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CLASSIFICATION OF CERTAIN IDEMPOTENT MATRICES OVER BINARY BOOLEAN ALGEBRA

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ABSTRACT. In this article we classify certain matrices over binary Boolean algebra to find out whether they are idempotent or not for $n \times n$ matrices. That is, we determine whether all 3×3 matrices are idempotent or not and extend this result to certain $n \times n$ binary Boolean matrices. We obtain these results by investigating the sums of cells of diagonal cells and off-diagonal cells. Consequently, we classify completely the matrices of the sums of mutually distinct four cells and obtain the cases of being idempotent.

1. Introduction.

If S is a set of one element, then the power set of S consists of two subsets of S. We denote the null set by $\mathbf{0}$ and S by $\mathbf{1}$. Let $\mathbb{B}$ be the power set of S. In $\mathbb{B}$, let us denote the union of the elements by $+$, intersection by juxtaposition, and complementation by $*$. Then $\mathbb{B} = \{0\ ,\ 1\}$ with these operations is an algebra and called the *binary Boolean algebra*([6]).

Let $\mathcal{M}_n(\mathbb{B})$ denote the set of $n \times n$ matrices with entries in $\mathbb{B}$ which are called the set of $n \times n$ *binary Boolean matrices.* If O is the zero matrix([6]), then $O^2 = O$. Also the identity matrix I([6]) satisfies $I^2 = I$. Thus there exist matrices A_α in $\mathcal{M}_n(\mathbb{B})$ such that ${A_\alpha}^2 = A_\alpha$. We call these matrices A_α as *idempotent matrices.* For these idempotent matrices, it is natural to ask the following questions: What are their forms and how many idempotent matrices exist in $\mathcal{M}_n(\mathbb{B})$? In this article, we study on these problems. That is, we investigate whether a given matrix is idempotent or not. First of all, we determine whether all matrices with only one nonzero entry are idempotent or not. And we determine whether all matrices with many nonzero entries are idempotent or not by using the above matrices.

In section 2, we give some definitions and some preliminaries. In section 3, we determine whether all matrices with only four nonzero entries are idempotent or not. In section 4, we research all 3×3 binary Boolean matrices and determine their idempotency. Thus we obtain all 123 idempotent matrices in $\mathcal{M}_3(\mathbb{B})$. We also show that the other binary Boolean 3×3 matrices are not idempotent and that the number of them is 389.

2. Definitions and Preliminaries.

We introduce some definitions and notations which we shall use in this article. Let $\mathbb{B}$ denote the binary Boolean algebra of two elements 0 and 1. Its arithmetic is the same as that of any ring, except that $1 + 1 = 1$([5]). In this article, the entries of all matrices are in the binary Boolean algebra. Addition and multiplication of matrices over $\mathbb{B}$ are defined as if they were over a field([5]). The matrix with all entries equal to 0 is called *zero matrix* and denoted by O . The matrix with all entries equal to 1 is denoted by J([5],[6]).

The zero-one $n \times n$ matrices with only one entry equal to 1 are called *cells*([1]). If the nonzero entry occurs in row i and column j, we denote the cell by E_{ij} and say that the cell is in *row* i and it is in *column* j([1]). A *line*([1]) is a row or column. A set of cells is *collinear*([1]) if they are all in the same line. When $i \neq j$, we say E_{ij} is an *off-diagonal* cell; E_{ii} is a *diagonal* cell([1]). we notice that any $n \times n$ matrix can be represented as the sum of the distinct cells.

The following proposition is an immediate consequence of the rules of matrix multiplication.

Proposition 2.1([1]). *For all indices i, j, u, and v, $E_{ij}E_{uv} = E_{iv}$ or O according as $j = u$ or $j \neq u$.*

Corollary 2.1.1([1]). *For all cells C, $C^2 = C$ or O according as C is a diagonal or off-diagonal cell.*

Proposition 2.2([1]). *Suppose C and D are cells and $CD \neq O$.*

(a) *If C and D are off-diagonal cells, then either*

(i) *CD is an off-diagonal cell distinct from C and D, and $DC = O$*

or

(ii) *$D = C^T$, and CD and DC are distinct diagonal cells.*

(b) *If C is a diagonal cell and D is not, then $CD = D$, and C and D are in the same row.*

If D is a diagonal cell and C is not, then $CD = C$, and C and D are in the same column.

(c) If C and D are diagonal, then $C = D$.

Proof. The proofs are all routine applications of Proposition 2.1. We prove (a). Suppose $C = E_{ij}$ and $D = E_{uv}$ where $i \neq j$ and $u \neq v$. Proposition 2.1 implies that $u = j$ and $CD = E_{iv}$ because $CD \neq O$. Then $v \neq j$ and $u \neq i$ because $u \neq v$ and $i \neq j$. Therefore CD is distinct from C and D. If $v \neq i$, then CD is off-diagonal and $DC = O$ by Proposition 2.1. If $v = i$, then $D = C^T$, $DC = E_{jj}$, and $CD = E_{ii}$ by the same proposition. The proofs of (b) and (c) are similar to (a). □

Let $\mathcal{M}_n(\mathbb{B})$ be the set of all $n \times n$ matrices whose entries are in $\mathbb{B} = \{0, 1\}$. Throughout this article, we assume that all matrices are in $\mathcal{M}_n(\mathbb{B})$.

We call a matrix E is *idempotent* if $E^2 = E$. If not, E is called *nonidempotent.*

Notice that all diagonal cells are idempotent and all off-diagonal cells are nonidempotent in $\mathcal{M}_n(\mathbb{B})$. Furthermore all matrices of sums of mutually distinct diagonal cells are idempotent.

Lemma 2.3. *Suppose E is a diagonal cell and F is an off-diagonal cell. Then their sum is idempotent if and only if they are collinear.*

Proof. $(\Longrightarrow)$: Suppose $(E+F)^2 = E+F$. By Proposition 2.1, $E^2 = E$ and $F^2 = O$. Therefore $E + EF + FE = E + F$ and so $EF + FE = F$. By Proposition 2.2-(b), E and F are collinear.

$(\Longleftarrow)$: Without loss of generality, we assume that $E = E_{ii}$ and $F = E_{ij}$ with $i \neq j$. Then

$$\begin{aligned}(E+F)^2 &= (E_{ii} + E_{ij})^2 = E_{ii}^2 + E_{ii}E_{ij} + E_{ji}E_{ii} + E_{ij}^2 \\ &= E_{ii} + E_{ij} + O + O = E + F\end{aligned}$$

Thus $E + F$ is idempotent. □

Lemma 2.4. *Suppose E and F are distinct off-diagonal cells. Then their sum is not idempotent.*

Proof. If E and F are in the same row, say $E = E_{ij}$ and $F = E_{ik}$ with $i \neq j, k$ and $j \neq k$, then $(E+F)^2 = O$ and so $E+F$ is not idempotent. If E and F are in the same column, say $E = E_{ij}$ and $F = E_{kj}$ with $i \neq j, k$ and $j \neq k$, then $(E+F)^2 = O$ and so $E+F$ is not idempotent. If E and F are not in the same line, say $E = E_{ij}$ and $F = E_{kl}$ with $i \neq j, k$ and $k \neq l$, then $(E+F)^2 = O$ or E_{il} or E_{kj} or $E_{il} + E_{kj}$ according as ($j \neq k$ and $l \neq i$) or ($j = k$ and $l \neq i$) or ($j \neq k$ and $l = i$) or ($j = k$ and $l = i$) and so $E + F$ is not idempotent. □

From the same method in the proof of lemma 2.4 we obtain the general result.

Corollary 2.4.1. *Suppose $E_1, E_2, \cdots, E_k$ are mutually distinct off-diagonal cells. Then their sum is not idempotent.*

Lemma 2.5. *Suppose E, F, and G are mutually distinct cells, E and F are diagonal but G is not. Then their sum is idempotent if and only if G is in the same line to E or F.*

Proof. The necessity is immediate and so we only prove the sufficiency. Suppose $(E+F+G)^2 = E+F+G$. Then by Proposition 2.1, $E^2 = E$, $F^2 = F$, $EF = FE = O$ and $G^2 = O$. So we have

$$E + F + (EG + GE) + (FG + GF) = E + F + G \tag{2.1}$$

Notice that $EG+GE = O$ or G, and $FG+GF = O$ or G. Therefore we obtain the equation (2.1) implies that $(EG + GE) + (FG + GF) = G$. Thus we have $EG + GE = G$ or $FG + GF = G$. By Proposition 2.2-(b), E and G are in the same line or F and G are in the same line. □

We can extend this Lemma 2.5 to the case of many diagonal cells.

Corollary 2.5.1. *Suppose $E_1, E_2, \cdots, E_k$, and F are mutually distinct cells, E_i's are diagonal but F is not. Then their sum is idempotent if and only if F is in the same line to at least one of $E_1, E_2, \cdots, E_k$.*

Lemma 2.6([1]). *Suppose E, F, and G are mutually distinct cells, E is diagonal but F and G are not. Then their sum is idempotent if and only if they are collinear.*

Proof. Suppose $S = E + F + G$ and $S^2 = S$. By corollary 2.1.1, $E^2 = E$ and $F^2 = G^2 = O$. Therefore we have

$$E + F + G = E + (EF + FE) + (EG + GE) + (FG + GF) \tag{2.2}$$

First we will show that $FG + GF = O$. By the equation (2.2), FG cannot be an off-diagonal cell distinct from F and G, nor can FG and GF be a pair of distinct diagonal cells. Therefore by Proposition 2.2-(a), $FG = O$. Similarly $GF = O$. Next we will show that $EF + FE$ is either O or F. If $EF \neq O$ or $FE \neq O$, then $EF = F$ or $FE = F$ by Proposition 2.2-(b). Similarly, $EG + GE$ is either O or G. Therefore we obtain the equation (2.2) implies that $EF + FE = F$ and $EG + GE = G$. By Proposition 2.2-(b), E and F are colliner, and E and G are colliner.

Without loss of generality, we assume that $E = E_{ii}$ and $F = E_{ij}$ for some $j \neq i$. Then for some $k \neq i$, G is of the form E_{ki} or E_{ik}. Now $G \neq E_{ki}$ because $GF = O$. So we have $G = E_{ik}$. Thus E, F, and G are collinear. □

3. All idempotent matrices that are the sums of four cells in $\mathcal{M}_n(\mathbb{B})$

In this section, we classify completely the matrices of the sums of mutually distinct four cells and obtain the cases of being idempotent.

Lemma 3.1. *Suppose E_1, E_2, E_3, and E_4 are mutually distinct diagonal cells in $\mathcal{M}_n(\mathbb{B})$ with $n \geq 4$. Then their sum is idempotent.*

Proof. It is trivial. □

Lemma 3.2. *Suppose E_1, E_2, E_3, and F_1 are mutually distinct cells, E_1, E_2, and E_3 are diagonal but F_1 is not. Then their sum is idempotent if and only if F_1 is in the same line to at least one of E_1, E_2 or E_3.*

Proof. This is a special case of corollary 2.5.1. □

Theorem 3.3. *Suppose E, F, G, and H are mutually distinct cells, E and F are diagonal but G and H are not. Then their sum is idempotent if and only if they satisty one of the following conditions;*

(1) G is in the same line to each E and F and $H = G^T$.

(2) G and H are collinear and they are in the same line to E or F.

(3) G and H are not collinear with $GH = HG = O$ and G is in the same line to either E or F and H is so.

Proof. The necessity is immediate and so we only prove the sufficiency. Suppose $(E + F + G + H)^2 = E + F + G + H$. By Proposition 2.1, $E^2 = E$, $F^2 = F$, $G^2 = H^2 = O$, and $EF = FE = O$. Thus we have

$$\begin{aligned} E + F + (EG + GE) + (EH + HE) + (FG + GF) \\ + (FH + HF) + (GH + HG) = E + F + G + H \end{aligned} \qquad (3.1)$$

Notice that $EG + GE = O$ or G, $EH + HE = O$ or H, $FG + GF = O$ or G, and $FH + HF = O$ or H. First we suppose that $GH + HG \neq O$. Then $GH \neq$ or $HG \neq O$, say $GH \neq O$. By Proposition 2.2-(a), $H = G^T$ and GH and HG are distinct diagonal cells. Therefore we obtain the equation (3.1) implies that $GH + HG = E + F$. Without loss of generality, we assume that $GH = E$ and $HG = F$. Let $E = E_{ii}$ and $F = E_{jj}$ with $i \neq j$. Then G is of the form E_{ik} with $i \neq k$ because $GH = E$. Similarly $H = E_{jt}$ with $j \neq t$. Since $H = G^T$, $E_{jt} = E_{ki}$ and so $j = k$ and $t = i$. That is, we obtain that $E = E_{ii}$, $F = E_{jj}$, $G = E_{ij}$, and $H = E_{ji}$ which satisfy the condition (1). Next we suppose that $GH + HG = O$. Then we have

$$(EG + GE) + (EH + HE) + (FG + GF) + (FH + HF) = G + H$$

Notice that ($EG+GE=G$ or $FG+GF=G$) and ($EH+HE=H$ or $FH+HF=H$). Without loss of generality, we assume that $E=E_{ii}$ and $F=E_{jj}$ with $i\neq j$.

We prove this theorem by three steps.

Step 1. Assume that $EG+GE=G$ and $FG+GF=G$.

By proprosition 2.2-(b), E and G are in the same line, and F and G are in the same line. Thus G is in the same line to each E and F. Therefore the form of G is either E_{ij} or E_{ji}.

Case 1.1) $EH+HE=H$ and $FH+HF=H$.

By Proposition 2.2-(b), E and H are in the same line, and F and H are in the same line. So H is in the same line to each E and F. Thus the form of H is either E_{ji} or E_{ij} according to $G=E_{ij}$ or $G=E_{ji}$. Therefore $GH+HG=E+F(\neq O)$ which is a contradiction.

Case 1.2) $EH+HE=H$ and $FH+HF=O$.

By Proposition 2.2-(b), E and H are in the same line, and F and H are not in the same line. Thus H is of the form E_{ik} or E_{ti} with $i\neq k,t$. If $G=E_{ij}$, then H is of the form E_{ik} with $k\neq j$ (If not, $H=E_{ti}$ and $t\neq j$ and so $HG=E_{tj}(\neq O)$ which is a contradiction.). Therefore E,G, and H are in the same column. If $G=E_{ji}$, then H is of the form E_{ti} with $t\neq j$ (If not, $H=E_{ik}$ and $k\neq j$ and so $GH=E_{tk}(\neq O)$ which is a contradiction.). Therefore E,G, and H are in the same row.

Case 1.3) $EH+HE=O$ and $FH+HF=H$.

By the similar method of case 1.2), F,G, and H are in the same line.

Step 2. Assume that $EG+GE=G$ and $FG+GF=O$.

By Proposition 2.2-(b), E and G are in the same line, and F and G are not in the same line. So the form of G is either E_{ik} or E_{ti} with $i\neq k,t$ and $j\neq k,t$.

Case 2.1) $EH+HE=H$ and $FH+HF=H$.

By the similar method of case 1.2), E,G, and H are in the same line.

Case 2.2) $EH+HE=H$ and $FH+HF=O$.

By Proposition 2.2-(b), E and H are in the same line, and F and H are not in the same line. So H is of the form either E_{ia} or E_{bi} with $i\neq a,b$ and $j\neq a,b$.

(a) $G=E_{ik}$ and $H=E_{ia}$.

We notice that $k\neq a$ because G and H are distinct. Thus E,G, and H are in the same row.

(b) $G=E_{ik}$ and $H=E_{bi}$.

Now $HG=E_{bi}E_{ik}=E_{bk}$. But this cell is distinct from G and H. This contradicts to $GH+HG=O$.

(c) $G=E_{ti}$ and $H=E_{ia}$.

Now $GH = E_{ti}E_{ia} = E_{ta}$. But this cell is distinct from G and H. This contradicts to $GH + HG = O$.

(d) $G = E_{ti}$ and $H = E_{bi}$.

We notice that $k \neq b$ because G and H are distinct. Thus E, G, and H are in the same column.

From (a),(b),(c), and (d) E, G, and H are in the same line.

Case 2.3) $EH + HE = O$ and $FH + HF = H$.

By Proposition 2.2-(b), E and H are not in the same line, and F and H are in the same line. Thus H is of the form either E_{ja} or E_{bj} with $a \neq i, j$ and $b \neq i, j$.

(e) $G = E_{ik}$ and $H = E_{ja}$.

Since $GH = HG = O$, $k \neq j$ and $a \neq i$. Thus G is in the same row only to E and H is in the same row only to F. If $a = k$, then they satisfy the condition (2). If $a \neq k$, they satisfy the condition (3).

(f) $G = E_{ik}$ and $H = E_{bj}$.

Notice that $k \neq b$ (If $k = b$, then $GH = E_{ij} (\neq O)$ which is a contradiction.). Thus G is in the same row only to E and H is in the same column only to F. Therefore they satisfy the condition (3).

(g) $G = E_{ti}$ and $H = E_{ja}$.

Notice that $t \neq a$ (If $t = a$, then $HG = E_{ji} (\neq O)$ which is a contradiction.). Thus G is in the same column only to E and H is in the same row only to F. Therefore they satisfy the condition (3).

(h) $G = E_{ti}$ and $H = E_{bj}$.

Since $GH + HG = O$, $b \neq i$ and $t \neq j$. Thus G is in the same column only to E and H is in the same column only to F. If $b = t$, then they satisfy the condition (2). If $b \neq t$, then they satisfy the condition (3).

Step 3. Assume that $EG + GE = O$ and $FG + GF = G$.

The proof is similar to Step 2. □

Definition. *Let S be a binary Boolean matrix in $\mathcal{M}_n(\mathbb{B})$. Then we call S a rectangle form if S has only four 1's and the four 1's constitute a rectangle with a 1 on diagonal and the other three 1's on off-diagonal.*

Theorem 3.4. *Suppose E, G, H, and K are mutually distinct cells, E is diagonal but G, H, and K are not. Then their sum is idempotent if and only if they satisfy one of the following conditions;*

(1) They are collinear

(2) They have the rectangle form.

Proof. The necessity is immediate and so we only prove the sufficiency. Suppose $(E + G + H + K)^2 = E + G + H + K$. By Proposition 2.1, $E^2 = E$ and

$G^2 = H^2 = K^2 = O$. Thus we have

$$\begin{aligned} E &+ (EG+GE) + (EH+HE) + (EK+KE) \\ &+ (GH+HG) + (GK+KG) + (HK+KH) \\ &= E+G+H+K \end{aligned} \tag{3.2}$$

We notice that $EG+GE = O$ or G, $EH+HE = O$ or H, and $EK+KE = O$ or K. First we show that $GH+HG = O$ or K. Suppose $GH+HG \neq O$. Then $GH \neq O$ or $HG \neq O$, say $GH \neq O$. By Proposition 2.2-(a), GH is an off-diagonal cell distinct from G and H with $HG = O$. Thus we have the equation (3.2) implies that $FG = H$ which is desired result. Similarly, $GK+KG = O$ or H and $HK+KH = O$ or G.

Step 1. Assume that $GH+HG = O$.

Without loss of generality, we assume that $E = E_{ii}$.

Case 1.1) $GK+KG = O$ and $HK+KH = O$.

We notice that the equation (3.2) implies that $EG+GE = G$, $EH+HE = H$, and $EK+KE = K$. By Proposition 2.2-(b), E and G are collinear, E and H are collinear, and E and K are collinear. Thus G is of the form E_{ia} or E_{bi}. Similarly, $H = E_{ic}$ or E_{di} and $K = E_{ie}$ or E_{fi} with $i \neq a, b, c, d, e$, and f.

(a) $G = E_{ia}$, $H = E_{ic}$, and $K = E_{ie}$.

Since G, H, and K are mutually distinct cells, i, a, c, and e are mutually distinct. Therefore E, G, H, and K are in the same row.

(b) $G = E_{ia}$, $H = E_{ic}$, and $K = E_{fi}$.

Now $KG = E_{fa} (\neq O)$ which is impossible.

(c) $G = E_{ia}$, $H = E_{di}$, and $K = E_{ie}$.

Now $HK = E_{de} (\neq O)$ which is impossible.

(d) $G = E_{ia}$, $H = E_{di}$, and $K = E_{fi}$.

Now $KG = E_{fa} (\neq O)$ which is impossible.

If $G = E_{bi}$, then by the above method, E, G, H, and K are in the same column.

Case 1.2) $GK+KG = H$ and $HK+KH = O$.

We notice that the equation (3.2) implies that $EG+GE = G$, $EH+HE = O$ or H, and $EK+KE = K$. By Proposition 2.2-(b), E and G are collinear, and E and K are collinear. Thus G is of the form E_{ia} or E_{bi}, and K is of the form E_{ic} or E_{di} with $i \neq a, b, c$, and d. Since $GK+KG = H$, ($GK = H$ and $KG = O$) or ($GK = O$ and $KG = H$).

(e) $GK = H$ and $KG = O$.

Since $KG = O$, G and K are of the forms ($G = E_{ia}$ and $K = E_{ic}$) or ($G = E_{bi}$ and $K = E_{ic}$) or ($G = E_{bi}$ and $K = E_{di}$). Let $G = E_{ia}$ and $K = E_{ic}$. Since $GK = H$, $a = i$ which is impossible. Let $G = E_{bi}$ and $K = E_{ic}$. Since

$GK = H$ and H is an off-diagonal cell, $H = E_{bc}$ and $b \neq c$. Since $i \neq b, c$, $EH + HE = O$. Thus E, G, H, and K have the rectangle form. Let $G = E_{bi}$ and $K = E_{di}$. Since $GK = H$, $d = i$ which is impossible.

(f) $GK = O$ and $KG = H$.

The proof is similar to the above (e).

Case 1.3) $GK + KG = O$ and $HK + KH = G$.

By the similar method of case 1.2), E, G, H, and K have the rectangle form.

Case 1.4) $GK + KG = H$ and $HK + KH = G$.

We notice that the equation (3.2) implies that $EK + KE = K$. By Proposition 2.2-(b), E and K are collinear. Thus K is of the form E_{ia} or E_{bi} with $i \neq a, b$. Now, we will only consider $K = E_{ia}$, $GK = H$, and $HK = G$. Since $GK = H$ and $HK = G$, $KG = KH = O$. Since $K = E_{ia}$ and $GK = H$, G is of the form E_{ci} and H is of the form E_{ca} with $i \neq c$ and $c \neq a$. Since $HK = G(\neq O)$, $a = c$ which is impossible.

Step 2. Assume that $GH + HG = K$.

The proof is similar to Step 1. □

4. All 3×3 idempotent matrices.

Now in this section we determine that each 3×3 binary Boolean matrix is idempotent or not. To determine them, we investigate the sums of cells according to diagonal cells and off-diagonal cells. Through this section, D and D_i's mean mutually distinct diagonal cells and F and F_j's off-diagonal cells.

0. Matrix of zero cell.

The zero matrix is trivially idempotent.

1. Matrices of one cell.

1) The diagonal cells are idempotent and the number of them is 3.

2) The off-diagonal cells are not idempotent and the number of them is 6.

2. Matrices of two cells.

1) The matrices of the forms $D_1 + D_2$ are idempotent and the number of them is 3.

2) The matrices of the forms $D + F$;

(a) If they are collinear, then the matrices are idempotent and the number of them is 12.

(b) If they are not collinear, then the matrices are not idempotent and the number of them is 6.

3) The matrices of the forms $F_1 + F_2$ are not idempotent and the number of them is 15.

3. Matrices of three cells.

1) The matrix of the form $D_1 + D_2 + D_3$ is idempotent and the number of it is 1.

2) The matrices of the forms $D_1 + D_2 + F$;
We note that F is in the same line to D_1 or D_2. So the matrices are idempotent and the number of them is 18.

3) The matrices of the forms $D + F_1 + F_2$;

(a) If they are collinear, then the matrices are idempotent and the number of them is 6.

(b) If they are not collinear, then the matrices are not idempotent and the number of them is 39.

4) The matrices of the forms $F_1 + F_2 + F_3$ are not idempotent and the number of them is 20.

4. Matrices of four cells.

1) The matrices of the forms $D_1 + D_2 + D_3 + F$;
We note that F is in the same line to at least one of D_1, D_2 or D_3. So the matrices are idempotent and the number is 6.

2) The matrices of the forms $D_1 + D_2 + F_1 + F_2$;

(a) If F_1 is in the same line to each D_1 and D_2 and $F_2 = {F_1}^T$, then the matrices are idempotent and the number of them is 3.

(b) If F_1 and F_2 are collinear and they are in the same line to D_1 or D_2, then the matrices are idempotent and the number of them is 18.

(c) If otherwise, then the matrices are not idempotent and the number of them is 24.

3) The matrices of the forms $D + F_1 + F_2 + F_3$;

(a) If they have the rectangle forms, then the matrices are idempotent and the number of them is 6.

(b) If otherwise, then the matrices are not idempotent and the number of them is 54.

4) The matrices of the forms $F_1 + F_2 + F_3 + F_4$ are not idempotent and the number of them is 15.

5. Matrices of five cells.

1) The matrices of the forms $D_1 + D_2 + D_3 + F_1 + F_2$;

(a) If F_1 and F_2 are collinear, then the matrices are idempotent and the number of them is 6.

(b) If $F_2 = {F_1}^T$, then the matrices are idempotent and the number of them is 3.

(c) If otherwise, the matrices are not idempotent and the number of them is 6.

2) The matrices of the forms $D_1 + D_2 + F_1 + F_2 + F_3$;

For $D_1 = E_{11}$ and $D_2 = E_{22}$ the forms of idempotent matrices are the following 3 matrices and their transposes only;

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Similarly, we have have the same results for the other cases. That is, we have 18 idempotent matrices and 42 nonidempotent matrices of these forms.

3) The matrices of the forms $D_1 + F_1 + F_2 + F_3 + F_4$ are not idempotent and the number of them is 45.

4) The matrices of the forms $F_1 + F_2 + F_3 + F_4 + F_5$ are not idempotent and the number of them is 6.

6. Matrices of six cells.

1) The matrices of the forms $D_1 + D_2 + D_3 + F_1 + F_2 + F_3$;

The forms of idempotent matrices are the following 3 matrices and their transposes only;

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

That is, we have 6 idempotent and 14 nonidempotent matrices of these forms.

2) The matrices of the forms $D_1 + D_2 + F_1 + F_2 + F_3 + F_4$;

The forms of idempotent matrices are the following 3 matrices and their transposes only;

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

That is, we have 6 idempotent and 39 nonidempotent matrices of these forms.

3) The matrices of the forms $D+F_1+F_2+F_3+F_4+F_5$ are not idempotent and the number of them is 18.

4) The matrix of the form $F_1 + F_2 + F_3 + F_4 + F_5 + F_6$ is not idempotent and the number of it is 1.

7. Matrices of seven cells.

1) The matrices of the forms $D_1 + D_2 + D_3 + F_1 + F_2 + F_3 + F_4$;
The forms of idempotent matrices are the following 3 matrices and their transposes only;

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

That is, we have 6 idempotent and 9 nonidempotent matrices of these forms.

2) The matrices of the forms $D_1 + D_2 + F_1 + F_2 + F_3 + F_4 + F_5$ are not idempotent and the number of them is 18.

3) The matrices of the forms $D_1 + F_1 + F_2 + F_3 + F_4 + F_5 + F_6$ are not idempotent and the number of them is 3.

8. Matrices of eight cells.

1) The matrices of the forms $D_1 + D_2 + D_3 + F_1 + F_2 + F_3 + F_4 + F_5$ are not idempotent and the number of them is 6.

2) The matrices of the forms $D_1 + D_2 + F_1 + F_2 + F_3 + F_4 + F_5 + F_6$ are not idempotent and the number of them is 3.

9. Matrix of nine cells.

The J is trivially idempotent.

Consequently, there exist 123 idempotent and 389 nonidempotent matrices in $\mathcal{M}_3(\mathbb{B})$.

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