-negative. Then the eigenvalue h_1 associated to the first eigenvalue μ_1 of Y_b can be chosen to be positive and then μ_1 is non-negative. Thus

$$h_1^{-1}Y_b h_1 = \mu_1. \tag{4.12}$$

Since $\sup\inf\{h^{-1}Y_b h\} \geq \mu_1$, we have the following corollary.

Corollary 4.2 *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and bundle-like metric g_M with $\kappa \in \Omega_B^1(\mathcal{F})$ and $\delta\kappa = 0$. If the transversal scalar curvature satisfies $\sigma^\nabla \geq 0$, then we have*

$$\lambda^2 \geq \frac{q}{4(q-1)}(\mu_1 + \inf_M |\kappa|^2). \tag{4.13}$$

5 Characterization of the foliation

In this section, we study the foliated Riemannian manifold M which admits a non-zero foliated spinor Ψ_1 such that $D_b\Psi_1 = \lambda_1\Psi_1$ with $\lambda_1^2 = \frac{1}{4}(q/(q-1))(\mu_1 + \inf|\kappa|^2)$. We define $Ric_\nabla^f : \Gamma Q \otimes S(\mathcal{F}) \to S(\mathcal{F})$ by

$$Ric_\nabla^f(X \otimes \Psi) = \sum_a E_a \cdot R^f(X, E_a)\Psi, \tag{5.1}$$

where R^f is the curvature tensor with respect to $\overset{f}{\nabla}$ defined by $\overset{f}{\nabla}_X \Psi = \nabla_X\Psi + f\pi(X)\cdot\Psi$. By long calculation, for $X \in \Gamma Q$ and $\Psi \in \Gamma S(\mathcal{F})$ we have ([8])

$$Ric_\nabla^f(X\otimes\Psi) = -\frac{1}{2}\rho^\nabla(X)\cdot\Psi + 2(q-1)f^2 X\cdot\Psi - qX(f)\Psi - grad_\nabla(f)\cdot X\cdot\Psi \tag{5.2}$$

for $X \in \Gamma Q$. Similarly, we obtain the formula for $Ric^{f}_{\bar\nabla}(X \otimes \bar\Psi)$ associated to $\bar S(\mathcal{F})$. Namely,

$$Ric^{f}_{\bar\nabla}(X\otimes\bar\Psi) = -\frac{1}{2}\rho^{\bar\nabla}(X)\bar\cdot\bar\Psi + 2(q-1)f^2X\bar\cdot\bar\Psi - qX(f)\bar\Psi - \overline{grad_\nabla(f)}\bar\cdot X\bar\cdot\bar\Psi, \quad (5.3)$$

where $\rho^{\bar\nabla}(X)$ is the transversal Ricci curvature with respect to $\bar\nabla$. From (5.2) and (5.3), we have the following facts.

Proposition 5.1 ([9]) *If M admits a non-zero foliated spinor Ψ with $\overset{f}{\bar\nabla}\bar\Psi = 0$, then f is constant and for any $X \in TM$*

$$\nabla_X\Psi = -fe^u\pi(X)\cdot\Psi + \frac{1}{2}\pi(X)\cdot grad_\nabla(u)\cdot\Psi + \frac{1}{2}g_Q(grad_\nabla(u),\pi(X))\Psi. \quad (5.4)$$

And also we have the following theorem.

Theorem 5.2 *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and bundle-like metric g_M such that $\kappa \in \Omega^1_B(\mathcal{F})$ and $\delta\kappa = 0$. Assume that $\sigma^\nabla \geq 0$. If there exists an eigenspinor field Ψ_1 of the basic Dirac operator D_b for the eigenvalue $\lambda_1^2 = \frac{q}{4(q-1)}(\mu_1 + \inf|\kappa|^2)$, then Ψ_1 is a transversal Killing spinor and $\mathcal{F}$ is minimal, transversally Einsteinian with positive constant transversal scalar curvature σ^∇.*

From the generalized Lichnerowicz and Obata theorem in section 3, we have the following theorem (cf. [7]).

Theorem 5.3 *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q = 3, 4, 7, 8$ and a bundle-like metric g_M with $\kappa \in \Omega^1_B(\mathcal{F})$. Assume that the mean curvature κ of $\mathcal{F}$ satisfies $\delta\kappa = 0$ and $\sigma^\nabla \geq 0$. If there exists an eigenspinor field Ψ_1 for λ_1 with $\lambda_1^2 = \frac{q}{4(q-1)}(\mu_1 + \inf|\kappa|^2)$, then*

(1) *$\mathcal{F}$ is minimal, transversally Einsteinian.*

(2) *$\mathcal{F}$ is transversally isometric to the action of discrete subgroup of $O(q)$ acting on the $q-$sphere, where $q = 3, 4, 7, 8$.*

Proof. (1) is trivial from Theorem 5.2. Next, we prove (2). Since $\mathcal{F}$ is minimal, $\rho^\nabla(X) = \frac{1}{q}\mu_1 X$. Let Ψ and Φ be the foliated spinors with $D_b\Psi = \lambda_1\Psi$ and $D_b\Phi = -\lambda_1\Phi$. Then we have the following equations. For any $X \in \Gamma Q$

$$\nabla_X\Psi = -\frac{\lambda_1}{q}X\cdot\Psi, \quad \nabla_X\Phi = \frac{\lambda_1}{q}X\cdot\Phi. \quad (5.5)$$

If we put $f = (\Psi, \Phi)$, then by direct calculation, we have

$$\Delta_B f = \frac{\mu_1}{q-1} f. \tag{5.6}$$

It is sufficient to prove that f does not vanish identically.

(1) In case $q = 4, 8$, it is well known [12] that the real spinor bundle $S(\mathcal{F})$ spilts as the two irreducible real representations:

$$S(\mathcal{F}) = S^+(\mathcal{F}) \oplus S^-(\mathcal{F}). \tag{5.7}$$

Then $\Psi = \Psi^+ + \Psi^-$ and $\Phi = \Psi^+ - \Psi^-$, where $\Psi^\pm \in S^\pm(\mathcal{F})$. Hence we have that for any $X \in \Gamma Q$,

$$X(f) = \frac{4\lambda_1}{q}(X \cdot \Psi^+, \Psi^-), \tag{5.8}$$

where $(\cdot,\cdot) = Re < \cdot, \cdot >$. Let us define the map $F : Q \to S^-(\mathcal{F})$ by $X \to X \cdot \Psi^+$. Then F is the $\mathbb{R}$-linear and injective. Since $d_4 = 8$ and $d_8 = 16$ from Proposition 3.4, $\dim_{\mathbb{R}} Q = \dim_{\mathbb{R}} S^-(\mathcal{F})$. Hence F is isomorphism and there exists $X \neq 0$ such that

$$(X \cdot \Psi^+, \psi^-) \neq 0, \tag{5.9}$$

which implies that $f \not\equiv 0$.

(2) In case $q = 3, 7$, if we define $F : Q \to S(\mathcal{F})$ by $X \to X \cdot \Psi$, then F is $\mathbb{R}$-linear and injective. Since $d_3 = 4$ and $d_7 = 8$ in Proposition 3.4, $\dim_{\mathbb{R}} Q = \dim_{\mathbb{R}} F(Q) = \dim_{\mathbb{R}} S(\mathcal{F}) - 1$. Since $(\Psi, X \cdot \Psi) = 0$, $F(X) \notin E_{\lambda_1}(D_b)$, where $E_{\lambda_1}(D_b)$ is the eigenspace corresponding to the eigenvalue λ_1. Hence $\dim E_{\lambda_1}(D_b) = 1$ and $F(Q) = E_{\lambda_1}(D_b)^\perp$. So $F : Q \to E_{\lambda_1}(D_b)^\perp$ is an isomorphism. Since $\Phi \in F(Q)$, there exists $X \neq 0$ such that

$$X(f) = \frac{-2\lambda_1}{q}(X \cdot \Psi, \Phi) \neq 0, \tag{5.10}$$

which implies that $f \not\equiv 0$. □

Theorem 5.4 *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q = 5$(resp. $q = 6$) and a bundle-like metric g_M. Assume that the mean curvature κ of $\mathcal{F}$ satisfies $\delta\kappa = 0$ and $\sigma^\nabla \geq 0$. If the dimension of the eigenspinor space of λ_1 with $\lambda_1^2 = \frac{q}{4(q-1)}(\mu_1 + \inf|\kappa|^2)$ is 3 (resp. 2), then*

(1) $\mathcal{F}$ is minimal, transversally Einsteinian.

(2) $\mathcal{F}$ is transversally isometric to the action of discrete subgroup of $O(5)$ (resp. $O(6)$) acting on the $5-$(resp. $6-$) sphere.

Proof. The proof is similar to the one of (2) in Theorem 5.3. □

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References

[1] J. A. Alvarez López, *The basic component of the mean curvature of Riemannian foliations*, Ann. Global Anal. Geom. 10 (1992), 179-194.

[2] J. Brüning and F. W. Kamber, *Vanishing theorems and index formulas for transversal Dirac operators*, A.M.S Meeting 845, Special Session on operator theory and applications to Geometry, Lawrence, KA; A.M.S. Abstracts, October 1988.

[3] D. Domínguez, *A tenseness theorem for Riemannian foliations*, C. R. Acad. Sci. Sér. I 320(1995), 1331-1335.

[4] T. Friedrich, *Der erste Eigenwert des Dirac operators einer kompakten,Riemannschen Mannigfaltigkeit nichtnegative skalarkrümmung,* Math. Nachr. 97 (1980), 117-146.

[5] J. F. Glazebrook and F. W. Kamber, *Transversal Dirac families in Riemannian foliations*, Comm. Math. Phys. 140 (1991), 217-240.

[6] O. Hijazi, *A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors*, Comm. Math. Phys. 104 (1986), 151-162.

[7] O. Hijazi, *Caractérisation de la sphère par les premières valeurs propres de l'opérateur de Dirac en dimensions 3,4,7 et 8.*, C. R. Acad. Sci. Paris 303, Série I (1986), 417-419.

[8] S. D. Jung, *The first eigenvalue of the transversal Dirac operator*, J. Geom. Phys. 39(2001), 253-264.

[9] S. D. Jung, B. H. Kim and J. S. Pak, *Lower bounds for the eigenvalues of the basic Dirac operator on a Riemannian foliation*, to appear in J. Geom. Phys.

[10] F. W. Kamber and Ph. Tondeur, *Harmonic foliations*, Proc. National Science Foundation Conference on Harmonic Maps, Tulane, Dec. 1980, Lecture Notes in Math. 949, Springer-Verlag, New York, 1982, 87-121.

[11] F. W. Kamber and Ph. Tondeur, *Foliated bundles and Characteristic classes*, Lecture Notes in Math. 493, Springer-Verlag, Berlin, 1975.

[12] H. B. Lawson, Jr. and M. L. Michelsohn, *Spin geometry,* Princeton Univ. Press, Princeton, New Jersey, 1989.

[13] J. Lee and K. Richardson, *Lichnerowicz and Obata theorems for foliations*, to appear in Pacific J. of Math.

[14] A. Lichnerowicz, *Spineurs harmoniques*, C. R. Acad. Sci. Paris Ser. A-B, 257 (1963).

[15] P. March, M. Min-Oo and E. A. Ruh, *Mean curvature of Riemannian foliations*, Canad. Math. Bull. 39(1996), 95-105

[16] A. Mason, *An application of stochastic flows to Riemannian foliations*, Houston J. Math. 26(2000), 481-515.

[17] E. Park and K. Richardson, *The basic Laplacian of a Riemannian foliation*, Amer. J. Math. 118(1996), 1249-1275.

[18] Ph. Tondeur, *Foliations on Riemannian manifolds*, Springer-Verlag, New-York, 1988.

[19] S. Yorozu and T. Tanemura *Green's theorem on a foliated Riemannian manifold and its applications*, Acta Math. Hungar. 56 (1990), 239-245.

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