The Number of Partitions of a set

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집합의 분할의 갯수

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Summary

In this paper, we obtained the general formula for the number of partation of a finite set.

In this paper, we find the number of partitions of a finite set.

The set $\{1,2,\ldots,m\}$ is denoted by Z_m, and the set of all positive integers is denoted by Z_+. For a finite set A, let $\#(A)$ denote the number of elements of A. A partition θ of the set Z_m is said to divide Z_m into n parts if $\#(\theta) = n$.

DEFINITION 1. For any $m \in Z_+$, $\mathcal{P}(m)$ denote the family of all partitions of Z_m.

DEFINITION 2 For any $m,n \in Z_+$, $\mathcal{P}(m,n) = \{\theta \in \mathcal{P}(m) : \#(\theta) = n\}$, and $\{{m \atop n}\} = \#(\mathcal{P}(m,n))$.

The following is immediate from definitions.

PROPOSITION 1. i) For any $m \in Z_+$, $\{{m \atop 1}\} = 1$.
ii) For any $n \in Z_+$ with $n \geq 2$, $[{1 \atop n}] = 0$.

We have the following important formula.

THEOREAM 2. For any $m,n \in Z_+$, $[{m+1 \atop n+1}] = [{m \atop n}] + (n+1)\{{m \atop n+1}\}$.

proof) Let $\mathcal{A} = \{\theta \in \mathcal{P}(m+1, n+1) : \{m+1\} \in \theta\}$ and $\mathcal{B} = \{\theta \in \mathcal{P}(m+1, n+1) : \{m+1\} \notin \theta\}$. Obviously, $\mathcal{P}(m+1, n+1)$ is the disjoint union of $\mathcal{A}$ and $\mathcal{B}$.

It is easy to see that $\#(\mathcal{A}) = [{m \atop n}]$. Thus it remains to show that $\#(\mathcal{B}) = (n+1)[{m \atop n+1}]$. Obviously, $\mathcal{P}(m,n+1) = \phi$ if and only if $\mathcal{B} = \phi$. Let $\mathcal{B} \neq \phi$, and let $\theta = \{A_1, A_2, \ldots, A_{n+1}\} \in \mathcal{P}(m,n+1)$. Define $\theta_i = \{A_1, A_2, \ldots, A_{i-1}, A_i \cup \{m+1\}, A_{i+1}, \ldots, A_n, A_{n+1}\}$, for all $i \in Z_{n+1}$. Then $\theta_1, \theta_2, \ldots, \theta_{n+1}$ are distinct partitions in $\mathcal{B}$ Let θ and Λ be distinct partitions in $\mathcal{P}(m, n+1)$. Then it is easy to see that $\theta_i \neq \Lambda_j$ for all $i,j \in Z_{n+1}$. Thus we get $(n+1)[{m \atop n+1}]$ distinct partitions in $\mathcal{B}$ from $[{m \atop n+1}]$ partitions in $\mathcal{P}(m,n+1)$. The fact that any partition in $\mathcal{B}$ is equal to θ_i for some $\theta \in \mathcal{P}(m,n+1)$ and for some $i \in Z_{n+1}$ completes the proof.

Note that $[{m \atop n}]$'s are completely determined by **Proposition 1 and Theorem 2.**

We define a function f on $Z_+ \times Z_+$, and prove that $f(m,n) = [{m \atop n}]$.

DEFINITION 3. For any $m,n \in Z_+$, $f(m,n) = \sum_{k=1}^{n} \frac{(-1)^{n-k} k^{m-1}}{(n-k)!\,(k-1)!}$

We have the following propositions which correspond to **Proposition 1 and Theorem 2.**

PROPOSITION 3. i) For any $m \in Z_+$, $f(m,1) = 1$.
ii) For any $n \in Z_+$ with $n \geq 2$, $f(1,n) = 0$.

proof) i) For any $m \in Z_+$, $f(m,1) = \frac{(-1)^0}{0!\,0!} 1^{m-1} = 1$.

ii)
$$\begin{aligned} f(1,n) &= \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(n-k)!\,(k-1)!} \\ &= \frac{1}{(n-1)!} \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!\,(k-1)!} (-1)^{n-k} \\ &= \frac{1}{(n-1)!} \sum_{k=1}^{n} \binom{n-1}{n-k} (-1)^{n-k} \\ &= \frac{1}{(n-1)!} (1-1)^{n-1} \\ &= 0, \text{ if } n \geq 2 \end{aligned}$$

PROPOSITION 4. For any $m,n \in Z_+$, $f(m+1,n+1) = f(m,n) + (n+1)f(m,n+1)$.

Proof) $f(m,n) + (n+1)f(m,n+1)$

$$= \sum_{k=1}^{n} \frac{(-1)^{n-k}\, k^{m-1}}{(n-k)!\,(k-1)!} + (n+1) \sum_{k=1}^{n+1} \frac{(-1)^{n+1-k}\, k^{m-1}}{(n+1-k)!\,(k-1)!}$$

$$= \sum_{k=1}^{n} \left(\frac{n+1}{(n+1-k)!\,(k-1)!} - \frac{1}{(n-k)!\,(k-1)!} \right) (-1)^{n+1-k} k^{m-1} + \frac{(n+1)^m}{o!n!}$$

$$= \sum_{k=1}^{n+1} \frac{(-1)^{n+1-k}\, k^{m}}{(n+1-k)!\,(k-1)!}$$

$$= f(m+1, n+1)$$

By the above propositions we have the following theorem, one of our main results.

THEOREM 5. For any $m, n \in Z_+$, $\left\{{m \atop n}\right\}$

$$= \sum_{k=1}^{n} \frac{(-1)^{n-k}\, k^{m-1}}{(n-k)!\,(k-1)!}$$

$$= \frac{1}{(n-1)!} \left\{ \binom{n-1}{0} n^{m-1} - \binom{n-1}{1} (n-1)^{m-1} + \ldots + (-1)^{n-1} \binom{n-1}{n-1} \right\}.$$

Proof) By Proposition 1, Theorem 2, Proposition 3, and Porporition 4, we can conclude that for any positive integer m and n, $\left[{m \atop n}\right] = f(m,n)$. The second equality is obvious.

REMARK. By Proposition 1 and Theorem 2, we have the following triangular array of positive integers, which is similar to Pascal's triange.

TABLE OF $\left[{m \atop n}\right]$

	n=1	n=2	n=3	n=4	n=5	n=6	.	.
m=1→	1	0*	0	0	0	0	.	.
m=2→	1	1	0	0	0	0	.	.
m=3→	1	3	1	0	0	0	.	.
m=4→	1*	7	6	1	0	0	.	.
m=5→	1	15	25**	10	1	0	.	.
.	.	.	.	.	.	.	.	.

EXAMPLE: (*) $\left[{1 \atop 2}\right] = 0$; $\left[{4 \atop 1}\right] = 1$ (Proposition 1),

(**) $\left[{5 \atop 3}\right] = 7 + 3 \cdot 6 = 25$ (Theorem 2).

We have some formulas.

COROLLARY 5. For any $m, n \in Z_+$,

i) $\binom{m-1}{0} m^{m-1} - \binom{m-1}{1} (m-1)^{m-1} + \ldots + (-1)^{m-1} = (m-1)!$,

ii) $\binom{n-1}{0} n^{m-1} - \binom{n-1}{1} (n-1)^{m-1} + \ldots + (-1)^{m-1} = 0$ if $m < n$.

Proof) i) $\left[{m \atop m}\right] = 1$, ii) $\left[{m \atop n}\right] = 0$ if $m < n$.

We now consider the number of all partitions which divide Z_m into less than or equal to n parts.

DEFINITION 4. For any $m,n \in Z_+$, $\left\{{m \atop n}\right\}$ denote the number of partitions which divide Z_m into less than or equal to n parts, that is, $\left\{{m \atop n}\right\} = \sum_{k=1}^{n} \left[{m \atop k}\right]$.

DEFINITION 5. For any nonnegative integer n, let

$$d_n = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} \ldots + (-1)^n \frac{1}{n!},$$

that is, a partial sum of the convergent series

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!} + \ldots .$$

The following are our main results.

THEOREM 6. For any $m, n \in Z_+$, $\left\{{m \atop n}\right\} = \sum_{k=1}^{n} \frac{k^{m-1}}{(k-1)!} d_{n-k}$

Proof) $\left\{{m \atop n}\right\} = \sum_{i=1}^{n} \left[{m \atop i}\right]$

$$= \sum_{i=1}^{n} \sum_{k=1}^{i} \frac{(-1)^{i-k}\, k^{m-1}}{(i-k)!\,(k-1)!}$$

$$= \sum_{k=1}^{n} \sum_{i=k}^{n} \frac{k^{m-1}\, (-1)^{i-k}}{(k-1)!\,(i-k)!}$$

$$= \sum_{k=1}^{n} \frac{k^{m-1}}{(k-1)!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!}$$

$$= \sum_{k=1}^{n} \frac{k^{m-1}}{(k-1)!} d_{n-k}$$

COROLLARY 6. For any $m \in Z_+$, $\#(\wp(m)) = \left\{{m \atop m}\right\}$

$$= \sum_{k=1}^{m} \frac{k^{m-1}}{(k-1)!} d_{m-k}$$

References

Mood, A.M. and Graybill, F.A. 1963 "Introduction to the theory of statistics," McGraw-Hill Book Company, Inc., New York.

Riordan, J. 1958 "An introduction to combinatorial analysis," John, Wiley.

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