On the Covariant Derivative of the Nonholonomic Vectors in Vn

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Vn 공간에서 Nonholonomic Vector 들의 공변미분에 관하여

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I. Introduction

Let V_n be a n-dimensional Riemannian space referred to a real coordinate system x^ν and defined by a fundamental metric tensor $T_{\lambda\mu}$, whose determinant

$$(1.1)\quad T \overset{def}{=} Det\,((T_{\lambda\mu}))\neq 0.$$

According to (1.1) there is a unique tensor $T^{\lambda\nu}=T^{\nu\lambda}$ defined by

$$(1.2)\quad T_{\lambda\mu}T^{\lambda\nu} \overset{def}{=} \delta^\nu_\mu$$

Let $\underset{i}{e}^\nu$ (i=1,2, ..., n) be a set of n linearly independent vectors. Then there is a unique reciprocal set of n linearly independent covariant vectors $\overset{i}{e}_\lambda$ (i=1,2, . . . , n) satisfying

$$(1.3)\quad \underset{i}{e}^\nu \overset{i}{e}_\lambda=\delta^\nu_\lambda{}^{**}$$
$$\underset{j}{e}^\lambda \overset{i}{e}_\lambda=\delta^i_j$$

With the vectors $\underset{i}{e}^\nu$ and $\overset{i}{e}_\lambda$ a nonholonomic frame of V_n defined in the following way

If $T^{\nu\cdots}_{\lambda\cdots}$ are holonomic components of a tensor, then its nonholonomic components are defined by

$$(1.4)\quad T^{i\cdots}_{j\cdots} \overset{def}{=} T^{\nu\cdots}_{\lambda\cdots}\,\overset{i}{e}_\nu\,\underset{j}{e}^\lambda\cdots$$

From (1.3) and (1.4)

$$(1.5)\quad T^{\nu\cdots}_{\lambda\cdots} \overset{def}{=} T^{i\cdots}_{j\cdots}\,\underset{i}{e}^\nu\,\overset{j}{e}_\lambda\cdots$$

II. Preliminary results

In this section, for our further discussion, results obtained in our previous paper will be introduced without proof.

Theorem 2.1. We have

$$(2.1)\quad T^{\lambda\mu}=\underset{i}{e}^\lambda\,T^{ij}\,\underset{j}{e}^\mu=\overset{i}{e}_\mu\,T_{ij}\,\overset{j}{e}_\mu$$

(**) Throughout the present paper, Greek indices take values 1,2, ⋯, n unles explicitly stated otherwise and follow the summation convention, while Roman indices are used for the nonholonomic componts of a tensor and run from 1 to n. Roman indices also follow the summation convention.

Theorem 2.2. The derivative of $\overset{\lambda}{\underset{i}{e}}$ is negative self-adjoint. That is

$$(2.2)\quad \partial_k(\overset{j}{e}_\lambda)\,\underset{j}{e}^{\mu} = -\partial_k(\underset{j}{e}^{\mu})\,\overset{j}{e}_\lambda.$$

Theorem 2.3. The nonholonomic components of the christoffel symbols of the second kind may be expressed as

$$(2.3)\quad \left\{ {i \atop jk} \right\} = \overset{i}{e}_\nu\, \underset{k}{e}^{\mu}(\nabla_\mu \underset{j}{e}^{\nu}) = -\underset{j}{e}^{\nu}\, \underset{k}{e}^{\mu}(\nabla_\mu \overset{i}{e}_\nu)$$

, where ∇_k is the symbol of the covariant derivative with respect to $\left\{ {i \atop jk} \right\}$

Theorem 2.4. The holonomic components of the christoffel symbols, as follows;

$$(2.4)\text{a}\quad [\lambda\mu, w] = [ij, m]\, \overset{i}{e}_\lambda\, \overset{k}{e}_\mu\, \overset{m}{e}_w + a_{jk}(\partial_\mu \overset{j}{e}_\lambda)\, \overset{k}{e}_\mu$$

$$(2.4)\text{b}\left\{ {\nu \atop \lambda\mu} \right\} = \left\{ {i \atop jk} \right\} \underset{i}{e}^{\nu}\, \overset{j}{e}_\lambda\, \overset{k}{e}_\mu - (\partial_\mu \underset{j}{e}^{\nu})\, \overset{j}{e}_\lambda$$

$$= \left\{ {i \atop jk} \right\} \underset{i}{e}^{\nu}\, \overset{j}{e}_\lambda\, \overset{k}{e}_\mu + (\partial_\mu \overset{j}{e}_\lambda)\, \underset{j}{e}^{\nu}.$$

Theorem 2.5. The holonomic components of the christoffel symbols of the second kind may be expressed as

$$(2.5)\quad \left\{ {\nu \atop \lambda\mu} \right\} = -\overset{i}{e}_\lambda\, \overset{k}{e}_\mu (\nabla_k \underset{j}{e}^{\nu}) = \overset{k}{e}_\mu\, \underset{j}{e}^{\nu} (\nabla_\mu \overset{j}{e}_\beta)$$

III. Covariant Derivatives of the Nonholonomic Covariant ana Contravariant Vectors in Vn

We see the partial derivatives of the holonomic components of a vector is not components of a tensor in Vn

In this paper, reconstruct and invastigate the relationships between the partial derivative of the holonomic and nonholonomic components of a vector.

Take a coordinate system y^i for which we have at a point p of V_n

$$(3.1)\quad \frac{\partial y^i}{\partial x^\lambda} = \overset{i}{e}_\lambda, \quad \frac{\partial x^\nu}{\partial y^i} = \underset{i}{e}^{\nu}.$$

We have

Theorem 3.1. The covariant derivative of the holonomic covariant vector, is given by

$$(3.2)\quad \nabla_\mu(a_\lambda) = \left[\frac{\partial a_j}{\partial y^k} - a_i\left\{ {i \atop jk} \right\}\right] \overset{k}{e}_\mu\, \overset{j}{e}_\lambda$$

$$= \nabla_k(a_j)\, \overset{k}{e}_\mu\, \overset{j}{e}_\lambda.$$

Proof. By means of the covariant derivative of holonomic vector

$$(3.3)\quad \nabla_\mu(a_\lambda) = \frac{\partial a_\lambda}{\partial x^\mu} - a_\nu\left\{ {\nu \atop \lambda\mu} \right\}$$

Using (1.5) and (2.4)b,

$$(3.4)\quad \nabla_\mu(a_\lambda) = \frac{\partial}{\partial x^\mu}(a_j\, \overset{j}{e}_\lambda) - a_i\, \overset{i}{e}_\nu\Big[\left\{ {i \atop jk} \right\} \underset{i}{e}^{\nu}\, \overset{j}{e}_\lambda\, \overset{k}{e}_\mu + \partial_\mu(\overset{j}{e}_\lambda)\, \underset{j}{e}^{\nu}\Big]$$

By virtiue of (1.3) and

$$(3.5)\quad a_i\, \overset{i}{e}_\nu (\partial_\mu \overset{j}{e}_\lambda)\, \underset{j}{e}^{\nu} = a_j\left(\frac{\partial}{\partial y^k}\, \overset{j}{e}_\lambda\right) \overset{k}{e}_\mu$$

Hence we obtain

$$(3.6)\quad \nabla_\mu(a_\lambda) = \left(\frac{\partial}{\partial y^k}\, a_j\right) \overset{j}{e}_\lambda\, \overset{k}{e}_\mu - a_i\left\{ {i \atop jk} \right\} \overset{j}{e}_\lambda\, \overset{k}{e}_\mu$$

$$= \nabla_k(a_j)\, \overset{j}{e}_\lambda\, \overset{k}{e}_\mu$$

, where $\nabla_k(a_j) = \dfrac{\partial a_j}{\partial y^k} - a_i\left\{ {i \atop jk} \right\}$

Theorem 3.2. We have the covariant derivative of the nonholonomic covariant vector is equivalent to

$$(3.7)\quad \nabla_k(a_j) = \left[\frac{\partial a_\lambda}{\partial x^\mu} - a_\nu\left\{ {\nu \atop \lambda\mu} \right\}\right] \underset{k}{e}^{\mu}\, \underset{j}{e}^{\lambda}$$

$$= \nabla_\mu(a_\lambda)\, \underset{k}{e}^{\mu}\, \underset{j}{e}^{\lambda}$$

Proof. Multiplying $e^{\mu}_{k}\ e^{\lambda}_{j}$ to both sides of (3.2) and using (2.1) and (3.3), we obtain (3.7).

Corollary 3.3. We have

$$(3.8)\ \nabla_\mu(a_\lambda) = \frac{\partial a}{\partial x^\mu} - a_j(\nabla_\mu \overset{j}{e}_\lambda)$$

Proof. Using (1.4), (2.4) and (3.3)

$$(3.9)\quad \nabla_\mu(a_\lambda) = \frac{\partial a_\lambda}{\partial x^\mu} - a_i \overset{i}{e}_\nu(\nabla_\mu \overset{j}{e}_\lambda)\overset{k}{e}_\mu \overset{\nu}{e}_j = \frac{\partial a_\lambda}{\partial x^\mu} - a_j(\nabla_\mu \overset{j}{e}_\lambda).$$

Corollary 3.4. We have

$$(3.10)\quad \nabla_\mu(a_\lambda) = \frac{\partial a_j}{\partial y^k}\overset{j}{e}_\lambda \overset{k}{e}_\mu + a_j(\nabla_\mu \overset{j}{e}_\beta).$$

Proof. From (3.2) and (2.3),

$$(3.11)\quad \nabla_\mu(a_\lambda) = \frac{\partial a_j}{\partial y^k}\overset{j}{e}_\lambda \overset{k}{e}_\mu - a_i(\nabla_\mu \overset{\alpha}{e}_j)\ \overset{i}{e}_\alpha \overset{r}{e}_k \overset{k}{e}_\mu \overset{j}{e}_\lambda$$

Making use of (1.3) and (2.2), we have (3.10).

Theorem 3.5. The covariant derivative of the holonomic contravariant vector may be expressed as following relation

$$(3.12)\ (\nabla_\mu a^\nu) = \nabla_k(a^i)\, e^\nu_i \overset{k}{e}_\mu .$$

Proof. By means of the covariant derivative of the holonomic contravariant vector

$$(3.13)\quad \nabla_\mu(a^\nu) = \frac{\partial a^\nu}{\partial x^\mu} + a^\lambda \left\{ {\nu \atop \lambda\mu} \right\}$$

From (1.5) and (2.4)b

$$(3.14.)\ \nabla_\mu(a^\nu) = \frac{\partial}{\partial x^\mu}(a^i e^\nu_i) + a^j e^\lambda_j \left[\left\{ {i \atop jk} \right\} e^\nu_i \overset{j}{e}_\lambda \overset{k}{e}_\mu + (\partial_\mu \overset{j}{e}_\lambda) e^\nu_j\right.$$

Using (2.2) and (3.1)

$$(3.15)\ \nabla_\mu(a^\nu) = \frac{\partial a^i}{\partial y^k} e^\nu_i \overset{k}{e}_\mu + a^j\left\{ {i \atop jk} \right\} e^\nu_i \overset{k}{e}_\mu + a^i\left(\frac{\partial}{\partial y^k} e^\nu_i\right)\overset{k}{e}_\mu - a^j(\partial_\mu e^\nu_j)$$

By virtiue of (1.3)

$$(3.16)\ a^i\left(\frac{\partial}{\partial y^k} e^\nu_i\right)\overset{k}{e}_\mu = a^j(\partial_\mu e^\nu_j).$$

We obtain

$$(3.17)\quad \nabla_\mu(a^\nu) \frac{\partial a^i}{\partial y^k} e^\nu_i \overset{k}{e}_\mu + a^j\left\{ {i \atop jk} \right\} e^\nu_i \overset{k}{e}_\mu = \nabla_\mu(a^i) e^\nu_j \overset{k}{e}_\mu$$

, where $\nabla_k(a^i) = \frac{\partial a^i}{\partial y^k} + a^j\left\{ {i \atop jk} \right\}$

Theorem 3.6. We have the covariant derivative of the nonholonomic contravariant vector, as follows

$$(3.18)\quad \nabla_k(a^i) = \nabla_\mu(a^\nu)\overset{i}{e}_\nu e^\mu_k$$

Proof. In order to prove (3.18), Multiplying $\overset{\ell}{e}_\nu\, e^\mu_m$ to both sides of (3.18) and using (1.3).

$$(3.19)\quad \nabla_\mu(a^\nu)\overset{j}{e}_\nu e^\mu_\ell = \nabla_\ell(a^j).$$

Replacing j by i and ℓ by k, we have (3.18).

Corollary 3.7. We have

$$(3.20)\quad \nabla_\mu(a^\nu) = \frac{\partial a^\nu}{\partial x^\mu} - a^i(\nabla_\mu e^\nu_i).$$

Proof. Making use of (2.5) and (3.13), (3.20) may be written in the form

$$(3.21)\quad \nabla_\mu(a^\nu) = \frac{\partial a^i}{\partial x^\mu} + a^j e^\lambda_j(\nabla_k \overset{j}{e}_\mu)\overset{k}{e}_\mu e^\nu_j = \frac{\partial a^\nu}{\partial x^\mu} - a^j(\nabla_\mu e^\nu_j).$$

Replacing i by j, we obtain (3.20).

Corollary 3.8. We have

$$(3.22)\quad \nabla_\mu(a^\nu) = \frac{\partial a^i}{\partial y^k} \, e_i^\nu \, e_\mu^k + a^i(\nabla_\mu e_i^\nu).$$

Proof. (3.23) can be also obtained from (3.17) by making use of (2.3) as follows

$$(3.23)\quad \nabla_\mu(a^\nu) = \frac{\partial a^j}{\partial y^k} \, e_i^\nu \, e_\mu^k + a^j(\nabla_\mu e_j^\nu)$$

$$e_\nu^i \, e_i^\nu \, e_k^\mu \, e_\mu^k$$

By means of (1.3) and the properties of the Kronecker deltas, obtained (3.22).

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國 文 抄 錄

Nonholonomic vector들의 derivative에 관한 성질은 이미 발표된바 있다. 본 논문에서는 Nonholonomic Tensor들의 성질을 Nonhnlonomic vector와 Nonholonomic정의 및 Holonomic Tensor들의 성질을 이용하여 보다 새로운 결과들을 얻으므로서 n-차원 Riemann 공간 V_n을 다른 각도에서 구성하고 연구할 수 있는 기초 이론을 정립코자 한다.