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STOCHASTIC DIFFERENTIAL INCLUSION ON FINITE DIMENSIONAL SPACE

YONG SIK YUN

Department of Mathematics, Cheju National University, Korea

ABSTRACT. For the stochastic differential inclusion of the form $dX_t \in \sigma(t, X_t)dB_t + b(t, X_t)dt$, where σ, b are set-valued maps, B is a standard Brownian motion, we prove the existence of solution under the assumption that σ and b satisfy the local Lipschitz property and linear growth.

1. INTRODUCTION

Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space with a right-continuous increasing family $(\mathfrak{F}_t)_{t\geq 0}$ of sub $\sigma-$fields of $\mathfrak{F}$ each containing all P-null sets. Let $B = (B_t)_{t\geq 0}$ be an r-dimensional $(\mathfrak{F}_t)$-Brownian motion. We consider the following stochastic differential inclusion.

$$dX_t \in \sigma(t, X_t)dB_t + b(t, X_t)dt, \tag{1.1}$$

where $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^r$, $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ are set-valued maps. In recent years the study of the existence and properties of solution for these stochastic differential inclusions have been developed by many authors ([4]). Furthermore the results for the viable solutions have been made ([2], [6]). For the stochastic differential equation associated with (1.1), many results for the existence, uniqueness, and properties of solutions have been done under various conditions that σ and b are continuous and bounded or Lipschitzean or Hölder continuous ([3]).

In this paper, we prove the existence of solution for stochastic differential inclusion (1.1) under the condition that σ and b satisfy the local Lipschitz property and linear growth.

2. PRELIMINARIES

We prepare the definition of solution for stochastic differential inclusion and some results for the stochastic differential equation and selection theorems.

Definition 2.1. An r-dimensional continuous process $B = (B_t)_{t\in[0,\infty)}$ is called an r-dimensional $(\mathfrak{F}_t)$-Brownian motion if it is $(\mathfrak{F}_t)$-adapted and satisfies

$$E[\exp[i < \xi, B_t - B_s >] \mid \mathfrak{F}_s] = \exp[-(t-s)|\xi|^2/2], \quad \text{a.s.}$$
$$\text{for every } \xi \in \mathbb{R}^r \text{ and } 0 \le s < t.$$

Let us consider the stochastic differential inclusion

$$dX_t \in \sigma(t, X_t)dB_t + b(t, X_t)dt \tag{1.1}$$

with the initial value $X_0 = x_0$, where $\sigma : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d\otimes\mathbb{R}^r$, $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ are set-valued maps and x_0 is a $\mathbb{R}^d$-valued $\mathfrak{F}_0$-measurable function.

Definition 2.2. A stochastic process $X = \{X_t,\ t \in [0,T]\} \in L^q(\Omega \to C([0,T] \to \mathbb{R}^d)), q \ge 2$, is said to be a solution of (1.1) on $[0,T]$ with the initial condition x_0 if there are predictable random processes $f : \Omega\times[0,T] \to \mathbb{R}^d\otimes\mathbb{R}^r$, $g : \Omega\times[0,T] \to \mathbb{R}^d$ such that $f(t) \in \sigma(t, X_t)$, $g(t) \in b(t, X_t)$ a.s. on $[0,T]$ and for every $t \in [0,T]$,

$$X_t = x_0 + \int_0^t f(s)\, dB_s + \int_0^t g(s)\, ds \quad \text{a.s.},$$

where

$$L^q(\Omega \to C([0,T] \to \mathbb{R}^d))$$
$$= \{X \mid X \text{ is predictable, continuous, and } E[\sup_{0\le s\le T} |X_s|^q] < \infty\}.$$

For the stochastic differential equation

$$X_t = \xi + \int_0^t \sigma(s, X_s)dB_s + \int_0^t b(s, X_s)ds, \tag{2.1}$$

where $\sigma : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d\otimes\mathbb{R}^r$, $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ are $\mathfrak{B}([0,T])\otimes\mathfrak{B}(\mathbb{R}^d)\otimes\mathfrak{F}_T$-measurable and $\mathfrak{F}_t$-progressively measurable for each $x \in \mathbb{R}^d$, ξ is $\mathfrak{F}_0$-measurable, the following theorems are well known.

Theorem 2.3. ([5]) We assume the followings.

(i) For each $N > 0$, there exists a constant $C_N > 0$ such that

$$\begin{cases} \|\sigma(t,x) - \sigma(t,y)\| \le C_N \cdot |x-y|, & x, y \in B_N \\ |b(t,x) - b(t,y)| \le C_N \cdot |x-y|, & x, y \in B_N, \end{cases}$$

where $B_N = \{x \in \mathbb{R}^d, |x| \leq N\}$ and $||\sigma||^2 = \sum_{j=1}^r \sum_{i=1}^d |\sigma_j^i|^2 \equiv \mathrm{tr}(\sigma\sigma^*)$.

(ii) There exists a constant $K > 0$ such that

$$\frac{1}{2}||\sigma(t,x)||^2 + x^* \cdot b(t,x) \leq K(r(t)^2 + |x|^2),$$

where $r(t)$ is a progressively measurable such that

$$E\left[\ |\xi|^2 + \int_0^T \{|b(s,0)|^2 + r(s)^2\}ds\ \right] < \infty.$$

Then (2.1) has unique solution X_t and

$$E[|X_t|^2] \leq E\left[\ |\xi|^2 + 2K\int_0^t r(s)^2 ds\ \right] e^{2Kt}, \quad \forall t \leq T.$$

3. Main Results

For a Banach space X with the norm $||\cdot||$ and for non-empty sets A, A' in X, we denote $||A|| = \sup\{||a|| \mid a \in A\}$, $d(a, A') = \inf\{d(a,a') \mid a' \in A'\}$, $d(A, A') = \sup\{d(a, A') \mid a \in A\}$ and $d_H(A, A') = \max\{d(A, A'), d(A', A)\}$, a Hausdorff metric. Given a family of sets $\{F_\alpha \mid \alpha \in A\}$, a selection is a map $\alpha \to f_\alpha$ in F_α. The most famous continuous selection theorem is the following result by Michael.

Theorem 3.1. ([1]) Let X be a metric space, Y a Banach space. Let F from X into the closed convex subsets of Y be lower semi-continuous. Then there exists $f : X \to Y$, a continuous selection from F.

Proof. Step 1. Let us given by proving the following claim : given any convex (not necessarily closed) valued lower semi-continuous map Φ and every $\varepsilon > 0$, there exists a continuous $\phi : X \to Y$ such that for ξ in $X, d(\phi(\xi), \Phi(\xi)) \leq \varepsilon$.

In fact, for every $x \in X$, let $y_x \in \Phi(x)$ and let $\delta_x > 0$ be such that $(y_x + \varepsilon\mathring{A}) \cap \Phi(x') \neq \emptyset$ for x' in $B(x, \delta_x)$, where $\mathring{A}$ denotes the open unit ball. Since X is metric, it is paracompact. Hence there exists a locally finite refinement $\{\mathfrak{U}_x\}_x \in X$ of $\{B(x,\delta_x)\}_x$. Let $\{\pi_x(\cdot)\}_x$ be a partition of unity subordinate to it. The mapping $\varphi : X \to Y$ given by $\varphi(\xi) = \sum \pi_x(\xi) y_x$ is continuous since it is locally a finite sum of continuous functions. Fix ξ. Whenever $\pi_x(\xi) > 0$, $\xi \in \mathfrak{U}_x \subset B(x, \delta_x)$, hence $y_x \in \Phi(\xi) + \varepsilon\mathring{A}$. Since this latter set is convex, any convex combination of such y's (in particular, $\varphi(\xi)$) belongs to it.

Step 2. Next we claim that we can define a sequence $\{f_n\}$ of continuous mappings from X into Y with the following properties

i) for each $\xi \in X, d(f_n(\xi), F(\xi)) \leq \frac{1}{2^n}$, $\quad n = 1, 2, \cdots$,

ii) for each $\xi \in X, ||f_n(\xi) - f_{n-1}(\xi)|| \leq \frac{1}{2^{n-2}}$, $\quad n = 2, \cdots$.

For $n = 1$ it is enough to take in the claim of part Step 1, $\Phi = F$ and $\varepsilon = 1/2$.

Assume we have defined mappings f_n satisfying i) up to $n = \nu$. We shall define $f_{\nu+1}$ satisfying i) and ii) as follows.

Consider the set $\Phi(\xi) \doteq (f_\nu(\xi) + \frac{1}{2^\nu}\mathring{A}) \cap F(\xi)$. By i) it is not empty, and it is a convex set. The map $\xi \to \Phi(\xi)$ is lower semicontinuous and by the claim of Step 1, there exists a continuous φ such that $d(\varphi(x), \Phi(x)) \leq \frac{1}{2^{\nu+1}}$.

Set $f_{\nu+1}(\xi) \doteq \varphi(\xi)$. A fortiori $d(f_{\nu+1}(\xi), F(\xi)) \leq \frac{1}{2^{\nu+1}}$, proving i). Also $f_{\nu+1}(\xi) \in \Phi(\xi) + \frac{1}{2^{\nu+1}}\mathring{A} \subset f_\nu(\xi) + (\frac{1}{2^\nu} + \frac{1}{2^{\nu+1}})\mathring{A}$ i.e.,

$$||f_{\nu+1}(\xi) - f_\nu(\xi)|| \leq \frac{1}{2^{\nu-1}}$$

proving ii).

Step 3. Since the series $\sum \frac{1}{2^n}$ converges, $\{f_n(\cdot)\}$ is a Cauchy sequence, uniformly converging to a continuous $f(\cdot)$. Since the values of F are closed, by i) of part Step 2, f is a selection from F.

Let $A \subset \mathbb{R}^n$ be a compact convex body, i.e., a compact set with nonempty interior, and let m_n be the n-dimensional Lebesgue measure. Since $m_n(A)$ is positive, we can define the barycenter of A as

$$b(A) = \frac{1}{m_n(A)} \int_A x \, dm_n.$$

Lemma 3.2. ([1]) The barycenter of A, $b(A)$, belongs to A.

Proof. Assume the contrary: $d(b(A), A)$ is positive. Set a to be $\pi_A(b(A))$, b to be $b(A)$ and $p \doteq b - a$.

By the characterization of the best approximation we have that for all x in A, $< x - a, p > \leq 0$. However from

$$p = b - a = \frac{1}{m_n(A)} \int_A (x - a) dm_n$$

we have

$$\begin{aligned} ||p||^2 &= < \frac{1}{m_n(A)} \int_A (x - a) dm_n, p > \\ &= \frac{1}{m_n(A)} \int_A < x - a, p > dm_n \leq 0, \end{aligned}$$

a contradiction; hence $b(A)$ belongs to A.

Lemma 3.3. ([1]) Let $A \subset \mathbb{R}^n$ be compact and convex and consider $A^1 \doteq A + B$, where B is the closed unit ball. Then $b(A^1)$ belongs to A.

Proof. As above assume it is not so. Set a to be $\pi_A(b(A^1))$, the point of A nearest to $b = b(A^1)$, set $p \doteq b - a$ and $\hat{p} = p/||p||$. Then

$$(3.1) \qquad ||p||^2 = \frac{1}{m_n(A^1)} \int_{A^1} < x - a, p > dm_n$$

and as, before, to reach a contradiction it is enough to show that the right hand side is non positive.

It is convenient to consider S_P, the linear transformation mapping x into its symmetric with respect to the hyperplane orthogonal to p through a:

$S_P(x) = a + (x - a) - 2 < x - a, \hat{p} > \hat{p}$.

Set $A^1_+ \doteq \{a \in A^1 | < x - a, p > > 0\}$, $A^1_- \doteq \{x \in A^1 | < x - a, p > \leq 0\}$.

We remark that $S_P(A^1_+) \subset A^1$. In fact fix x in A^1_+ and consider $S_P(x)$: Set x' to be the projection of $\pi_A(x)$ on the line through x and $S_P(x)$. By the Pythagorean theorem to show that

$||x - \pi_A(x)|| \geq ||S_P(x) - \pi_A(x)||$ it is enough to show that

$||x - x'|| \geq ||S_P(x) - x'||$. We have that

$||x - x'|| = < x - x', \hat{p} > = < x - a, \hat{p} > - < x' - a, \hat{p} >$

and

$$\begin{aligned} ||S_P(x) - x'|| &= - < S_P(x) - x', \hat{p} > = - < S_P(x) - a, \hat{p} > + < x' - a, \hat{p} > \\ &= < x - a, \hat{p} > + < x' - a, \hat{p} > . \end{aligned}$$

Since, again by the characterization of the best approximation, x' belongs to A^1_-,

$d(S_P(x), A) \leq ||S_P(x) - \pi_A(x)|| \leq ||x - \pi_A(x)|| = d(x, A) \leq 1$,

Then $S_P(x)$ belongs to A^1.

Write A^1 as $(A^1_+ \cup S_P(A^1_+)) \cup (A^1 \backslash (A^1_+ \cup S_P(A^1_+)))$ and consider the integral in (3.1) separately on these two subsets. Remark that the first is invariant with respect to the transformation S_P, that the determinant of the Jacobian of the transformation S_P is one and that the map $x \to < x - a, \hat{p} >$ is antisymmetric with respect to S_P. The change of variables formula hence yields

$$\begin{aligned} \int_{S_P(A^1_+ \cup S_P(A^1_+))} < x - a, p > &= \int_{(A^1_+ \cup S_P(A^1_+))} < S_P(x) - a, p > \\ &= - \int_{S_P(A^1_+ \cup S_P(A^1_+))} < x - a, p > . \end{aligned}$$

Hence this integral is zero.

Since $A^1 \backslash (A^1_+ \cup S_P(A^1_+))$ is contained in A^1_-,

$$\int_{A^1} < x - a, p > \leq 0$$

the desired contradiction.

Using Lemma 3.2 and 3.3, we have the following local Lipschitz barycentric selection theorem.

Theorem 3.4. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a local Lipschitz set-valued map with compact convex images, i.e., there exists a constant $K_N > 0$ such that

$$d_H(F(x), F(y)) \leq K_N \cdot |x - y|, \quad \forall x, y \in B_N = \{x \in \mathbb{R}^n, |x| \leq N\}.$$

Assume moreover that there exists a constant $C > 0$ such that $||F(x)|| \leq C \cdot (1 + |x|)$, for every $x \in \mathbb{R}^n$. Then there exist a constant $\hat{C}_N > 0$ and a single valued map $f : \mathbb{R}^n \to \mathbb{R}^n$, local Lipschitzean with constant $\hat{C}_N$, a selection from F.

Proof. By Lemma 3.2 and 3.3, the single valued map $b^1 = x \to b(F(x) + B)$ is a selection from F. We have to prove that it is a local Lipschitzean selection.

Fix $x, x' \in B_N$. Call $\Phi(x) \doteq F(x) + B, \Phi'(x') \doteq F(x') + B$. Since $||\Phi(x)|| \leq ||F(x) + B|| \leq ||F(x)|| + 1 \leq C \cdot (1 + |x|) + 1 \leq C \cdot (1 + N) + 1 = C_{N'}$ and $m_n(\Phi(x)) \leq C_{N''}$, we have

$$\begin{aligned}
&\frac{1}{m_n(\Phi(x))} \int_{\Phi(x)} x \, dm_n - \frac{1}{m_n(\Phi'(x'))} \int_{\Phi'(x')} x \, dm_n \\
&\leq \left| \left(\frac{1}{m_n(\Phi(x))} - \frac{1}{m_n(\Phi'(x'))} \right) \int_{\Phi(x) \cap \Phi'(x')} x \, dm_n \right| \\
&\quad + \left| \frac{1}{m_n(\Phi(x))} \int_{\Phi(x) \backslash \Phi'(x')} x \, dm_n - \frac{1}{m_n(\Phi'(x'))} \int_{\Phi'(x') \backslash \Phi(x)} x \, dm_n \right| \\
&\leq |m_n(\Phi(x)) - m_n(\Phi'(x'))| \cdot C_{N'} \cdot C_{N''}/(m_n(B))^2 \\
&\quad + \{m_n(\Phi(x) \setminus \Phi'(x')) + m_n(\Phi'(x') \setminus \Phi(x))\} \cdot C_{N'} \cdot C_{N''}/m_n(B).
\end{aligned} \tag{3.2}$$

We with to express the above estimate in terms of $d_H(\Phi, \Phi')$. For this purpose, we begin to compare $m_n(\Phi + \delta B)$, $\delta > 0$, and $m_n(\Phi)$. Since the unit ball of $\mathbb{R}^n$ is contained in the unit cube $\{|x_i| \leq 1, i = 1, \cdots, n\}$, we can as well estimate

$$m_n\{\varphi + \sum \delta_i e_i \mid \varphi \in \Phi, |\delta_i| \leq \delta\}$$

where $\{e_i\}$ is an orthonormal basis.

¿From elementary calculus we have that when S is a convex set and ν a unit vector, the measure of $\{S+\delta_x\nu \mid |\delta_x| \leq \delta\}$ is $m_n(S)+|\delta| m_{n-1}(P_\nu(S))$ where P_ν is the projection of S into the hyperplane normal to ν through the origin ($P_\nu(S)$ is the "shadow" of S).

Denote by

$$\Phi_\nu \doteq \{\varphi + \sum_{i=1}^{\nu} \delta_i e_i | \varphi \in \Phi, \delta_i \leq \delta\}$$

and by P_i the projection along the direction e_i.

Recursively we obtain

$$m_n(\Phi_n) \leq m_n(\Phi) + \delta \sum_{j=0}^{n-1} m_{n-1}(P_{n-j}(\Phi_{n-j})).$$

Since Φ is contained in $(M+1)B$, each element of each $P_j(\Phi_j)$ has a distance from the origin of at most $(M+1)+\delta\sqrt{n}$, so that, setting B_{n-1} the unit ball in $\mathbb{R}^{n-1}$,

$$\begin{aligned} m_n(\Phi+\delta B) &\leq m_n(\Phi_n) \\ &\leq m_n(\Phi) + \delta n m_{n-1}((M+1+\delta\sqrt{n})B_{n-1}) \\ &\leq m_n(\Phi) + \delta K \end{aligned}$$

for some constant K.

Set δ to be $d_H(\Phi, \Phi')$. Then $\Phi' \subset \Phi + \delta B$ and $\Phi \subset \Phi' + \delta B$, hence $m_n(\Phi \backslash \Phi') \leq m_n(\Phi' + \delta B) - m_n(\Phi')$, and $m_n(\Phi' \backslash \Phi) \leq m_n(\Phi + \delta B) - m_n(\Phi)$. Analogously, $|m_n(\Phi) - m_n(\Phi')| \leq K\delta$. Hence by (3.2), we obtain

$$|b(F(x)+B) - b(F(x')+B)| \leq C'_N \cdot d_H(F(x)+B, F(x')+B)$$

for a suitable C'_N. Finally, since K_N is the local Lipschitz constant of F and set $\hat{C}_N$ to be $K_N \cdot K$. We have

$$\begin{aligned} |b^1(x) - b^1(x')| &\leq K \cdot d_H(F(x)+B, F(x')+B) \\ &\leq K \cdot d_H(F(x), F(x')) \leq \hat{C}_N \cdot d(x, x'), \end{aligned}$$

i.e. $f = b^1$ is the required Lipschitzean selection.

Thus we have the following another main theorem by the above lemmas and Theorem 3.4.

Theorem 3.5. Assume that

(i) for each $N > 0$, there exist constants $C > 0$ and $C_N > 0$ such that

$$\begin{cases} d_H(\sigma(t,x)-\sigma(t,y)) \le C_N \cdot |x-y|, & x,y \in B_N, \\ d_H(b(t,x)-b(t,y)) \le C_N \cdot |x-y|, & x,y \in B_N, \\ ||\sigma(t,x)|| + |b(t,x)| \le C \cdot (1+|x|), & x \in \mathbb{R}^n, \end{cases}$$

where $B_N = \{x \in \mathbb{R}^d, |x| \le N\}$,

(ii) there exists a constant $K > 0$ such that

$$\frac{1}{2}||\sigma(t,x)||^2 + |x| \cdot |b(t,x)| \le K(r(t)^2 + |x|^2),$$

where $r(t)$ is a progressively measurable such that

$$E\left[\, |x_0|^2 + \int_0^T \{|b(s,0)|^2 + r(s)^2\} \,\right] ds < \infty.$$

Then (1.1) has a solution X_t and

$$E[|X_t|^2] \le E\left[\, |x_0|^2 + 2K \int_0^t r(s)^2 ds \,\right] e^{2Kt}, \ \ \forall t \le T.$$

Proof. By the hypothesis i) and Theorem 3.4, σ and b have local Lipschitzean selection. Thus the proof is complete by Theorem 2.3.

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