Minimum Permanents on Certain Faces of Matrices Containing an Identity Submatrix

Seok-Zun Song
Department of Mathematics
Cheju National University
Cheju City, 690, Republic of Korea

Submitted by Richard A. Brualdi

ABSTRACT

We determine the minimum permanents on certain faces of Ω_n for the fully indecomposable (0,1) matrices containing an identity submatrix of some order. We also determine whether the given fully indecomposable (0,1) matrices are either cohesive and barycentric.

1. INTRODUCTION AND PRELIMINARIES

The recent solution [3] of the van der Waerden conjecture for the minimum permanent of matrices in Ω_n, the polytope of n-square doubly stochastic matrices, suggests the possiblity of determining the minimum permanent of matrices for faces of Ω_n. Several authors have already considered this problem for some faces [1–8].

Let $D=[d_{ij}]$ be an n-square (0,1) matrix, and let

$$\Omega(D)=\left\{X=[x_{ij}]\in\Omega_n \,\middle|\, x_{ij}=0 \text{ whenever } d_{ij}=0\right\}.$$

Then $\Omega(D)$ is a face of the polytope Ω_n, and hence, being a compact subset of a finite dimensional Euclidean space, contains a matrix A such that $\operatorname{per}A\leqslant\operatorname{per}X$ for all $X\in\Omega(D)$. Such a matrix A will be called a minimizing matrix on $\Omega(D)$.

52 Vanderbilt Ave., New York, NY 10017 0024-3795/88/$3.50

Brualdi [1] defined an n-square $(0,1)$-matrix D to be cohesive if there is a matrix Z in the interior of $\Omega(D)$ for which

$$\operatorname{per} Z = \min\{\operatorname{per} X : X \in \Omega(D)\}.$$

And he defined an n-square $(0,1)$-matrix D to be barycentric if

$$\operatorname{per} b(D) = \min\{\operatorname{per} X : X \in \Omega(D)\},$$

where the barycenter $b(D)$ of $\Omega(D)$ is given by

$$b(D) = \frac{1}{\operatorname{per} D} \sum_{P \leqslant D} P,$$

where the summation extends over the set of all permutation matrices P with $P \leqslant D$ and $\operatorname{per} D$ is their number.

In this paper we consider faces $\Omega(D)$, where D is a $(0,1)$ matrix having I_k as a submatrix, for some k. For some of these faces we are able to determine the minimum permanent and whether D is cohesive or barycentric. We also provide an example of a cohesive, nonbarycentric matrix in Theorem 2.3. Another example has been given by Foregger [12].

Let A be an n-square nonnegative matrix. If column k of A contains exactly two nonzero entries, say in rows i and j, then the $(n-1)$-square matrix $C(A)$ obtained from A by replacing row i with the sum of rows i and j and deleting row j and column k is called a *contraction* of A. If A has a row with exactly two nonzero entries, then $C(A^t)^t$ is also a contraction of A, where A^t is the transpose of A.

Lemma 1.1 (Foregger [4]). *Let $D = [d_{ij}]$ be an n-square fully indecomposable $(0,1)$ matrix, and let $A = [a_{ij}]$ be a minimizing matrix on $\Omega(D)$. Then A is fully indecomposable, and for (i, j) such that $d_{ij} = 1$,*

$$\operatorname{per} A(i|j) = \operatorname{per} A \qquad \text{if} \quad a_{ij} > 0, \tag{1.1}$$

$$\operatorname{per} A(i|j) \geqslant \operatorname{per} A \qquad \text{if} \quad a_{ij} = 0. \tag{1.2}$$

Lemma 1.2 (Foregger [4]). *Suppose $A \in \Omega_n$ is fully indecomposable and has a column (row) with exactly two positive entries. Then $\overline{C(\bar{A})}$ is $(n-1)$-square doubly stochastic and fully indecomposable, and*

$$2 \operatorname{per} A \geqslant 2 \operatorname{per} \bar{A} = \operatorname{per} C(\bar{A}) \geqslant \operatorname{per} \overline{C(\bar{A})},$$

where $\bar{A}$ $(\overline{C(\bar{A})})$ *is a minimizing matrix on* $\Omega(A)$ *(on* $\Omega(C(\bar{A}))$*, respectively) and* $C(\bar{A})$ *is a contraction of* $\bar{A}$.

Now, Lemma 1.1 *has been strengthened by Minc* [8], *with the aid of Egorycev's reformulation* [3] *of Alexandrov's inequality*

$$(\operatorname{per} A)^2 \geqslant \operatorname{per}[a_1,\dots,a_{n-1},a_{n-1}] \times \operatorname{per}[a_1,\dots,a_n,a_n]$$

for any nonnegative matrix $A=[a_1,\dots,a_n]$, *as follows.*

Lemma 1.3 (Minc [8]). *Let* $A=[a_{ij}]$ *be a minimizing matrix on* $\Omega(D)$, *where* $D=[d_1,\dots,d_n]$ *is an n-square* (0,1) *matrix. If, for some* $k \leqslant n$, $d_{j_1}=\cdots=d_{j_k}$, *and if, for some i,* $a_{ij_1}+\cdots+a_{ij_k} \neq 0$, *then* $\operatorname{per} A(i|j_t)=\operatorname{per} A$ *for* $t=1,\dots,k$.

By the linearity, with respect to each column, of the permanent function, Lemma 1.3 implies the averaging method, namely: If $A=[a_1,\dots,a_n]$ is a minimizing matrix on $\Omega(D)$, $D=[d_1,\dots,d_n]$, and if $d_1=d_2$, then

$$\operatorname{per}[ua_1+va_2, va_1+ua_2, a_3,\dots,a_n]=\operatorname{per} A$$

for any $u, v \geqslant 0$ with $u+v=1$.

Throughout this paper, $K_{p,q}$, for a pair (p,q) of positive integers, will denote the $p \times q$ matrix all of whose entries are 1, which will be denoted by K_p in case that $p=q$; and I_k will stand for the identity matrix of order k.

2. RESULTS

Proposition 2.1. *Let*

$$W_{m,n}=\left[\begin{array}{c|c} K_m & \begin{array}{c} 0_{m-1,n} \\ K_{1,n} \end{array} \\ \hline K_{n,m} & I_n \end{array}\right] \tag{2.1}$$

be an $(m+n)$*-square* (0,1) *matrix, for* $n \geqslant 2$. *Then* $W_{m,n}$ *is not cohesive, and the minimum permanent on* $\Omega(W_{m,n})$ *is*

$$\frac{(m-1)!}{m^{m-1}} \cdot \frac{(n-1)^{n-1}}{n^n}. \tag{2.2}$$

Proof. Choose A so that it has the minimum permanent on the face $\Omega(W_{m,n})$. Then A is fully indecomposable by Lemma 1.1. Since the first m columns of $W_{m,n}$ are the same, we can replace each of the first m columns by their average, by Lemma 1.3. Then the resulting matrix Z has the same permanent as A and has the following form:

$$Z = \left[\begin{array}{ccc|cccc} & \frac{1}{m}K_{m-1,m} & & & 0_{m-1,n} & & \\ \hline a & \cdots & a & mb_1 & mb_2 & \cdots & mb_n \\ \hline b_1 & \cdots & b_1 & x_1 & 0 & & \\ \vdots & & \vdots & & \ddots & & \\ b_n & \cdots & b_n & 0 & & x_n & \end{array}\right],$$

where $mb_j = 1 - x_j$ for $j = 1,\ldots,n$. Since Z is fully indecomposable, b_j and x_j are not zero fro $j = 1,\ldots,n$. Therefore

$$\operatorname{per} Z = \operatorname{per} Z(1|1) = \operatorname{per} Z(m|i)$$

for $i = m+1,\ldots,m+n$, by Lemma 1.1. In order to know the relation between b_1 and b_2, we calculate

$$\operatorname{per} Z(m|m+1) = m!\left(\frac{1}{m}\right)^{m-1} b_1 x_2 x_3 \cdots x_n,$$

$$\operatorname{per} Z(m|m+2) = m!\left(\frac{1}{m}\right)^{m-1} b_2 x_1 x_3 \cdots x_n.$$

Then their equality implies that $x_1 = x_2$. Similarly, we have $x_1 = x_j = x$ and $b_1 = b_j = (1-x)/m$ for all $j = 2,\ldots,n$. Using $a = [1 - n(1-x)]/m$, we have $x \neq \frac{1}{2}$ for $n \geqslant 3$ and

$$\begin{aligned} 0 &= \operatorname{per} Z(1|1) - \operatorname{per} Z(m|m+1) \\ &= (m-1)!\left(\frac{1}{m}\right)^{m-1} x^{n-1}\left[2nx^2 + (1-3n)x + n - 1 + x\right] \\ &= (m-1)!\left(\frac{1}{m}\right)^{m-1} x^{n-1}(2x-1)(nx-n+1). \end{aligned}$$

Hence we have $x=(n-1)/n$ for $n \geqslant 2$ and $a=0$. Therefore $W_{m,n}$ is not cohesive, and we calculate

$$\begin{aligned}\operatorname{per} Z &= \operatorname{per} Z(m|m+1)\\ &= \frac{(m-1)!}{m^{m-1}} \cdot \frac{(n-1)^{n-1}}{n^n},\end{aligned}$$

as required.

Brualdi [1] found the minimum permanent on $\Omega(W_{1,(n-1)})$. Hence we have generalized his result in Proposition 2.1.

LEMMA 2.2. *For $m \geqslant 2$, let*

$$V'_{m,3}=\left[\begin{array}{c|ccc} K_m & & K_{m,3} & \\ \hline & 1 & 0 & 0\\ K_{3,m} & 0 & 1 & 0\\ & 0 & 0 & 0\end{array}\right]. \tag{2.3}$$

Then $V'_{2,3}$ is not cohesive, and the minimum permanent on the face $\Omega(V'_{2,3})$ is $\frac{1}{16}$. For $m \geqslant 3$, $V'_{m,3}$ is cohesive and the minimum permanent on the face $\Omega(V'_{m,3})$ is

$$(m-1)!\left(\frac{m-1-2mb}{m^2}\right)^{m-2}\left((m-1)b^2+\frac{1-mb}{m^2}(m-1-2mb)\right), \tag{2.4}$$

where b is the unique real root of the equation:

$$m^3(m^2+m+2)b^3-2(m+1)m^3b^2+2m(m^2-1)b-(m-1)^2=0. \tag{2.5}$$

Proof. Using the averaging method on the first m rows and first m columns of a minimizing matrix on the face $\Omega(V'_{m,3})$, we may write a minimizing matrix A as follows:

$$A = \left[\begin{array}{ccc|ccc} & & & b_1 & b_2 & \frac{1}{m} \\ & aK_m & & \vdots & \vdots & \vdots \\ & & & b_1 & b_2 & \frac{1}{m} \\ \hline b_1 & \cdots & b_1 & x_1 & 0 & 0 \\ b_2 & \cdots & b_2 & 0 & x_2 & 0 \\ \frac{1}{m} & \cdots & \frac{1}{m} & 0 & 0 & 0 \end{array}\right].$$

From Theorem 4.4 of [6], it follows that $x_1 > 0$ and $x_2 > 0$. Then

$$\begin{aligned} 0 &= \operatorname{per} A(m+1|m+1) - \operatorname{per} A(m+2|m+2) \\ &= (m-1)!a^{m-2}[(m-1)(b_2-b_1)(b_2+b_1)+(x_2-x_1)a] \\ &= (m-1)!a^{m-2}(x_1-x_2)\left(\frac{(m-1)^2}{m^2} - am\right). \end{aligned}$$

Hence we know that $x_1 = x_2$ or $am^3 = (m-1)^2$.

For $m = 2$, we have $x_1 = x_2 = \frac{1}{2}$ and $a = 0$ from $\operatorname{per} A(1|3) = \operatorname{per} A(1|4) = \operatorname{per} A(1|5)$. Thus the minimum permanent is $\frac{1}{16}$, and $V'_{2,3}$ is not cohesive.

Let $m \geqslant 3$. Assume that $x_1 \neq x_2$, so that $am^3 = (m-1)^2$. Then we have

$$\begin{aligned} 0 &= \operatorname{per} A(1|m+1) - \operatorname{per} A(1|m+2) \\ &= (x_1-x_2)(m-2)!\frac{(m-1)^2}{m^2}a^{m-3}[(m-2)b_1b_2 - a]. \end{aligned}$$

The assumption $x_1 \neq x_2$ implies that $(m-2)b_1b_2 = a$. Since b_1, b_2 are less than $1/m$, b_1b_2 must be less than $1/m^2$. But then

$$b_1b_2 = \frac{a}{m-2} = \frac{(m-1)^2}{(m-2)m^3} > \frac{1}{m^2} > b_1b_2.$$

This is a contradiction. So we conclude $x_1 = x_2$. Therefore,

$$\begin{aligned} 0 &= \operatorname{per} A(m+2|m+2) - \operatorname{per} A(1|m+2) \\ &= \frac{(m-1)!}{m}a^{m-3}[ma\{(m-1)b_1^2 + ax_1\} \end{aligned}$$

$$-(m-1)b_1\{(m-2)b_1^2+ax_1\}\Big].$$

Thus, the quantity in the brackets is zero. Using

$$x_1=1-mb_1 \quad \text{and} \quad a=\frac{1}{m}\left(\frac{m-1}{m}-2b_1\right),$$

we obtain

$$f(b_1)=(m^2+m+2)b_1^3-2(m+1)b_1^2+\frac{2(m^2-1)}{m^2}b_1-\frac{(m-1)^2}{m^3}=0.$$

Since $f'(b_1)>0$ and $f(0)<0$, and since

$$f\left(\frac{1}{m}\right)=\frac{m-1}{m^3}>0,$$

$f(b_1)=0$ has a unique real root in $(0,1/m)$. Therefore the minimum value on the face $\Omega(V'_{m,3})$ is $\operatorname{per}A=\operatorname{per}A(m+2|m+2)$, which is the value given by (2.4) and (2.5). ∎

Theorem 2.3. *For $m\geqslant 2$, let*

$$V_{m,3}=\begin{bmatrix} K_m & K_{m,3} \\ K_{3,m} & I_3 \end{bmatrix} \tag{2.6}$$

be an $(m+3)$-square $(0,1)$ matrix which contains I_3 as a submatrix. Then we have a minimizing matrix form on the face $\Omega(V_{m,3})$ as follows:

$$A=\left[\begin{array}{c|ccc} aK_m & \bar{b}_1 & \bar{b}_2 & \bar{b}_2 \\ \hline \bar{b}'_1 & x_1 & 0 & 0 \\ \bar{b}'_2 & 0 & x_2 & 0 \\ \bar{b}'_2 & 0 & 0 & x_2 \end{array}\right]$$

where $\bar{b}_i$ $(\bar{b}'_i)$ is a column (row) vector with b_i as all its entries for $i=1,2$. This form A shows that $V_{m,3}$ is cohesive and not barycentric. In particular,

(1) *the minimum permanent on the face $\Omega(V_{2,3})$ is*

$$\tfrac{1}{2}(1-2b)^2(1-5b+12b^2), \tag{2.7}$$

where b is the unique real root of the equation

$$44b^3 - 16b^2 + 9b - 1 = 0; \qquad (2.7\text{-}1)$$

(2) *the minimum permanent on the face* $\Omega(V_{m,3})$ *is*

$$m!a^{m-2}\left[(m-1)mb^4 + 2maxb^2 + x^2a^2\right], \qquad (2.8)$$

where $ma = 1 - 3b$, $x = 1 - mb$, *and b is a real root of*

$$\left(m^2 + 6m + 20 + \frac{27}{m}\right)b^5 - \left(3m + 21 + \frac{57}{m} + \frac{54}{m^2}\right)b^4 + \left(5 + \frac{31}{m} + \frac{62}{m^2} + \frac{27}{m^3}\right)b^3$$

$$- \left(\frac{5}{m} + \frac{24}{m^2} + \frac{27}{m^3}\right)b^2 + \left(\frac{3}{m^2} + \frac{9}{m^3}\right)b - \frac{1}{m^3} = 0 \qquad (2.8\text{-}1)$$

for $m \geqslant 5$.

Proof. Using the averaging method on the first m rows and the first m columns of a minimizing matrix on the face $\Omega(V_{m,3})$, we may write a minimizing matrix A as follows:

$$A = \left[\begin{array}{c|ccc} aK_m & \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \\ \hline \bar{b}_1' & x_1 & 0 & 0 \\ \bar{b}_2' & 0 & x_2 & 0 \\ \bar{b}_3' & 0 & 0 & x_3 \end{array}\right], \qquad (2.9)$$

where $\bar{b}_i$ ($\bar{b}_i'$) is a column (row) vector with b_i as all its entries for $i = 1, 2, 3$. If x_3 were zero, then the minimum permanent would equal (2.4) in Lemma 2.2 for $m \geqslant 3$ or $\frac{1}{16}$ for $m = 2$. For $m = 2$, $\operatorname{per} A(5|5) = \frac{1}{64} < \frac{1}{16} = \operatorname{per} A$. For $m \geqslant 3$,

$$\begin{aligned} &\operatorname{per} A - \operatorname{per} A(m+3|m+3) \\ &= \operatorname{per} A(m+2|m+2) - \operatorname{per} A(m+3|m+3) \\ &= (m-1)!a^{m-2}\big[b_1^2\{(m-1) - m^2(m-1)b_1^2 - m^2x_1a\} \\ &\qquad + x_1a(1 - m^2b_1^2 - mx_1a)\big] \\ &> 0, \end{aligned}$$

since

$$(m-1)-m^2(m-1)b_1^2-m^2x_1a=m(m+1)b_1(1-mb_1)>0$$

and

$$1-m^2b_1^2-mx_1a=-(m+2)mb_1^2+(m+1)b_1+\frac{1}{m}>0$$

for $0<b_1<1/m$. This contradicts Lemma 1.1. Hence x_3 is not zero. Similarly, x_1 and x_2 are not zero. Since A is fully indecomposable, a and b_1, b_2, b_3 are not zero. So $V_{m,3}$ is a cohesive matrix.

The barycenter of $B_{m,3}$ is

$$b(V_{m,3})=\begin{bmatrix} aK_m & bK_{m,3} \\ bK_{3,m} & xI_3 \end{bmatrix}, \qquad (2.10)$$

where

$$a=\frac{m^3-3m^2+5m-2}{m(m^3+2m+1)}, \qquad b=\frac{m^2-m+1}{m^3+2m+1}, \qquad x=\frac{m^2+m+1}{m^3+2m+1}. \qquad (2.10\text{-}1)$$

Since $\operatorname{per} b(V_{m,3})(1|m+3)-\operatorname{per} b(V_{m,3})(m+3|m+3)<0$, $b(V_{m,3})$ is not a minimizing matrix. So $V_{m,3}$ is not barycentric.

In order to find a minimizing matrix, we calculate

$$\begin{aligned} 0 &= \operatorname{per}A(m+1|m+1)-\operatorname{per}A(m+2|m+2) \\ &= m!ma^{m-2}(b_2-b_1)\big[(b_1+b_2)\{(m-1)b_3^2-mab_3+a\} \\ &\qquad -a(a-mab_3+mb_3^2)\big]. \end{aligned}$$

Since $b_1+b_2=1-ma-b_3$, the equation becomes

$$\begin{aligned} &m!ma^{m-2}(b_1-b_2)\big[(m-1)b_3^3-(m-1)(1-ma)b_3^2 \\ &\qquad +a(m+1)(1-ma)b_3-a(1-a-ma)\big]=0. \qquad (2.11) \end{aligned}$$

Similarly, we have

$$0 = \operatorname{per} A(m+1|m+1) - \operatorname{per} A(m+3|m+3)$$
$$= m!ma^{m-2}(b_1 - b_3)\big[(m-1)b_2^3 - (m-1)(1-ma)b_2^2$$
$$+ a(m+1)(1-ma)b_2 - a(1-a-ma)\big], \quad (2.11\text{-}1)$$
$$0 = \operatorname{per} A(m+2|m+2) - \operatorname{per} A(m+3|m+3)$$
$$= m!ma^{m-2}(b_2 - b_3)\big[(m-1)b_1^3 - (m-1)(1-ma)b_1^2$$
$$+ a(m+1)(1-ma)b_1 - a(1-a-ma)\big]. \quad (2.11\text{-}2)$$

If b_1, b_2, and b_3 are all distinct, then they are the real roots of

$$g(b) = (m-1)b^3 - (m-1)(1-ma)b^2 + a(m+1)(1-ma)b$$
$$- a(1-a-ma) = 0,$$

from (2.11), (2.11-1), and (2.11-2). Therefore, $b_1 + b_2 + b_3 = 1 - ma$, $b_1b_2 + b_2b_3 + b_3b_1 = [1/(m-1)]a(m+1)(1-ma)$. Hence

$$0 < b_1^2 + b_2^2 + b_3^2 = (b_1 + b_2 + b_3)^2 - 2(b_1b_2 + b_2b_3 + b_3b_1)$$
$$= -(1-ma)\left(\frac{m^2+m+2}{m-1}a - 1\right).$$

Since ma is less than 1, we have

$$0 < a < \frac{m-1}{m^2+m+2}. \quad (2.12)$$

Since b_1, b_2, and b_3 must be in $(0, 1/m)$, we have $g(0)g(1/m) < 0$. That is, $a[(m+1)a - 1][a - (m^2 - 2m + 1)/m^3] < 0$. Since $a > 0$, we have

$$\frac{m^2 - 2m + 1}{m^3} < a < \frac{1}{m+1}. \quad (2.13)$$

From (2.12) and (2.13), we have a contradiction that $(m-1)/(m^2+m+2) < (m^2 - 2m + 1)/m^3$. Hence b_1, b_2, and b_3 are not all distinct. Therefore a minimizing matrix on the face $\Omega(V_{m,3})$ is of the form A in (2.9) with $b_2 = b_3$.

(1) Now, let us consider the case $m = 2$. Since b_1, b_2 are not zero, we have

$$0 = \operatorname{per} A(1|3) - \operatorname{per} A(1|4)$$
$$= (b_1 - b_2)\left[2(2b_1+3)b_2^2 - 4b_2 + 1 - b_1\right].$$

But the quantity in the brackets above is positive for arbitrary b_1, b_2 in $(0, \frac{1}{2})$. Hence we have $b_1 = b_2$ and $x_1 = x_2$. Since $a \neq 0$,

$$0 = \operatorname{per} A(1|1) - \operatorname{per} A(1|3)$$
$$= (1 - 2b_1)\left(-22b_1^3 + 16b_1^2 - \tfrac{9}{2}b_1 + \tfrac{1}{2}\right).$$

Since $0 < b_1 < \frac{1}{2}$, we have $h(b_1) = 22b_1^3 - 16b_1^2 + \frac{9}{2}b_1 - \frac{1}{2} = 0$. Since $h'(b_1) > 0$, $h(0) = -\frac{1}{2} < 0$, and $h(\frac{1}{2}) = \frac{1}{2} > 0$, $h(b_1)$ has a unique real root in $(0, \frac{1}{2})$. Hence we have the minimum permanent on the face $\Omega(V_{2,3})$ from $\operatorname{per} A = \operatorname{per} A(1|1)$, in agreement with (2.7) and (2.7-1). [We remark that this minimum permanent on the face $\Omega(V_{2,3})$ is about 0.0478105 when $b_1 \approx 0.295134$, $a \approx 0.057299$, and $x_1 \approx 0.409732$.]

(2) Let $m \geqslant 5$. A minimizing matrix is of the form A in (2.9) with $b_2 = b_3$. Assume $b_1 \neq b_2$. Then $g(b_3) = 0$ must have at least one real root in $(0, 1/m)$ from (2.11).

Case 1. $g(b_3) = 0$ *has one real root in* $(0, 1/m)$ *and two real roots in* $[1/m, \infty)$. This case cannot hold, from (2.12) and (2.13).

Case 2. $g(b_3) = 0$ *has two real roots in* $(0, 1/m)$ *and one real root in* $[1/m, \infty)$. Let us change ma by $1 - b_1 - 2b_2$ at $g(b_3) = 0$. Then we have

$$F(b_3) = -\left(\frac{m^2+3m+4}{m}\right)b_3^3 + \left(-\frac{m^2+3m+4}{m}b_1 + \frac{2m^2+6m+4}{m^2}\right)b_3^2$$
$$+\left(-\frac{m+1}{m}b_1^2 + \frac{m^2+5m+4}{m^2}b_1 - \frac{2(m+2)}{m^2}\right)b_3$$
$$+\frac{1}{m^2}(1-b_1)\left[1-(m+1)b_1\right] = 0.$$

Since the product of three real roots of $F(b_3) = 0$ is positive, we have $0 < b_1 < 1/(m+1)$. Hence $F(0) > 0$ and $F(1/m) = -(b_1/m + 4/m^3) < 0$. Therefore $F(b_2) = 0$ cannot have two real roots in $(0, 1/m)$, so $g(b_3) = 0$. This case cannot hold.

Case 3. *$g(b_3) = 0$ has three real roots in $(0,1/m)$.* This case cannot hold, from (2.12) and (2.13).

Case 4. *$g(b_3) = 0$ has one real root in $(0,1/m)$ and two imaginary roots.* Then we have $(m-1)^2/m^3 < a < 1/(m+1)$, since $f(0)f(1/m) < 0$. Consider

$$0 = \operatorname{per} A(1|m+3) - \operatorname{per} A(m+3|m+3)$$
$$= m!a^{m-3}\big[(m-1)b_1^2 b_2^2\{(m-2)b_3 - ma\}$$
$$+ ax_1 b_2^2\{(m-1)b_3 - ma\}$$
$$+ ax_2 b_1^2\{(m-1)b_3 - ma\} + x_1 x_2 a^2 (b_3 - a)\big].$$

From this equation, we have that

$$\frac{m-2}{m} b_3 < a < b_3.$$

And similarly, we have

$$\frac{m-2}{m} b_i < a < b_i \qquad \text{for} \quad i = 1,2,3.$$

So $1 = ma + b_1 + b_2 + b_3 > (m+3)a$ and hence $a < 1/(m+3)$. Since a satisfies (2.13), we have a contradiction as follows:

$$\frac{1}{m+3} < \frac{(m-1)^2}{m^3} < a < \frac{1}{m+1}$$

for $m \geqslant 5$.

By cases 1 to 4, we have $b_1 = b_2 = b_3$. Hence we have (2.8-1) from the equality of $\operatorname{per} A(1|m+3)$ and $\operatorname{per} A(m+3|m+3)$. And the minimum permanent on the face $\Omega(V_{m,3})$ is $\operatorname{per} A = \operatorname{per} A(m+3|m+3)$, in agreement with (2.8) and (2.8-1). ■

We remark that cases 1–3 cannot hold for $m = 3,4$. But we do not know whether or not case 4 holds for $m = 3,4$.

PROPOSITION 2.4. *For $n > 1$, let*

$$U = \begin{bmatrix} 0_{n-1} & K_{n-1,n} \\ K_{n,n-1} & I_n \end{bmatrix}$$

be a $2n-1$ *square* $(0,1)$ *matrix. Then the minimum permanent on the face* $\Omega(U)$ *is* $((n-1)!/n^{n-1})^2$, *and* U *is barycentric.*

Proof. Using the averaging method, we may write a minimizing matrix A on the face $\Omega(U)$ as follows:

$$A = \left[\begin{array}{c|cccc} 0_{n-1} & \bar{b}_1 & \bar{b}_2 & \cdots & \bar{b}_n \\ \hline \bar{b}'_1 & x_1 & & 0 & \\ \bar{b}'_2 & & x_2 & & \\ \vdots & & & \ddots & \\ \bar{b}'_n & 0 & & & x_n \end{array}\right],$$

where $\bar{b}_i$ ($\bar{b}'_i$) is a column (row) vector of order $n-1$ with all entries b_i. Since A is fully indecomposable, each b_i is not zero. If some x_i were zero, say $x_1 = 0$, then $b_1 = 1/(n-1)$. Since not all x_i are zero, we may assume that x_2 is not zero without loss of generality. Then $b_2 = (1-x_2)/(n-1) < b_1$, and

$$\begin{aligned} \operatorname{per} A(n|n) - \operatorname{per} A &= \operatorname{per} A(n|n) - \operatorname{per} A(n+1|n+1) \\ &= [(n-1)!b_3 \cdots b_n]^2 (b_2 - b_1)(b_2 + b_1) < 0. \end{aligned}$$

This contradicts Lemma 1.1. Hence x_1 is not zero. Similarly, not all x_i are zero. Therefore $b_1 = b_2$ from the equation

$$\begin{aligned} 0 &= \operatorname{per} A(n|n) - \operatorname{per} A(n+1|n+1) \\ &= [(n-1)!b_3 \cdots b_n]^2 (b_2 - b_1)(b_2 + b_1). \end{aligned}$$

Similarly, we have that all $x_i = b_i = 1/n$ for $i = 1, \ldots, n$. Then A is the barycenter of $\Omega(U)$. And the minimum permanent on the face $\Omega(U)$ is $\operatorname{per} A = \operatorname{per} A(n|n) = [n-1)!/n^{n-1}]^2$, as required. ■

For $n \geqslant 3$, let

$$U_{2,n} = \begin{bmatrix} 0_2 & K_{2,n} \\ K_{n,2} & I_n \end{bmatrix}, \qquad V_{2,n} = \begin{bmatrix} K_2 & K_{2,n} \\ K_{2,n} & I_n \end{bmatrix} \tag{2.14}$$

be $(n+2)$-square $(0,1)$ matrices that contain the identity submatrix of order n.

THEOREM 2.5. *For $n \geqslant 4$, if*

$$A = \left[\begin{array}{cc|cccc} \multicolumn{2}{c|}{\multirow{2}{*}{0_2}} & b_1 & b_2 & \cdots & b_n \\ & & b_1 & b_2 & \cdots & b_n \\ \hline b_1 & b_1 & x_1 & & & 0 \\ \vdots & \vdots & & \ddots & & \\ b_n & b_n & 0 & & & x_n \end{array}\right] \tag{2.15}$$

is a minimizing matrix on the face $\Omega(U_{2,n})$, then x_i and b_i are nonzero for $=1,\dots,n$ (i.e., $U_{2,n}$ is cohesive). Moreover, a local minimum for the permanent on the face $\Omega(U_{2,n})$ occurs at the barycenter $b(U_{2,n})$, and

$$\operatorname{per} b(U_{2,n}) = \frac{2(n-1)(n-2)^{n-2}}{n^{n+1}}.$$

Proof. Since $U_{2,n}$ is fully indecomposable, b_i is not zero for $i=1,\dots,n$. If some $x_i=0$, say $x_1=0$ without loss of generality, then the 3rd column has only two nonzero entries b_1. Then the contraction $C(A)$ of A on the 3rd column is

$$C(A) = \left[\begin{array}{cc|ccccc} 0 & 0 & 2b_2 & 2b_3 & \cdots & 2b_n \\ \hline b_1 & b_1 & 0 & 0 & \cdots & 0 \\ b_2 & b_2 & x_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & b_n & 0 & 0 & \cdots & x_n \end{array}\right].$$

From Lemma 1.2, we have

$$2\operatorname{per} A \geqslant \operatorname{per} \overline{C(A)}, \tag{2.16}$$

where $\overline{C(A)}$ is a minimizing matrix on the face $\Omega(C(A))$. Also we have that

$$\operatorname{per} \overline{C(A)} = \frac{(n-2)^{n-2}}{2(n-1)^{n-1}} \tag{2.17}$$

by Proposition 2.1. But the barycenter $b(U_{2,n})$ of $\Omega(U_{2,n})$ equals A in (2.15) with $b_i = 1/n$, $x_i = (n-2)/n$ for $i=1,\dots,n$. By (2.16) and (2.17), we have a contradiction as follows:

$$\operatorname{per}\overline{C(A)} = \frac{(n-2)^{n-2}}{2(n-1)^{n-1}} > 2\frac{2(n-1)(n-2)^{n-2}}{n^{n+1}}$$

$$= 2\operatorname{per} b(U_{2,n}) \geqslant 2\operatorname{per} A \geqslant \operatorname{per}\overline{C(A)}$$

for $n \geqslant 4$. Therefore x_i is not zero for $i = 1,\ldots,n$. That is, $U_{2,n}$ is a cohesive matrix.

Now, in order to obtain a local minimum permanent at the barycenter, we assume that

$$(n-3)b_i < x_i < (n-1)b_i \tag{2.18}$$

for $i = 1,\ldots,n$. Then we obtain $1/(n+1) < b_i < 1/(n-1)$ from the doubly stochastic property for $i = 1,\ldots,n$. And

$$\begin{aligned} 0 &= \operatorname{per} A(1|3) - \operatorname{per} A(1|4) \\ &= 2(b_1 - b_2)\big(b_3^2 x_4 \cdots x_n + x_3 b_4^2 x_5 \cdots x_n \\ &\qquad + \cdots + x_3 x_4 \cdots x_{n-1} b_n^2 - b_1 b_2 x_3 \cdots x_n\big). \end{aligned} \tag{2.19}$$

But the interior of the second parenthesis is greater than

$$\begin{aligned} b_3 x_4 \cdots x_n\left(b_3 - \frac{n-1}{n-2}b_1 b_2\right) + x_3 b_4 x_5 \cdots x_n\left(b_4 - \frac{n-1}{n-2}b_1 b_2\right) \\ + \cdots + x_3 x_4 \cdots x_{n-1} b_n\left(b_n - \frac{n-1}{n-2}b_1 b_2\right) > 0, \end{aligned}$$

since

$$b_i - \frac{n-1}{n-2}b_1 b_2 > \frac{1}{n+1} - \frac{1}{(n-2)(n-1)} > 0$$

for $i = 3,4,\ldots,n$ and $n \geqslant 4$. Hence $b_1 = b_2$ from (2.19). Similarly, we have that $b_1 = b_i = 1/n$ and $x_1 = x_i = (n-2)/n$ for all $i = 2,\ldots,n$. In this case A is the barycenter of $\Omega(U_{2,n})$ and a local minimum permanent on $\Omega(U_{2,n})$ is obtained as required. ■

REMARK 2.6. For $n = 3$, 4, and 5, it can be shown that U_{2+n} is in fact barycentric. We omit the proof.

THEOREM 2.7. *For $n \geqslant 4$, a local minimum permanent on the face*

$\Omega(V_{2,n})$ for $V_{2,n}$ in (2.14) is

$$\frac{2(n-1)(n-2)^{n-2}}{n^{n+1}}, \tag{2.20}$$

which occurs at the barycenter $b(U_{2,n})$ of the face $\Omega(U_{2,n})$ in Theorem 2.5.
In particular, the value in (2.20) is the global minimum permanent for $n=4$ or 5.

Proof. Assume that

$$Z = \left[\begin{array}{cc|cccc} a & a & b_1 & b_2 & \cdots & b_n \\ a & a & b_1 & b_2 & \cdots & b_n \\ \hline b_1 & b_2 & x_1 & 0 & \cdots & 0 \\ b_2 & b_2 & 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & & & \ddots & \\ b_n & b_n & & 0 & & x_n \end{array}\right] \tag{2.21}$$

is a minimizing matrix on the face $\Omega(V_{2,n})$. Then all x_i and b_i are not zero for $i=1,2,\ldots,n$, by the same method as in the proof of Theorem 2.5. And there exists some i such that $x_i \geqslant 2b_i$. We may assume that $x_n \geqslant 2b_n$ without loss of generality. If $a \neq 0$, then $\operatorname{per} Z(1|1) = \operatorname{per} Z$ and hence

$$\begin{aligned} 0 &= \operatorname{per} Z(1|1) - \operatorname{per} Z(1|n+2) \\ &= (x_n - 2b_n)\big(ax_1x_2\cdots x_{n-1} + b_1^2x_2x_3\cdots x_{n-1} \\ &\qquad + \cdots + b_{n-1}^2x_1x_2\cdots x_{n-2}\big) \\ &\quad + b_n^2x_1x_2\cdots x_{n-1} \\ &> 0. \end{aligned}$$

This is a contradiction. So a must be zero in (2.21), and the form A in (2.15) becomes a minimizing matrix on $\Omega(V_{2,n})$. Hence we have a local minimizing matrix $b(U_{2+n})$, and have a local minimum permanent in (2.20) on the face $\Omega(V_{2+n})$ by Theorem 2.5.

In particular, Remark 2.6 implies that the value in (2.20) is the global minimum permanent for $n=4$ and 5. ■

REFERENCES

1 R. A. Brualdi, An interesting face of the polytope of doubly stochastic matrices, *Linear and Multilinear Algebra* 15:5–18 (1985).

2 D. K. Chang, Minimum permanents of doubly stochastic matrices with one fixed entry, *Linear and Multilinear Algebra* 15:313–317 (1984).

3 G. P. Egorycev, The solution of the van der Waerden problem for permanents, *Dokl. Akad. Nauk. SSSR* 258:1041–1044 (1981).

4 T. H. Foregger, On the minimum value of the permanent of a nearly decomposable doubly stochastic matrix, *Linear Algebra Appl.* 32:75–85 (1980).

5 S. Friedland, A proof of a generalized van der Waerden conjecture on permanents, *Linear and Multilinear Algebra* 11:107–120 (1982).

6 S. G. Hwang, Minimum permanent on faces of staircase type of the polytope of doubly stochastic matrices. *Linear and Multilinear Algebra* 18:271–306 (1985).

7 P. Knopp and R. Sinkhorn, Minimum permanents of doubly stochastic matrices with at least one zero entry, *Linear and Multilinear Algebra* 11:351–355 (1982).

8 H. Minc, Minimum permanents of doubly stochastic matrices with prescribed zero entries, *Linear and Multilinear Algebra* 15:225–243 (1984).

9 H. Minc, *Permananets, Encyclopedia of Mathematics and Its Applications*, Vol. 6, Addison-Wesley, 1978.

10 H. Minc, Theory of permanents 1978–1981, *Linear and Multilinear Algebra* 12:227–263 (1983).

11 H. Minc, Theory of permanents 1982–1985, *Linear and Multilinear Algebra*, to appear.

12 T. Foregger, Minimum permanents of multiplexes, *Linear Algebra Appl.* 87:197–211 (1987).

Received 10 July 1987; final manuscript accepted 7 January 1988