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EXISTENCE OF THE TRANSVERSE SPIN STRUCTURE ON FOLIATED RIEMANNIAN MANIFOLD

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ABSTRACT. We prove that a transversally oriented Riemannian foliation M admits a transverse spin structure if and only if the second Stiefel-Whitney class on the normal bundle is trivial

1. Introduction

A spin structure on any vector bundle E over a Riemannian manifold M is one of the important structures. Its existence is expressed by characteristic class $\omega_2(E) \in H^2(M; Z_2)$, called the second Stiefel-Whitney class of E; unlike the Euler, Chern, and Pontrjagin characteristic class, the Stiefel-Whitney classes of E are not de-Rham cohomology classes of M, and hence are not represented in terms of the curvature operator of a connection in E. As an example of the sort of information carried by the Stiefel-Whitney classes, a vector bundle E is orientable if and only if $\omega_1(E) \in H^1(M; Z_2)$ is trivial ([Mi]) and an oriented Riemannian vector bundle E over M admits a spin structure if and only if $\omega_2(E) \in H^2(M; Z_2)$ is trivial ([BH]).

In this paper, we prove that a transversally oriented Riemannian foliation M admits a transverse spin structure if and only if $\omega_2(Q)$ is trivial. The proof is similar to the case of Riemannian manifold.

2. Preliminaries

Let $(M, g_M, \mathcal{F})$ be an $n(= p+q)$ dimensional Riemannian manifold with the bundle-like metric g_M and foliation $\mathcal{F}$ with codimension q. Let $Q =$

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TM/L be the normal bundle of $\mathcal{F}$, where L is an integrable subbundle of TM which defining the foliation $\mathcal{F}$. Let $Cl(q)$ be the Clifford algebra of R^q with the standard inner product. Put

$$Pin(q) = \{\tilde{g} \in Cl(q) \mid \tilde{g} = u_1 \cdots u_r, \quad |u_j| = 1\},$$
$$Spin(q) = \{\tilde{g} \in Pin(q) \mid \tilde{g} = u_1 \cdots u_{2r}\},$$

where "$\cdot$" denotes the Clifford multiplication. These are called the *Pin group* and *Spin group* respectively. It is well known that $Pin(q)$ and $Spin(q)$ are Lie groups([LM]). We define a homomorphism $\tau : Pin(q) \longrightarrow GL(R^q)$ by, for $\tilde{g} \in Pin(q)$ and $u \in R^q$,

$$\tau(\tilde{g})u = \tilde{g} \cdot u \cdot \tilde{g}^t,$$

where $\tilde{g}^t = u_j \cdots u_1$ for $\tilde{g} = u_1 \cdots u_j$.

Then there is an exact sequence

$$0 \longrightarrow Z_2 \longrightarrow Spin(q) \xrightarrow{\tau_0} SO(q) \longrightarrow 0, \tag{2.1}$$

where $\tau_0 = \tau|_{Spin(q)}$. It is well known that $Spin(q)(q > 2)$ is simply connected and the universal cover of $SO(q)$([G],[LM]). Let $\pi : P_{SO}(Q) \longrightarrow M$ be the foliated principal $SO(q)$-bundle of (oriented) transverse orthonormal frames. Note that if M is transversally orientable, then choosing an orientation on Q is equivalent to choosing a foliated principal $SO(q)$-bundle $P_{SO}(Q) \subset P_O(Q)$. This embedding is, of course, compatible with the action of $SO(q) \subset O(q)$. Having thereby made the structure group of Q 0-connected, one might ask whether it is possible to make the struture group simply connected. This leads us to the concept of a transverse spin structure.

Definition. Suppose $q \geq 3$. Then a *transverse spin structure* on foliated Riemannian manifold M is a foliated principal $Spin(q)$-bundle $P_{Spin}(Q)$ together with a 2-sheeted covering

$$\xi : P_{Spin}(Q) \longrightarrow P_{SO}(Q) \tag{2.2}$$

such that $\xi(p \cdot \tilde{g}) = \xi(p) \cdot \tau_0(\tilde{g})$ for all $p \in P_{spin}(Q)$ and all $\tilde{g} \in Spin(q)$. When $q = 2$, a transverse spin structure on M is defined analogously, with $Spin(q)$ replaced by $SO(2)$ and $\tau_0 : SO(2) \longrightarrow SO(2)$ the connected 2-fold covering.

When $q = 1$, $P_{SO}(Q) \cong M$ and a transverse spin structure is simply defined to be a 2-fold covering of M.

Let $\{U_\alpha\}$ be a distinguished open cover of M so $P_{SO}(Q)$ is local trivial over U_α and let s_α be a local transverse orthonormal frames over U_α. On the overlap, we express $s_\alpha = g_{\alpha\beta}s_\alpha$, where $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow SO(q)$ is basic (that is, locally constant along the leaves). These satisfy the cocycle condition:

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = I \quad \text{and} \quad g_{\alpha\alpha} = I.$$

Since the foliated principal bundle $P_{SO}(Q)$ of (oriented) transverse orthonormal frames has transition functions $g_{\alpha\beta}$ acting on $SO(q)$ from the left, a transverse spin structure on M can be considered as lifting of the transition functions to $Spin(q)$ preserving the cocycle condition. Hence the existence of the transverse spin structure on M is equivanlent to the existence of the lifting $\tilde{g}_{\alpha\beta}$ of $g_{\alpha\beta}$ satisfying the cocycle condition. That is,

$$(2.3) \qquad \tau_0(\tilde{g}_{\alpha\beta}) = g_{\alpha\beta},\ \tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} = I \quad \text{and} \quad \tilde{g}_{\alpha\alpha} = I.$$

To do our work, we introduce a Čech-cohomology theory. Now we fix $\{U_\alpha\}$ a distinguished simple cover of $(M, g_M, \mathcal{F})$ and let Z_2 be the multiplicative group $\{-1, 1\}$. A *transverse Čech j-cochain* is a basic function $f : U_{\alpha_0} \cap \cdots \cap U_{\alpha_j} \longrightarrow Z_2$ defined for (j+1)-tuples of indices on $U_{\alpha_0} \cap \cdots \cap U_{\alpha_j} \neq \phi$, which is totally symmetric, that is,

$$f(\alpha_{\sigma(0)}, \cdots, \alpha_{\sigma(j)}) = f(\alpha_0, \cdots, \alpha_j)$$

for any permutation σ. If $C^j_{tr}(M, Z_2)$ denotes the multiplicative group of all the transversal Čech j-cochains, the coboundary $\delta : C^j_{tr}(M, Z_2) \longrightarrow C^{j+1}_{tr}(M, Z_2)$ is defined by:

$$(2.4) \qquad (\delta f)(\alpha_0, \cdots, \alpha_{j+1}) = \prod_{i=0}^{j+1} f(\alpha_0, \cdots, \hat{\alpha}_i, \cdots, \alpha_{j+1}).$$

The multiplicative identity of $C^j_{tr}(M, Z_2)$ is the function 1 and clearly $\delta^2 f = 1$. Hence we can define a cohomology group $H^j_{tr}(M; Z_2) = Ker\ \delta / Im\ \delta$ which are independent of the particular distinguished simple cover chosen. This is called the *j-th transversal Čech cohomology group*.

3. Main results

Consider the vector bundle $Q = TM/L \longrightarrow M$, not necessarily orientable. Let s_α be a transverse orthonormal frame on Q over U_α. On $U_\alpha \cap U_\beta \neq \phi$, $s_\alpha = g_{\alpha\beta} s_\beta$, where $g_{\alpha\beta}$ is the transition functions. Now, we define a 1-cochain f as

$$(3.1) \qquad f(\alpha,\beta) = det(g_{\alpha\beta}) = \pm 1.$$

This is well-defined since $U_\alpha \cap U_\beta$ is contractible and hence connected. Since $f(\alpha,\beta) = f(\beta,\alpha)$, this defines an element of $C^1_{tr}(M, Z_2)$. If the transition functions $g_{\alpha\beta}$ satisfy the cocycle condition, then from (3.1),

$$(3.2) \qquad (\delta f)(\alpha,\beta,\gamma) = f(\beta,\gamma)f(\alpha,\gamma)f(\alpha,\beta) = det(g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}) = 1.$$

Hence f defines an element of $H^1_{tr}(M; Z_2)$. If we replace s_α by $s'_\alpha = h_\alpha s_\alpha$, the new transition functions become $g'_{\alpha\beta} = h_\alpha g_{\alpha\beta} h_\beta^{-1}$. If $f_0(\alpha) = det(h_\alpha)$, then

$$(3.3) \qquad f'(\alpha,\beta) = det(g'_{\alpha\beta}) = det(h_\alpha)det(g_{\alpha\beta})det(h_\beta) = ((\delta f_0)f)(\alpha,\beta).$$

This proves that the element in cohomology defined by f is independent of the particular frame chosen and we will denote this by $[f] \equiv \omega_1(Q) \in H^1_{tr}(M; Z_2)$, called the *first Stiefel-Whitney class*. Hence if M is transversally orientable, we can choose a transverse frames such that $det(g_{\alpha\beta}) = 1$. This implies that $\omega_1(Q)$ is trivial. Conversely, if $\omega_1(Q)$ is trivial, then $f = \delta f_0$ for some $f_0 \in C^0_{tr}(M, Z_2)$. If we choose h_α with $det(h_\alpha) = f_0(\alpha)$, then the new frames $s'_\alpha = h_\alpha s_\alpha$ will have transition functions $g'_{\alpha\beta}$ with $det(g'_{\alpha\beta}) = 1$ and define an orientation of Q. Hence we have

Theorem 3.1. *Let $(M, g_M, \mathcal{F})$ be the Riemannian foliation of codimension q with bundle-like metric g_M. Then M is a transversally orientable if and only if $\omega_1(Q) \in H^1_{tr}(M; Z_2)$, is trivial.*

Now, let $(M, g_M, \mathcal{F})$ be the transversally oriented Riemannian manifold with bundle-like metric g_M and foliation $\mathcal{F}$ with codimension q and let $g_{\alpha\beta} \in SO(q)$ the transition functions of $P_{SO}(Q)$. We choose any lifting $\tilde{g}_{\alpha\beta}$ to Spin(q) by τ_0 in (2.1) such that

$$(3.4) \qquad \tau_0(\tilde{g}_{\alpha\beta}) = g_{\alpha\beta} \quad \text{and} \quad \tilde{g}_{\alpha\beta}\tilde{g}_{\beta\alpha} = I.$$

Since the U_α is contractible, such lifts always exist. Since $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = I$, $\tau_0(\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha}) = I$. Hence $\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} = \pm I$. Now, define the transverse 2-cochain f as

$$f(\alpha,\beta,\gamma)I = \tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha}. \tag{3.5}$$

Clearly $f(\alpha,\beta,\gamma) \in Z_2$. Furthermore, f is symmetric and $\delta f = 1$. Hence if we change the choice of the transverse frame s_α or change the choice of lifts, then f changes by a coboundary. This implies f defines an element in $H^2_{tr}(M;Z_2)$, which is independent of the choices made. We will denote this by $[f] \equiv \omega_2(Q) \in H^2_{tr}(M;Z_2)$, called the *second Stiefel-Whitney class*. Since from (3.5), M admits a transverse spin structure if and only if we can choose the lifting such that $f(\alpha,\beta,\gamma) = 1$, we have

Theorem 3.2. *Let $(M, g_M, \mathcal{F})$ be the transversally oriented Riemannian foliation with codimension q and with bundle-like metric g_M. Then M admits a transverse spin structure if and only if $\omega_2(Q) \in H^2_{tr}(M;Z_2)$ is trivial.*

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