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SYMMETRIC TRIANGULAR FUZZY NUMBER

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Abstract. For various fuzzy numbers, many operations have been calculated. We generalize about triangular fuzzy number and calculate four operations, addition $A(+)B$, subtraction $A(-)B$, multiplication $A(\cdot)B$ and division $A(/)B$ for two generalized symmetric triangular fuzzy numbers.

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1. Introduction

Let $(\Omega, \mathfrak{F}, P)$ be a probability space, where Ω denotes the sample space, $\mathfrak{F}$ the $\sigma-$algebra on Ω, and P a probability measure. A fuzzy set A on Ω is called a fuzzy event. Let $\mu_A(\cdot)$ be the membership function of the fuzzy event A. Then the probability of the fuzzy event A is defined by Zadeh([5]) as the integral for $\mu_A(\cdot)$ on Ω with respect to dP.

$$\widetilde{P}(A) = \int_\Omega \mu_A(\omega)\, dP(\omega), \qquad \mu_A(\omega) : \Omega \to [0,1]$$

The operations of two fuzzy numbers (A, μ_A) and (B, μ_B) are based on the Zadeh's extension principle([6], [7], [8]). We consider four operations, addition $A(+)B$, subtraction $A(-)B$, multiplication $A(\cdot)B$ and division $A(/)B$.

We defined the normal fuzzy probability using the normal distribution and calculated the normal fuzzy probability for quadratic fuzzy number([2]). And then we had the explicit formula for the normal fuzzy probability for trigonometric fuzzy number. Furthermore we calculated the normal fuzzy probability for trigonometric fuzzy numbers driven by the above four operations([3]).

In this paper, we generalize the triangular fuzzy number. There are many generalized triangular fuzzy numbers. But, we study only symmetric triangular fuzzy numbers and calculate four operations, addition $A(+)B$, subtraction $A(-)B$, multiplication $A(\cdot)B$ and division $A(/)B$ for two generalized symmetric triangular fuzzy numbers.

2. Preliminaries

Let $(\Omega, \mathfrak{F}, P)$ be a probability space, and X be a random variable defined on it. Let g be a real-valued Borel-measurable function on $\mathbb{R}$. Then $g(X)$ is also a random variable.

Definition 2.1. We say that the mathematical expectation of $g(X)$ exists if

$$E[g(X)] = \int_\Omega g(X(\omega))\, dP(\omega) = \int_\Omega g(X)\, dP$$

is finite.

Example 2.2. ([4]) Let the random variable X have the normal distribution given by the probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-m)^2}{2\sigma^2}}, \quad x \in \mathbb{R},$$

where $\sigma^2 > 0$ and $m \in \mathbb{R}$. Then $E[|X|^\gamma] < \infty$ for every $\gamma > 0$, and we have

$$E[X] = m \quad \text{and} \quad E[(X-m)^2] = \sigma^2.$$

The induced measure P_X is called the normal distribution.

Example 2.3. ([4]) Let the random variable X have the exponential distribution given by the probability density function

$$f(x) = \lambda e^{-\lambda x}$$

for $x \geq 0$ and $\lambda > 0$. Then we have

$$E[X] = \frac{1}{\lambda} \quad \text{and} \quad E[(X - \frac{1}{\lambda})^2] = \frac{1}{\lambda^2}.$$

The induced measure P_X is called the exponential distribution.

A fuzzy set A on Ω is called a *fuzzy event*. Let $\mu_A(\cdot)$ be the membership function of the fuzzy event A. Then the probability of the fuzzy event A is defined by Zadeh([5]) as

$$\widetilde{P}(A) = \int_\Omega \mu_A(\omega)\, dP(\omega), \quad \mu_A(\omega) : \Omega \to [0,1].$$

Definition 2.4. The normal fuzzy probability $\widetilde{P}(A)$ of a fuzzy set A on $\mathbb{R}$ is defined by

$$\widetilde{P}(A) = \int_{\mathbb{R}} \mu_A(x)\, dP_X,$$

where P_X is the normal distribution.

Definition 2.5. The exponential fuzzy probability $\widetilde{P}(A)$ of a fuzzy set A on $\mathbb{R}$ is defined by

$$\widetilde{P}(A) = \int_{\mathbb{R}} \mu_A(x)\, dP_X,$$

where P_X is the exponential distribution.

Definition 2.6. The set $A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$ is said to be the *α-cut* of a fuzzy set A.

The membership function of a fuzzy set A can be expressed in terms of the characteristic functions of its α-cuts according to the formula

$$\mu_A(x) = \sup_{\alpha \in (0,1]} \min(\alpha, \mu_{A_\alpha}(x)),$$

where

$$\mu_{A_\alpha}(x) = \begin{cases} 1, & x \in A_\alpha, \\ 0, & \text{otherwise.} \end{cases}$$

It is easily checked that the following properties hold

$$(A \cup B)_\alpha = A_\alpha \cup B_\alpha, \quad (A \cap B)_\alpha = A_\alpha \cap B_\alpha.$$

Definition 2.7. A triangular fuzzy number is a fuzzy number A having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, \ a_3 \leq x, \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x < a_2, \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x < a_3. \end{cases}$$

The above triangular fuzzy number is denoted by $A = (a_1, a_2, a_3)$.

Definition 2.8. The addition, subtraction, multiplication, and division of two fuzzy numbers are defined as

1. Addition $A(+)B$:

$$\mu_{A(+)B}(z) = \sup_{z=x+y} \min\{\mu_A(x), \mu_B(y)\}, \ x \in A, y \in B.$$

2. Subtraction $A(-)B$:

$$\mu_{A(-)B}(z) = \sup_{z=x-y} \min\{\mu_A(x), \mu_B(y)\}, \ x \in A, y \in B.$$

3. Multiplication $A(\cdot)B$:

$$\mu_{A(\cdot)B}(z) = \sup_{z=x\cdot y} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B.$$

4. Division $A(/)B$:

$$\mu_{A(/)B}(z) = \sup_{z=x/y} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B.$$

Example 2.9. ([1]) For two triangular fuzzy numbers $A = (1,2,4)$ and $B = (2,4,5)$, we have

1. Addition : $A(+)B = (3,6,9)$.
2. Subtraction : $A(-)B = (-4,-2,2)$.
3. Multiplication :

$$\mu_{A(\cdot)B}(x) = \begin{cases} 0, & x < 2,\ 20 \le x, \\ \frac{-2+\sqrt{2x}}{2}, & 2 \le x < 8, \\ \frac{7-\sqrt{9+2x}}{2}, & 8 \le x < 20. \end{cases}$$

Note that $A(\cdot)B$ is not a triangular fuzzy number.

4. Division :

$$\mu_{A(/)B}(x) = \begin{cases} 0, & x < \frac{1}{5},\ 2 \le x, \\ \frac{5x-1}{x+1}, & \frac{1}{5} \le x < \frac{1}{2}, \\ \frac{-x+2}{x+1}, & \frac{1}{2} \le x < 2. \end{cases}$$

Note that $A(/)B$ is not a triangular fuzzy number.

3. Main results

We generalize the triangular fuzzy number. A generalized triangular fuzzy number is symmetric and may not have value 1.

Definition 3.1. A symmetric triangular fuzzy number is a fuzzy set A having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1,\ a_2 \le x, \\ \frac{2c(x-a_1)}{a_2-a_1}, & a_1 \le x < \frac{a_1+a_2}{2}, \\ \frac{-2c(x-a_2)}{a_2-a_1}, & \frac{a_1+a_2}{2} \le x < a_2, \end{cases}$$

where $a_1, a_2 \in \mathbb{R}$ and $0 < c \le 1$.

The above symmetric triangular fuzzy number is denoted by $A = ((a_1, c, a_2))$.

Theorem 3.2. For two symmetric triangular fuzzy numbers $A = ((a_1, c_1, a_2))$ and $B = ((b_1, c_2, b_2))$, we have

1. $A(+)B = ((a_1 + b_1, \dfrac{c_1c_2(a_2 - a_1 + b_2 - b_1)}{c_2(a_2 - a_1) + c_1(b_2 - b_1)}, a_2 + b_2)).$

2. $A(-)B = ((a_1 - b_2, \dfrac{c_1c_2(a_2 - a_1 + b_2 - b_1)}{c_2(a_2 - a_1) + c_1(b_2 - b_1)}, a_2 - b_1)).$

3. $A(\cdot)B$ is a fuzzy set on (a_1b_1, a_2b_2), but need not to be symmetric triangular fuzzy number.

4. $A(/)B$ is a fuzzy set on $(\frac{a_1}{b_2}, \frac{a_2}{b_1})$, but need not to be symmetric triangular fuzzy number.

Proof. Note that

$$\mu_A(x) = \begin{cases} 0, & x < a_1,\ a_2 \le x, \\ \frac{2c_1(x-a_1)}{a_2-a_1}, & a_1 \le x < \frac{a_1+a_2}{2}, \\ -\frac{2c_1(x-a_2)}{a_2-a_1}, & \frac{a_1+a_2}{2} \le x < a_2, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < b_1,\ b_2 \le x, \\ \frac{2c_2(x-b_1)}{b_2-b_1}, & b_1 \le x < \frac{b_1+b_2}{2}, \\ -\frac{2c_2(x-b_2)}{b_2-b_1}, & \frac{b_1+b_2}{2} \le x < b_2, \end{cases}$$

we calculate exactly the above four operations using $\alpha-$ cuts.

Let A_α and B_α be the α-cuts of A and B, respectively. Let $A_\alpha = [a_1^{(\alpha)},\ a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)},\ b_2^{(\alpha)}]$. Since $\alpha = \frac{2c_1(a_1^{(\alpha)}-a_1)}{a_2-a_1}$ and $\alpha = -\frac{2c_1(a_2^{(\alpha)}-a_2)}{a_2-a_1}$, we have $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\frac{a_2-a_1}{2c_1}\alpha + a_1, \frac{a_1-a_2}{2c_1}\alpha + a_2]$. Since $\alpha = \frac{2c_2(b_1^{(\alpha)}-b_1)}{b_2-b_1}$ and $\alpha = -\frac{2c_2(b_2^{(\alpha)}-b_2)}{b_2-b_1}$, we have $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [\frac{b_2-b_1}{2c_2}\alpha + b_1, \frac{b_1-b_2}{2c_2}\alpha + b_2]$.

1. Addition :

By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [(\frac{a_2-a_1}{2c_1} + \frac{b_2-b_1}{2c_2})\alpha + a_1 + b_1, (\frac{a_1-a_2}{2c_1} + \frac{b_1-b_2}{2c_2})\alpha + a_2 + b_2]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[a_1 + b_1, a_2 + b_2]^c$ and $\mu_{A(+)B}(x) = \frac{c_1c_2(a_2-a_1+b_2-b_1)}{c_2(a_2-a_1)+c_1(b_2-b_1)}$ at $x = \frac{a_1+b_1+a_2+b_2}{2}$. By the routine calculation, we have

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < a_1 + b_1,\ a_2 + b_2 \le x, \\ \frac{2c_1c_2(x-a_1-b_1)}{c_2(a_2-a_1)+c_1(b_2-b_1)}, & a_1 + b_1 \le x < \frac{a_1+b_1+a_2+b_2}{2}, \\ -\frac{2c_1c_2(x-a_2-b_2)}{c_2(a_2-a_1)+c_1(b_2-b_1)}, & \frac{a_1+b_1+a_2+b_2}{2} \le x < a_2 + b_2, \end{cases}$$

i.e., $A(+)B = ((a_1 + b_1, \dfrac{c_1c_2(a_2 - a_1 + b_2 - b_1)}{c_2(a_2 - a_1) + c_1(b_2 - b_1)}, a_2 + b_2)).$

2. Subtraction :

By the above facts, $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [(\frac{a_2-a_1}{2c_1} - \frac{b_1-b_2}{2c_2})\alpha + a_1 - b_2, (\frac{a_1-a_2}{2c_1} - \frac{b_2-b_1}{2c_2})\alpha + a_2 - b_1]$. Thus $\mu_{A(-)B}(x) = 0$ on the interval $[a_1 - b_2, a_2 - b_1]^c$ and $\mu_{A(+)B}(x) = \frac{c_1c_2(a_2-a_1+b_2-b_1)}{c_2(a_2-a_1)+c_1(b_2-b_1)}$ at $x = \frac{a_1-b_1+a_2-b_2}{2}$. By the routine calculation, we have

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < a_1 - b_2,\ a_2 - b_1 \le x, \\ \frac{2c_1c_2(x-a_1+b_2)}{c_2(a_2-a_1)+c_1(b_2-b_1)}, & a_1 - b_2 \le x < \frac{a_1-b_1+a_2-b_2}{2}, \\ -\frac{2c_1c_2(x-a_2+b_1)}{c_2(a_2-a_1)+c_1(b_2-b_1)}, & \frac{a_1-b_1+a_2-b_2}{2} \le x < a_2 - b_1, \end{cases}$$

i.e., $A(-)B = ((a_1 - b_2, \dfrac{c_1c_2(a_2-a_1+b_2-b_1)}{c_2(a_2-a_1)+c_1(b_2-b_1)}, a_2 - b_1))$.

3. Multiplication :

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_1^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] = [\frac{(a_2-a_1)(b_2-b_1)}{4c_1c_2}\alpha^2 + (a_1+b_1)\alpha + a_1b_1, \frac{(a_1-a_2)(b_1-b_2)}{4c_1c_2}\alpha^2 + (a_2+b_2)\alpha + a_2b_2]$, $\mu_{A(\cdot)B}(x) = 0$ on the interval $[a_1b_1, a_2b_2]^c$. Thus $A(\cdot)B$ is a fuzzy set on (a_1b_1, a_2b_2), but need not to be symmetric triangular fuzzy number by Example 3.3.

4. Division :

Since $A_\alpha(/)B_\alpha = [\frac{a_1^{(\alpha)}}{b_2^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}] = [\frac{c_2}{c_1} \cdot \frac{(a_2-a_1)\alpha+2a_1c_1}{(b_1-b_2)\alpha+2b_2c_2}, \frac{c_2}{c_1} \cdot \frac{(a_1-a_2)\alpha+2a_2c_1}{(b_2-b_1)\alpha+2b_1c_2}]$, $\mu_{A(/)B}(x) = 0$ on the interval $[\frac{a_1}{b_2}, \frac{a_2}{b_1}]^c$. Thus $A(/)B$ is a fuzzy set on $(\frac{a_1}{b_2}, \frac{a_2}{b_1})$, but need not to be symmetric triangular fuzzy number by Example 3.3. □

Example 3.3. Let $A = ((1, \frac{4}{5}, 5))$ and $B = ((2, \frac{1}{2}, 8))$ be symmetric triangular fuzzy numbers, i.e.,

$$\mu_A(x) = \begin{cases} 0, & x < 1,\ 5 \le x, \\ \frac{2}{5}(x-1), & 1 \le x < 3, \\ -\frac{2}{5}(x-5), & 3 \le x < 5, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < 2,\ 8 \le x, \\ \frac{1}{6}(x-2), & 2 \le x < 5, \\ -\frac{1}{6}(x-8), & 5 \le x < 8, \end{cases}$$

we calculate exactly the above four operations using $\alpha-$ cuts.

Let A_α and B_α be the α-cuts of A and B, respectively. Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$. Since $\alpha = \frac{2}{5}(a_1^{(\alpha)} - 1)$ and $\alpha = -\frac{2}{5}(a_2^{(\alpha)} - 5)$, we have $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\frac{5}{2}\alpha + 1, -\frac{5}{2}\alpha + 5]$. Since $\alpha = \frac{1}{6}(b_1^{(\alpha)} - 2)$ and $\alpha = -\frac{1}{6}(b_2^{(\alpha)} - 8)$, $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [6\alpha + 2, -6\alpha + 8]$.

1. Addition :

By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [\frac{17}{2}\alpha + 3, -\frac{17}{2}\alpha + 13]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[3, 13]^c$ and $\mu_{A(+)B}(x) = \frac{10}{17}$ at $x = 8$. By the routine calculation, we have

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < 3,\ 13 \le x, \\ \frac{2}{17}(x-3), & 3 \le x < 8, \\ -\frac{2}{17}(x-13), & 8 \le x < 13, \end{cases}$$

i.e., $A(+)B = ((3, \frac{10}{17}, 13))$.

2. Subtraction :

Since $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [\frac{17}{2}\alpha - 7, -\frac{17}{2}\alpha + 3]$, we have $\mu_{A(-)B}(x) = 0$ on the interval $[-7, 3]^c$ and $\mu_{A(-)B}(x) = \frac{10}{17}$ at $x = -2$. By the routine calculation, we have

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < -7,\ 3 \le x, \\ \frac{2}{17}(x+7), & -7 \le x < -2, \\ -\frac{2}{17}(x-3), & -23 \le x < 3, \end{cases}$$

i.e., $A(-)B = ((-7, \frac{10}{17}, 3))$.

3. Multiplication :

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_1^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] = [15\alpha^2 + 11\alpha + 2, 15\alpha^2 - 50\alpha + 40]$, $\mu_{A(\cdot)B}(x) = 0$ on the interval $[2, 40]^c$ and $\mu_{A(\cdot)B}(x) = \frac{38}{61}$ at $x = \frac{54600}{3721}$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x) = \begin{cases} 0, & x < 2,\ 40 \le x, \\ \frac{-11+\sqrt{1+60x}}{30}, & 2 \le x < \frac{54600}{3721}, \\ \frac{25-\sqrt{25+15x}}{15}, & \frac{54600}{3721} \le x < 40. \end{cases}$$

Thus $A(\cdot)B$ is not a symmetric triangular fuzzy number.

4. Division :

Since $A_\alpha(/)B_\alpha = [\frac{a_1^{(\alpha)}}{b_2^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}] = [\frac{5\alpha+2}{-4(3\alpha-4)}, \frac{-5(\alpha-2)}{4(3\alpha+1)}]$, $\mu_{A(/)B}(x) = 0$ on the interval $[\frac{1}{8}, \frac{5}{2}]^c$ and $\mu_{A(/)B}(x) = \frac{38}{61}$ at $x = \frac{3}{5}$. By the routine calculation, we have

$$\mu_{A(/)B}(x) = \begin{cases} 0, & x < \frac{1}{8},\ \frac{5}{2} \le x, \\ \frac{16x-2}{12x+5}, & \frac{1}{8} \le x < \frac{3}{5}, \\ \frac{10-4x}{12x+5}, & \frac{3}{5} \le x < \frac{5}{2}. \end{cases}$$

Thus $A(/)B$ is not a symmetric triangular fuzzy number.

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