On the Mizohata Operator

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Mizohata 연산자에 대하여

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Summary

In this paper we prove:

1) $\frac{\partial u}{\partial t} + it\frac{\partial u}{\partial x} = f(x,t)$ has a unique c^∞ solution when f is analytic

2) Let $f(x,t) \in C_c^\infty(R^2)$ have the following properties:
$f(x,t) = f(x,-t)$ for all $(x,t) \in R^2$; the supp $f \cap \{x\text{-axis}\} = \{(0,0)\}$; $\iint_{R^2} f(x,t)\,dxdt \neq 0$.

Then $\frac{\partial u}{\partial t} + it\frac{\partial u}{\partial x} = f$ does not have c' solution.

I. Introduction

Throughout this paper Ω will denote an open subset of R^2, $C_c^\infty(\Omega)$ the space of C^∞ complex-valued functions in Ω having compact supports. We will denote a point in R^2 by (x, t).

Let L be a smooth complex vector field in Ω defined by

$$L = \frac{\partial}{\partial t} + ib(x,t)\frac{\partial}{\partial x}$$

where $b(x,t)$ is a real-valued C^∞ function in Ω.

When f and b are analytic, we know by the Cauchy-Kovalevska Theorem that

(1,1) $Lu = f$

has always a solution locally in the neighborhood of any point $p \in \Omega$. For the details, see II.

But, in 1957, H. Lewy showed that, under some restrictions of $f(x,t)$, the equation (1.1) does not have a c' solution for the generic C^∞ function f in any neighborhood of P. The simplest case of (1.1) without local solution is the Mizohata operator:

$$M = \frac{\partial}{\partial t} + i\frac{\partial}{\partial x}$$

That is, the equation

(1.2) $$Mu = \frac{\partial u}{\partial t} + i\frac{\partial u}{\partial x} = f$$

is not locally solvable for some function f.

A partial result on this questions was obtained by F. Treves; namely,

Theorem. Let $f(x,t) \in C_c^\infty(R^2)$ have the following properties:

$f(x,t) = f(x,-t)$ for all (x,t);

the supp f does not intersect the axis $t = 0$;

$\iint_{R^2} f(x,t)dxdt \neq o$.

Then the equation in R^2 Mu=f does not have any local solution.

The proof will be found in [9, §3]. In this paper, we shall remove the condition 'The supp f does not intersect the axis t=o' in the above theorem. Instead of the theorem, we will prove;

Theorem. Let $f(x,t) \in C_c^\infty(R^2)$ have the following properties:

$f(x,t) = f(x,-t)$ for all (x,t); the supp $f \cap \{x\text{-axis}\}$ is a nonempty finite set $\{(o,o)\}$;

$\iint_{R^2} f(x,t)\, dxdt \neq o$.

Then the equation in R^2 Mu = f does not have any solution.

This theorem is a generalization of Treves' result.

II. The Solvability of The Mizohata's Partial Differential Equations.

In this section, we will give the solution existence theorem when f is analytic on Ω.

Theorem. Let Mu = f, $u|_{t=o} = u_o$ be the Mizohata's partial differential equation with initial value u_o, $f \in C_c^\infty(R^2)$. Then there is a unique solution $u \in C_c^\infty(R^2)$ where u_o is considered to be C^∞ on R.

Proof. Set

$$u(t,x) = u_o(x) + \int_0^t f(x,s)ds + \int_o^t -is\frac{\partial u}{\partial x}\, ds.$$

This is a required solution. For the uniqueness, it is sufficient to prove that if $-it\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}$, $u(x,o) \equiv o$, then $u \equiv o$. Note that $u(x,t) = \int_{t_o}^t -is\frac{\partial u}{\partial x}(x,s)ds$. We may assume u is analystic on $R^2 = C$. Since $Z(u) = \{(x,t) \in R^2 : u(x,t) = o\}$ has límit points in R^2, $Z(u) = R^2$. Therefore $u \equiv o$ on R^2.

III. The Unsolvability of the Mizohata's Partial Differential Equations.

We need some preliminary results.

Theorem 3.1 Let f be holomorphic on the open subset Ω^+ of the upper half plane; assume that a segment (a, b) of the real axis forms part of the boundary of Ω^+, and that f is continuous on $\Omega^* \cup (a,b)$ and real-valued on (a, b). Let Ω^- be the reflection of Ω^+.

$$\Omega^- = \{z : \bar{z} \in \Omega^+\}.$$

Define

$$h(z) = \begin{cases} f(z) \text{ for } z \in \Omega^+ \cup (a,b) \\ \overline{f(\bar{z})} \text{ for } z \in \Omega^- \end{cases}$$

Then h(Z) is holomorphic on $\Omega = \Omega^+ \cup (a,b) \cup \Omega^-$.

Proof. If $D(z_o, r) \subset \Omega^-$, then $D(\overline{z_o}, r) \subset \Omega^+$, so for every $Z \in D(z_o, r)$ we have

$$f(\bar{z}) = \sum_{n=1}^{\infty} c_n (\bar{z} - \bar{z}_o)^n$$

$$\text{Hence } h(z) = \sum_{n=1}^{\infty} \bar{c}_n (z - z_o)^n \qquad (z \in D(z,r)).$$

Since h(z) is representable by power series in Ω^-, h(z) is holomorphic on $\Omega^+ \cup \Omega^-$ Let $z \in (a,b)$. If $\epsilon > 0$, there is a $\delta > 0$ such that if $w \in \Omega^+$ and $|w - z| < \delta$, then $|f(w) - f(z)| < \epsilon$. If $w \in \Omega^-$ and $|w - z| < \delta$, then $|\bar{w} - z| = |\overline{\bar{w} - \bar{z}}| = |w - z| < \delta$, hence $|f(\bar{w}) - f(z)| < \epsilon$. Since f is real-valued on (a,b),

$$|h(w) - h(z)| = |\overline{f(\bar{w})} - \overline{f(z)}| = |f(\bar{w}) - f(z)| < \epsilon$$

Thus h(z) is continuous on Ω.

Now assume $z \in (a,b)$, and let $D(z,r) \subset \Omega$. If ∇ is a triangle in D(z, r), then $\int_\nabla h = 0$ by the Cauchy's Theorem for a triangle. Hence by the

Morera's Theorem, $h(z)$ is holomorphic on $D(z,r)$.

Theorem 3.2. Let f be holomorphic on the open connected set $\Omega \subset C$. Suppose that f has a limit point of zeros in Ω, that is, there is a point $z_0 \epsilon \Omega$ and a sequence of points $z_n \epsilon \Omega$, $z_n \neq z_0$, such that $z_n \to z_0$ and $f(z_n)=0$ for all n=0, 1, 2, ... Then f is identically 0 on Ω.

Proof. Expand f in a Taylor series about z_0, say

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad |z-z_0|<r.$$

We will show that all $a_n=0$. If not, let m be the smallest integer such that $a_m \neq 0$. Then $f(z)=(z-z_0)^m g(z)$, where $g(z)$ is holomorphic at z_0 and $g(z_0)\neq 0$. By continuity, g is nonzero in a neighborhood of z_0, contradicting the fact that z_0 is a limit point of zeros.

Let $A=\{Z\epsilon\Omega$: there is a sequence of points $z_n \epsilon \Omega$, $z_n \neq z_0$, $z_n \to z$ with $f(z_n) = 0$ for all n $\}$. Since $z_0 \epsilon A$ by hypothesis, A is not empty. If $z \epsilon A$, then by the above argument f is zero on a disc $D(z,r)$ for some $r>0$ and it follows that $D(z,r) \subset A$. Thus A is open. If we can show that A is also closed in Ω, the connectedness of Ω gives $A=\Omega$, and the result will follow..

Let $z_n \to z\epsilon\Omega$, $z_n \epsilon A$. If $z_n=z$, there is nothing to prove; thus assume $z_n \neq z$ for all n=1,2, ... But since $z_n \neq z$ we have $f(z_n)=0$, and hence $z\epsilon A$ by the definition of A. Thus A is closed in Ω.

Theorem 3.3 (Stokes' Theorem) If ω is a (k−1) −form on an open set $A \subset R^n$ and c is a k-chain in A, then

$$\int_c d\omega = \int_{\partial c} \omega.$$

In particular, if $\omega=f\,dx+g\,dy$ is a 1-form on R^2, and $\phi: T \to S \subset R^2$ is a continuously differentiable mapping of a closed rectangle T, then

$$\iint_S \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dy \wedge dx = \int_{\partial S} f dx + g dy$$

Proof will be given in [6] p.102.

Now let's prove the following generalization of F. Treves' Theorem.

Theorem 3.4 Let $f(x,t) \epsilon C_c^\infty (R^2)$ have the following properties:

(3.1) $f(x,t)=f(x,-t)$ for all (x,t);

(3.2) the supp $f \cap [x\text{-axis}]$ is a finite set $\{(0,0)\}$;

(3.3) $\iint_{R^2} f(x,t)dx\,dt \neq 0$.

Then the equation in R^2 $Mu=f$ does not have any c' solution.

Proof. We may choose a $c>o$ so that supp $f \subset \{(x,t): t \geq c|x|\} \cup \{(x,t): t \leq -c|x|\}$. By (3.1) we may write

$$f(x,t)=F(x,s), \quad s=\frac{1}{2}t^2 >0.$$

For $s \leq 0$, we define $F(x,s)=0$. Suppose that there exists a solution u of $Mu=f$ where u us a c' function. Since we can put

$$u(x,t)=\phi(x,s)+t\,\Psi(x,s) \quad (s \geq 0)$$

for some even functions ϕ, Ψ,

$$Mu=\left(\frac{\partial}{\partial t}+it\frac{\partial}{\partial x}\right)(\phi+t\Psi) = t(\phi_s+i\phi_x) + (\Psi+2s\Psi_s + 2is\Psi_x)=F(x,s)=f(x,t).$$

Since $f(x,t)=f(x,-t)$, $s=\frac{1}{2}t^2$, it is proved that $Mu=f$ is equivalent to

(3.5) $\phi_s+i\phi_x=0$

(3.6) $\Psi+2s\Psi_s+2is\Psi_x=F$

But equation (3.6) can be rewritten

(3.7) $(\sqrt{s}\,\Psi)_s+i(\sqrt{s}\,\Psi)_x=\dfrac{F}{2\sqrt{s}}$ $(s>0)$

Put $\sqrt{s}\,\Psi(x,s) = h(z)$ $(s>0)$, where $z = x+is$. As

$\sqrt{s}\ \Psi$ vanishes when s=0, we define h(x,o)=0. Due to Theorem 3.1, h(z) can be extended as a holomorphic function, say h(z) again. Obviously

$$h(z) \equiv 0 \quad \text{on} \quad R^2 \setminus (\text{supp } F) \cap (\text{supp } F)^-,$$

where $(\text{Supp } F)^- = \{(x, -t): (x,t) \in \text{supp}F\}$. By (3.7) we have then $h \in C_c^\infty (R^2 \setminus \{0\})$.

Let C be a circle with center o enclosing supp F and C_n be a small circle with center 0, radius approaching 0 as $n \to \infty$. Let D_n be the annulus surrounded by C and C_n. Then using the Green's Theorem,

$$\frac{1}{\sqrt{2}} \iint_{R^2} f(x,t)\, dx\, dt \quad = \lim_{n\to\infty} \iint_{D_n} \frac{F(x,s)}{2\sqrt{s}} dx\, ds$$

$$= \lim_{n\to\infty} \iint_{D_n} [(\sqrt{s}\,\Psi)_s + i(\sqrt{s}\,\Psi)_x]\, dx\, ds$$

$$= [-\int_C \sqrt{s}\,\Psi\, dx + \int_C i\sqrt{s}\,\Psi\, ds]$$

$$+ \lim [\int_{C_n} \sqrt{s}\,\Psi\, dx - \int_{C_n} i\sqrt{s}\,\Psi\, ds].$$

But

$$\lim_{n\to\infty} \int_{C_n} \sqrt{s}\,\Psi\, dx = \lim_{n\to\infty} \int_{C_n} i\sqrt{s}\,\Psi\, ds = 0.$$

Since $\sqrt{s}\,\Psi \equiv 0$ on C,

$$\int_C \sqrt{s}\,\Psi\, dx = \int_C i\sqrt{s}\,\Psi\, ds = 0.$$

Hence $\frac{1}{\sqrt{2}} \iint_{R^2} f(x,t)\, dx\, dt = 0$,

contrary to the hypothesis (3.3).

This completes our theorem.

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국 문 초 록

Mizohata 연산자에 대하여

초기조건이 주어진 Mizohata 편미방은 유일한•해를 가짐을 증명하고, 해를 가지지 않을 조건을 Treves 의 결과보다 일반화하여 다음을 증명하였다.

$f(x,t) \in C_C^\infty(R^2)$가 다음 성질들을 갖는다고 하자.

i) $f(x,t) = f(x,-t) \quad (x,t) \in R^2$

ii) f 의 Support $\cap$ { x축 } = {(0,0)}

iii) $\iint_{R^2} f(x,t)\,dxdt \neq 0$

그러면, Mizohata 편미방은 해를 가지지 못한다.