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기초과학연구 제주대학교
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FIBREWISE REGULAR CONVERGENCE SPACES

JIN WON PARK AND RYANG GYU HARM

ABSTRACT. In this paper, we define a notion of fibrewise regular convergence spaces and investigate some properties of these spaces. And we find a condition for which the function space $C_B(X, Y)$ is fibrewise regular.

0. Introduction

The fibrewise viewpoint is standard in the theory of fibre bundles. However, it has been recognized only recently that the same viewpoint is also great value in other theories, such as general topology. I. M. James has been promoting the fibrewise viewpoint systematically in topology [2,3]. Many of the familiar definitions and theorems of ordinary topology can be generalized,in a natural way, so that one can develop a theory of topology over a base. On the other hand, as a convenient category, the category of convergence spaces was introduced which contains the category of topological spaces as a bireflective subcategory. So many familiar definitions of topological spaces were introduced in the convergence spaces. In this point of view, K. C. Min and S. J. Lee developed a general fibrewise theory in the category of convergence spaces, including the fibrewise notion of Hausdorffness [5,6].

In this paper, we define the notion of fibrewise regular convergence space as a generalization of the regular convergence spaces and investigate some properties of the fibreise regular convergence space.

1. Preliminaries

In this section, we collect some basic definitions and known results on convergence spaces over a base [1,5,6].

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For any set X, we denote by $F(X)$ the set of all filters on X, and by $P(F(X))$ the power set of $F(X)$.

DEFINITION 1.1. Let X be a set. A map $c : X \to P(F(X))$ is said to be a *convergence structure* if the following properties hold for any point $x \in X$:
(1) $\dot{x} \in c(x)$;
(2) if $\mathcal{F} \in c(x)$ and $\mathcal{F} \subseteq \mathcal{G}$, then $\mathcal{G} \in c(x)$;
(3) if $\mathcal{F}, \mathcal{G} \in c(x)$, then $\mathcal{F} \cap \mathcal{G} \in c(x)$.
Here $\dot{x}$ stands for the ultrafilter on X generated by $\{x\}$ and the pair (X, c) is called a *convergence space.*

If $f : X \to Y$ is a map and $\mathcal{F} \in F(X)$ then $f(\mathcal{F})$ is a filter base. In general, $f(\mathcal{F})$ is not a filter but the filter generated by $f(\mathcal{F})$ is also denoted by $f(\mathcal{F})$.

DEFINITION 1.2. Let (X, c) and (Y, c') be convergence spaces and $f : X \to Y$ a map. Then f is said to be *continuous at* $x \in X$ if for any $\mathcal{F} \in c(x), f(\mathcal{F}) \in c'(f(x))$. And f is said to be *continuous* if f is continuous at each point $x \in X$.

The class of all convergence spaces and continuous maps forms a category, which will be denoted by **Conv**.

For a given space B in **Conv**, the category $\mathbf{Conv}_B$ is defined as follows. An *object over* B is a pair (X, p) consisting of an object X of **Conv** and a morphism $p : X \to B$ of **Conv**. If (X, p) and (Y, q) are convergence spaces over B then a morphism $f : X \to Y$ of **Conv** is a *morphism over* B if $q \circ f = p$. In this case X is called a *convergence space over* B, p is called the *projection* and f is called a *continuous map over* B.
Composition in $\mathbf{Conv}_B$ is defined by the composition in **Conv**.

PROPOSITION 1.3. *The category* $\mathbf{Conv}_B$ *has an initial structure over* $\mathbf{Set}_B$.

In fact, for $X \in \mathbf{Set}_B$, a family $\{(X_i, p_i)\}_{i \in I}$ of convergence spaces over B and a source $\{f_i : X \to X_i\}_{i \in I}$, the initial structure on X is defined as follows. A filter $\mathcal{F}$ converges to x in X if and only if for each $i \in I$, $f_i(\mathcal{F})$ converges to $f_i(x)$ in X_i.

COROLLARY 1.4. *The category* $\mathbf{Conv}_B$ *has a final structure over* $\mathbf{Set}_B$.

In fact, for $X \in \mathbf{Set}_B$, a family $\{(X_i, p_i)\}_{i \in I}$ of convergence spaces over B and a sink $\{f_i : X_i \to X\}_{i \in I}$, the final structure on X is defined as follows. A filter $\mathcal{F}$ on X converges to x in X if and only if either $\mathcal{F} = \dot{x}$ or there are $i_1, \cdots, i_n \in I$ and a filter $\mathcal{G}_{i_k}$ on X_{i_k} which converges to y such that $f_{i_k}(y) = x$ and $\bigcap_{k=1}^{n} f_{i_k}(\mathcal{G}_{i_k}) \subseteq \mathcal{F}$.

PROPOSITION 1.5. *For a convergence spaces* $(X, p), (Y, q)$ *over* B, *let* $X \times_B Y$ *be the fibre product in* $\mathbf{Set}_B$ *endowed with the initial convergence structure* *with respect to the* $\{\pi_1 : X\times_B Y \to X, \pi_2 : X\times_B Y \to Y\}$. *Then* $(X\times_B Y, p\circ\pi_1)$ *is the product of* X *and* Y *in* $\mathbf{Conv}_B$.

Similarly for a family $\{X_i\}_{i\in I}$ of convergence spaces over B, we obtain the product $\prod_B X_i$ in the category $\mathbf{Conv}_B$.

Now we will extend a property of convergence spaces to convergence spaces over B, in a natural way. Specifically we aim to define, for a convergence space B, a property P_B of convergence spaces over B such that the following three conditions are satisfied :

(Condition 1) If X, Y are homeomorphic convergence spaces over B and if X has property P_B then so does Y.
(Condition 2) A convergence space X has property P if and only if the convergence space X over the point $*$ has property P_*.
(Condition 3) If a convergence space X over B has property P_B then the convergence space ξ^*X over B' has property $P_{B'}$ for each convergence space B' and continuous map $\xi : B' \to B$, where ξ^*X is the convergence space $(B'\times_B X, \pi_1)$ over B'.

In this case, the property P_B of convergence spaces over B is said to be *well-behaved* [2].

2. Fibrewise regular convergence spaces

In this section, we introduce a notion of fibrewise regular convergence spaces and investigate some properties of these spaces.

DEFINITION 2.1. Let X be a convergence space over B. X is said to be *fibrewise regular* (or *regular over* B) if for any filter $\mathcal{F}$ on X converging to $x \in X_b$ with $X_b \cap \overline{F} \neq \emptyset$ for all $F \in \mathcal{F}$, $\{X_b \cap \overline{F} | \ F \in \mathcal{F}\}$ converges to x.

THEOREM 2.2. *The property "fibrewise regular" is well-behaved.*

Proof. (Condition 1) Let X and Y be convergence spaces over B and $f : X \to Y$ be a homeomorphism. Let Y be fibrewise regualr. Suppose $\mathcal{F}$ converges to $x \in X_b$ such that $X_b \cap \overline{F} \neq \emptyset$ for all $F \in \mathcal{F}$. Then $f(\mathcal{F})$ converges to $f(x) \in Y_b$. Since Y is fibrewise regular and

$$\emptyset \neq f(X_b \cap \overline{F}) = f(X_b) \cap f(\overline{F}) = Y_b \cap \overline{f(F)},$$

$\{Y_b \cap \overline{f(F)} | \ F \in \mathcal{F}\}$ converges to $f(x)$. Moreover, it is easy to show that

$$f^{-1}(\{Y_b \cap \overline{f(F)} | \ F \in \mathcal{F}\}) \subseteq \{X_b \cap \overline{F} | \ F \in \mathcal{F}\}.$$

Thus $\{X_b \cap \overline{F} \mid F \in \mathcal{F}\}$ converges to x. Therefore X is fibrewise regular.

(Condition 2) It is obvious.

(Condition 3) Let X be fibrewise regular and $\xi : B' \to B$ be a continuous map. Let $\mathcal{F}$ converge to $(b', x) \in (\xi^* X)_{b'} = \{b'\} \times X_{\xi(b')}$ such that $(\xi^* X)_{b'} \cap \overline{F} \neq \emptyset$ for all $F \in \mathcal{F}$. Since π_2 is continuous, $\pi_2(\mathcal{F})$ converges to x. And since X is fibrewise regular and

$$\emptyset \neq \pi_2((\xi^* X)_{b'} \cap \overline{F}) \subseteq \pi_2((\xi^* X)_{b'}) \cap \pi_2(\overline{F}) \subseteq X_{\xi(b')} \cap \overline{\pi_2(F)},$$

$\{X_{\xi(b')} \cap \overline{\pi_2(F)} \mid F \in \mathcal{F}\}$ converges to x. It is easy to show that $\{X_{\xi(b')} \cap \overline{\pi_2(F)} \mid F \in \mathcal{F}\} \subseteq \pi_2(\{(\xi^* X)_{b'} \cap \overline{F} \mid F \in \mathcal{F}\})$. So $\pi_2(\{(\xi^* X)_{b'} \cap \overline{F} \mid F \in \mathcal{F}\})$ converges to x. Since $\pi_1((\xi^* X)_{b'} \cap \overline{F}) = \{b'\}$, $\pi_1(\{(\xi^* X)_{b'} \cap \overline{F} \mid F \in \mathcal{F}\})$ is the filter generated by $\{b'\}$ and thus $\pi_1(\{(\xi^* X)_{b'} \cap \overline{F} \mid F \in \mathcal{F}\})$ converges to b'. Hence $\{(\xi_* X)_{b'} \cap \overline{F} \mid F \in \mathcal{F}\}$ converges to (b', x). Therefore $\xi^* X$ is fibrewise regular.

In all, the property "fibrewise regular" is well-behaved.

REMARK. A convergence space X is said to be regular if for any filter $\mathcal{F}$ which converges to $x \in X$, the filter $\overline{\mathcal{F}} = \{\overline{F} \mid F \in \mathcal{F}\}$ converges to x [1]. So, by the condition 2 in the above theorem, the definition 2.1 can be regarded as a generalization of the regular convergence spaces.

PROPOSITION 2.3. *Let X be a fibrewise T_1 convergence space. Then if X is fibrewise regular, X is fibrewise Hausdorff.*

Proof. Let $\mathcal{F}$ converge to x and y with $x, y \in X_b$. Then for each $F \in \mathcal{F}$, $x \in X_b \cap \overline{F}$ and $y \in X_b \cap \overline{F}$. Hence the filter $\{X_b \cap \overline{F} \mid F \in \mathcal{F}\}$ is contained in $\dot{x}$ and $\dot{y}$, simultaneously. Since X is fibrewise regular, $\{X_b \cap \overline{F} \mid F \in \mathcal{F}\}$ converges to x and y. Thus $\dot{x}$ converges to x and y and $\dot{y}$ converges to x and y, simultaneously. Since X is fibrewise T_1, $x = y$. Hence X is fibrewise Hausdorff.

COROLLARY 2.4. *Let X be a regular T_1 convergence space. Then X is Hausdorff.*

PROPOSITION 2.5. *Let X and Y be convergence spaces over B and $f : X \to Y$ be an embedding. Then if Y is fibrewise regular, so is X.*

Proof. Let $\mathcal{F}$ converge to x in X and $X_b \cap \overline{F} \neq \emptyset$ for all $F \in \mathcal{F}$. Then $f(\mathcal{F})$ converges to $f(x)$ in Y, since f is continuous. Since Y is fibrewise regular and $\emptyset \neq f(\overline{F} \cap X_b) \subseteq \overline{f(F)} \cap Y_b$, $\{\overline{f(F)} \cap Y_b \mid F \in \mathcal{F}\}$ converges to $f(x)$ in Y. Thus $\{f^{-1}(\overline{f(F)} \cap Y_b) \mid F \in \mathcal{F}\}$ converges to x in X, since f is an embedding. Also, note that

$$\overline{F} \cap X_b \subseteq f^{-1}(f(\overline{F})) \cap f^{-1}(Y_b) \subseteq f^{-1}(\overline{f(F)} \cap Y_b).$$

Thus

$$\{f^{-1}(\overline{f(F)} \cap X_b) |\ F \in \mathcal{F}\} \subseteq \{\overline{F} \cap X_b |\ F \in \mathcal{F}\}.$$

Hence $\{\overline{F} \cap X_b |\ F \in \mathcal{F}\}$ converges to x in X, and so X is fibrewise regular.

COROLLARY 2.6. *Let $f : X \to Y$ be an embedding and Y be a regular convergence space. Then X is regular.*

COROLLARY 2.7. *Let X be a fibrewise regular convergence space. Then a subspace of X is also fibrewise regular.*

PROPOSITION 2.8. *Let $\{X_i |\ i \in I\}$ be a family of fibrewise regular convergence spaces. Then $\prod_B X_i$ is fibrewise regular.*

Proof. Let $\mathcal{F}$ converge to (x_i) in $\prod_B X_i$ and $(\prod_B X_i) \cap \overline{F} \neq \emptyset$ for all $F \in \mathcal{F}$. Then for each $i \in I$,

$$\emptyset \neq \pi_i(\textstyle\prod_B X_i \cap \overline{F}) \subseteq \pi_i(\prod_B X_i) \cap \pi_i(\overline{F}) \subseteq (X_i)_b \cap \overline{\pi_i(F)}.$$

For each $i \in I$, since $\pi_i(\mathcal{F})$ converges to x_i and since X_i is fibrewise regular, $\{(X_i)_b \cap \overline{\pi_i(F)} |\ F \in \mathcal{F}\}$ converges to x_i. Since $F \subseteq \prod \pi_i(F)$, $\overline{F} \subseteq \overline{\prod \pi_i(F)} = \prod \overline{\pi_i(F)}$ and thus $(\prod_B X_i)_b \cap \overline{F} \subseteq (\prod_B X_i)_b \cap \prod \overline{\pi_i(F)} = \prod((X_i)_b \cap \overline{\pi_i(F)})$. Therefore

$$\{\textstyle\prod((X_i)_b \cap \overline{\pi_i(F)}) |\ F \in \mathcal{F}\} \subseteq \{(\prod_B X_i)_b \cap \overline{F} |\ F \in \mathcal{F}\}.$$

Thus $\{(\prod_B X_i)_b \cap \overline{F} |\ F \in \mathcal{F}\}$ converges to (x_i). Hence $\prod_B X_i$ is fibrewise regular.

COROLLARY 2.9. *Let $\{X_i | i \in I\}$ be a class of convergence spaces. Then the product space $\prod_{i \in I} X_i$ is regular if and only if X_i is regular for all $i \in I$.*

Proof. The if part is immediate from the above proposition. For the only if part, we note that X_i is homeomorphic to a subspace of $\prod_{i \in I} X_i$ for all $i \in I$. Since any subspace of a regular convergence space is regular, X_i is regular for all $i \in I$.

PROPOSITION 2.10. *Let X and Y be convergence spaces over B and $f : X \to Y$ be an initial. Suppose f is surjective. Then if X is fibrewise regular, Y is fibrewise regular.*

Proof. Let $\mathcal{G}$ converge to y in Y and $\overline{G} \cap Y_b \neq \emptyset$ for all $G \in \mathcal{G}$. Note that $f^{-1}(\mathcal{G})$ converges to x for some $x \in f^{-1}(y)$ in X, since f is initial and surjective. We want to show that $f^{-1}(\overline{G}) \subset \overline{f^{-1}(G)}$. Let $p \in f^{-1}(\overline{G})$, then $f(p) \in \overline{G}$. Thus there exists an ultrafilter $\mathcal{U}$ containing G converging to $f(p)$.

Let $\mathcal{V}$ be an ultrafilter containing $f^{-1}(\mathcal{U})$. Then $\mathcal{V}$ converges to p in X and contains $f^{-1}(G)$. Thus $p \in \overline{f^{-1}(G)}$, and hence

$$\emptyset \neq f^{-1}(\overline{G} \cap Y_b) = f^{-1}(\overline{G}) \cap f^{-1}(Y_b) \subseteq \overline{f^{-1}(G)} \cap X_b.$$

Since X is fibrewise regular, $\{\overline{f^{-1}(G)} \cap X_b | \ G \in \mathcal{G}\}$ converges to x in X. Moreover, since f is continuous, $\{f(\overline{f^{-1}(G)} \cap X_b) | \ G \in \mathcal{G}\}$ converges to y in Y. So we have

$$\overline{G} \cap Y_b = f(f^{-1}(\overline{G} \cap Y_b)) = f(f^{-1}(\overline{G}) \cap f^{-1}(Y_b)) \subseteq f(\overline{f^{-1}(G)} \cap X_b),$$

and hence

$$\{f(\overline{f^{-1}(G)} \cap X_b) | \ G \in \mathcal{G}\} \subseteq \{\overline{G} \cap Y_b | \ G \in \mathcal{G}\}.$$

Therefore $\{\overline{G} \cap Y_b | \ G \in \mathcal{G}\}$ converges to $y \in Y$. In all, Y is fibrewise regular.

COROLLARY 2.11. *Let $f : X \to Y$ be an initial. Suppose f is surjective. Then if X is regular, then Y is regular.*

PROPOSITION 2.12. *Let X and Y be convergence spaces over B and $f : X \to Y$ be a final injection. Then if Y is fibrewise regular, X is fibrewise regular.*

Proof. Let $\mathcal{F}$ converge to x in X and $F \cap X_b \neq \emptyset$ for all $F \in \mathcal{F}$. Then $f(\mathcal{F})$ converges to $f(x)$ in Y. Note that $\overline{f(F)} \cap Y_b \neq \emptyset$ for all $F \in \mathcal{F}$. Since Y is fibrewise regular, $\{\overline{f(F)} \cap Y_b | \ F \in \mathcal{F}\}$ converges to $f(x)$ in Y. Since f is a final injection, there exists a filter $\mathcal{G}$ on X converging to x such that $f(\mathcal{G}) \subset \{\overline{f(F)} \cap Y_b | \ F \in \mathcal{F}\}$. So, for each $G \in \mathcal{G}$ there exists an $F \in \mathcal{F}$ such that $\overline{f(F)} \cap Y_b \subseteq f(G)$. Thus

$$\overline{F} \cap X_b = f^{-1}(f(\overline{F} \cap X_b)) \subseteq f^{-1}(\overline{f(F)} \cap Y_b) \subseteq f^{-1}(f(G)) = G.$$

Hence $\mathcal{G} \subseteq \{\overline{F} \cap X_b | \ F \in \mathcal{F}\}$. Since $\mathcal{G}$ converges to x, $\{\overline{F} \cap X_b | \ F \in \mathcal{F}\}$ converges to x in X. Therefore X is fibrewise regular.

COROLLARY 2.13. *Let $f : X \to Y$ be a final injection. Then if Y is regular, X is regular.*

3. Fibrewise continuous convergence structure

In this section, we introduce the fibrewise continuous convergence structure on $C_B(X,Y)$ and obtain a condition for the fibrewise regularity of the function space $C_B(X,Y)$.

Let X and Y be convergence spaces over B and $C_B(X,Y) = \cup_{b\in B}C(X_b, Y_b)$ as a set, where $C(X_b, Y_b)$ is the set of all continuous functions from X_b to Y_b. Define a filter $\mathcal{F}$ converges to f in $C_B(X,Y)$, where $f \in C(X_b, Y_b)$ if and only if

(1) for any filter $\mathcal{A}$ in X which converges to $x \in X_b$, $(\mathcal{F}\cap\dot{f})(\mathcal{A}\cap\dot{x})$ converges to $f(x)$ in Y and

(2) $p(\mathcal{F})$ converges to $p(f)$ in B, where $p : C_B(X,Y) \to B$ is defined by $p(g) = b$ if $g \in C(X_b, Y_b)$.

Then it is well known that $C_B(X,Y)$ with this structure is a convergence space and this structure is called *the fibrewise continuous convergence structure* on $C_B(X,Y)$ [5].

PROPOSITION 3.1. *Let X and Y be convergence spaces over B. If Y is fibrewise regular, then $C_B(X,Y)$ is fibrewise regular.*

Proof. Suppose $\mathcal{F}$ converges to $f \in C(X_b, Y_b)$ and $C(X_b, Y_b) \cap \overline{F} \neq \emptyset$ for all $F \in \mathcal{F}$. Then we have to show that $\mathcal{G} = \{C(X_b, Y_b) \cap \overline{F} |\ F \in \mathcal{F}\}$ converges to f in $C_B(X,Y)$. Let $\mathcal{A}$ converge to $x \in X_b$, then it is enough to show that $(\mathcal{G} \cap \dot{f})(\mathcal{A} \cap \dot{x})$ converges to $f(x)$ in Y. Since $\mathcal{F}$ converges to f in $C_B(X,Y)$, $(\mathcal{F}\cap\dot{f})(\mathcal{A}\cap\dot{x})$ converges to $f(x) \in Y_b$. Hence $f(x) \in \overline{(F \cup \{f\})(A \cup \{x\})}$ for all $F \in \mathcal{F}$ and $A \in \mathcal{A}$. So $Y_b \cap \overline{(F \cup \{f\})(A \cup \{x\})} \neq \emptyset$ for all $F \in \mathcal{F}$ and $A \in \mathcal{A}$. Therefore $\{Y_b \cap \overline{(F \cup \{f\})(A \cup \{x\})}|\ F \in \mathcal{F}, A \in \mathcal{A}\}$ converges to $f(x)$, since Y is fibrewise regular. We note that $(\overline{F}\cup\{f\}) \subseteq \overline{F \cup \{f\}}$ and $(A\cup\{x\}) \subseteq \overline{A \cup \{x\}}$. Thus $\{Y_b \cap \overline{(F \cup \{f\})(A \cup \{x\})}|\ F \in \mathcal{F}, A \in \mathcal{A}\}$ is contained in $\{Y_b \cap \overline{(\overline{F} \cup \{f\})(A\cup\{x\})}|\ F \in \mathcal{F}, A \in \mathcal{A}\}$, and hence $\{Y_b\cap\overline{(\overline{F}\cup\{f\})(A\cup\{x\})}|\ F \in \mathcal{F}, A \in \mathcal{A}\}$ converges to $f(x)$. We want to show that $\{Y_b \cap \overline{(\overline{F} \cup \{f\})(A \cup \{x\})}|\ F \in \mathcal{F}, A \in \mathcal{A}\}$ is contained in $\{(C(X_b, Y_b)\cap\overline{F})\cup\{f\}|\ F \in \mathcal{F}\}(\mathcal{A}\cap\dot{x})$. It is equivalent to show that $((C(X_b, Y_b) \cap \overline{F}) \cup \{f\})(A \cup \{x\}) \subseteq Y_b \cap \overline{(\overline{F} \cup \{f\})(A \cup \{x\})}$ for all $F \in \mathcal{F}$ and $A \in \mathcal{A}$. In fact, if $g \in C(X_b, Y_b) \cap \overline{F}$, then $g(A \cup \{x\}) \subseteq Y_b$ and $g(A\cup\{x\}) \in \overline{(\overline{F}\cup\{f\})(A\cup\{x\})}$. So $\{Y_b\cap\overline{(\overline{F}\cup\{f\})(A\cup\{x\})}|\ F \in \mathcal{F}, A \in \mathcal{A}\} \subseteq \{(C(X_b, Y_b)\cap\overline{F})\cup\{f\}|\ F \in \mathcal{F}\}(\mathcal{A}\cap\dot{x})$, and hence $\{(C(X_b, Y_b)\cap\overline{F})\cup\{f\}|\ F \in \mathcal{F}\}(\mathcal{A} \cap \dot{x})$ converges to $f(x)$. But,

$$\{(C(X_b, Y_b) \cap \overline{F}) \cup \{f\}|\ F \in \mathcal{F}\} = \{C(X_b, Y_b) \cap \overline{F}|\ F \in \mathcal{F}\} \cap \dot{f}.$$

In all, $C_B(X,Y)$ is fibrewise regular.

It is also well known that the category $\mathbf{Conv}_B$ is catesian closed [5]. So, for any convergence space Z over B and a function $f : Z \to C_B(X,Y)$, f is continuous if and only if $ev \circ (1_X \times_B f) : X \times_B Z \to Y$ is continuous, where $ev : X \times_B C_B(X,Y) \to Y$ is an evaluation map which is defined by $ev(x, f) = f(x)$.

PROPOSITION 3.2. *Let X and Y be convergence spaces over B and suppose the projection $p : X \to B$ is surjective. Let $K = \{f \in C_B(X,Y) | \ f :$ constant map$\}$. Then K is homeomorphic to Y.*

Proof. Define $\phi : Y \to C_B(X,Y)$ by, for $y \in Y_b$, $\phi(y) = c_y$, where c_y is the constant map from X_b to Y_b with value $y \in Y_b$. Clearly, ϕ is well-defined and injective. Note that $\phi(Y) = K$. Let $\psi : Y \to K$ be the corestriction of ϕ. Consider the following diagram

$$\begin{array}{ccc} X \times_B C_B(X,Y) & \xrightarrow{\ ev\ } & Y \\ {\scriptstyle id_X \times_B \phi}\uparrow & \nearrow{\scriptstyle \pi_2} & \\ X \times_B Y & & \end{array}$$

Note that $\pi_2 = ev \circ (id_X \times_B \phi)$, since $\pi_2(x,y) = y = c_y(x) = ev(x, c_y)$. Since π_2 is continuous, $ev \circ (id_X \times_B \phi)$ is continuous. Hence ϕ is continuous, by the cartesian closedness of the category of convergence spaces over B. Therefore, $\psi : Y \to K$ is continuous. Now, pick an $x_b \in X_b$ for all $b \in B$ and let $A = \{x_b | \ b \in B\}$. Then we know that

$$\psi^{-1} : K \xrightarrow{\ \ j\ \ } A \times_B K \xrightarrow{\ ev|_{A \times_B K}\ } Y$$

where $j(f) = (x_b, f)$ for $f \in C(X_b, Y_b)$. Since j and ev are continuous, ψ^{-1} is continuous. In all, K is isomorphic to Y.

By the above proposition, we have the following proposition which is the partial converse of the proposition 3.1.

PROPOSITION 3.3. *Let X and Y be convergence spaces over B and suppose the projection $p : X \to B$ is surjective. Then if $C_B(X,Y)$ is fibrewise regular, Y is fibrewise regular.*

Proof. By the above proposition and corollary 2.7, the proof follows immediately.

It is easily proved that the subspace structure on $C(X_b, Y_b)$ with respect to $C_B(X,Y)$ is the same as the continuous convergence structure on $C(X_b, Y_b)$. Hence we have the following corollary.

COROLLARY 3.4. *Let X and Y be convergence spaces. Then Y is regular if and only if $C(X,Y)$ is regular.*

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Department of Mathematics Education,
Cheju National University,
Cheju, 690-756,
Korea