스펙트랄 윈도우에 대한 고찰

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〈CONTENTS〉

要 約

시계열 분석의 목적은 원함수인 x(t)의 통계적 특징을 찾는 것이다. 연속형 함수인 x(t)에서 컴퓨터에 의한 계산을 위하여 이산형 자료인 x_r을 취하는 방법으로 표본함수에서 유한의 길이를 설정하여 취한 이산형 시계열에 대하여 임의의 확률 과정을 갖는 x(t)의 스펙트럼을 평가하면 원 스펙트럼을 계산하는 가운데 발생하는 함수의 주기성에 의한 겹침 현상은 aliasing을 발생시키므로, 이러한 오차를 smoothing 기법 등에 의한 스펙트랄 윈도우 방식으로 감소시킬 수 있다. 본 논문에서는 기본적인 스펙트랄 윈도우와 aliasing 스펙트랄 윈도우를 비교하여 스펙트랄 폭의 변화를 조사하였다.

1. Introduction

The objectives of time series analysis are to determine the statistical charcteristics of the original function x(t) by manipulating the series of the discrete numbers x_r. We are concerned with estimating the spectrum of a random process x(t) by the discrete time series obtained by sampling a finite length of a sample function. For the calculation procedure the true spectrum the aliasing is arised by difference the aperiodic function x(t). We can overcome these by the spectral window of the smoothing version the true spectrum. To deal with this problem, this paper shows the sketch of the aliasing spectral window and compare these with the basic spectral window about the bandwidth of graphs.

2. Discrete Fourier transforms

If $x(t)$ is a periodic function with period T, then it is always possible to write.

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos\frac{2\pi k}{T} t + b_k \sin\frac{2\pi k}{T} t\right)$$

where

$$\underset{k\geq 0}{a_k} = \frac{1}{T}\int_0^T x(t) \cos\frac{2\pi k}{T} t \, dt$$

$$\underset{k\geq 0}{b_k} = \frac{1}{T}\int_0^T x(t) \sin\frac{2\pi k}{T} t \, dt$$

By using complex notation with $X_k = a_k - ib_k$,

$$\underset{k\geq 0}{X_k} = \frac{1}{T}\int_0^T x(t) \, e^{-i(2\pi kt/T)} \, dt$$

Suppose that the discrete series $\{x_r\}$, $r=0,1,2,\cdots,(N-1)$, $t=r\Delta$, $\Delta=T/N$

$$X_k = \frac{1}{T}\sum_{r=0}^{N-1} x_r \, e^{-i(2\pi kt/T)(r\Delta)} \Delta$$

$$= \frac{1}{N}\sum_{r=0}^{N-1} x_r \, e^{-i(2\pi kr/N)} \qquad (2.1)$$

These transform is called the Discrete Fourier Transform (DFT) of x_r.

Theory 1. The value x_r of the series $\{x_r\}$ is given by

$$x_r = \sum_{k=0}^{N-1} X_k \, e^{i(2\pi kr/N)}.$$

proof) In the r.h.s. of (2.1),

$$\sum_{k=0}^{N-1} X_k \, e^{i(2\pi kr/N)} = \sum_{k=0}^{N-1} \left[\frac{1}{N}\sum_{s=0}^{N-1} x_s \, e^{-i(2\pi ks/N)}\right] e^{i(2\pi kr/N)}$$

$$= \sum_{k=0}^{N-1}\sum_{s=0}^{N-1}\frac{1}{N}\, x_s\, e^{-i(2\pi k/N)(s-r)}$$

$$= \sum_{s=0}^{N-1}\left[\sum_{k=0}^{N-1} e^{-i(2\pi k/N)(s-r)}\right]\frac{1}{N}\, x_s$$

$$\sum_{k=0}^{N-1} e^{-i(2\pi k/N)(s-r)} = \begin{cases} N, & s=r \\ 0, & s\neq r \end{cases}$$

and hence

$$\sum_{s=0}^{N-1}\left[\sum_{k=0}^{N-1} e^{-i(2\pi k/N)(s-r)}\right]\frac{1}{N}\, x_s = x_r$$

Similary, we may consider the discrete series $\{y_r\}$.

For calculation of spectral estimates, we write R_r as an estimate for the correlation function when $\tau = r\Delta$ and consider that $\{x_r\}$, $\{y_r\}$ are periodic series, then we can define

$$R_r = \frac{1}{N}\sum_{s=0}^{N-1} x_s\, y_{s+r},\quad r = 0,1,2,\cdots,(N-1) \qquad (2.2)$$

where $y_{s+r} = y_{s+r-N}$ when $s + r \geq N$ (2.3)

We define the DFT of the discrete series $\{R_r\}$ by

$$S_k = \frac{1}{N}\sum_{r=0}^{N-1} R_r\, e^{-i(2\pi kr/N)}$$

The circular spectrum is then given by

$$S_c(w) = \sum_k S_k\, \delta\left(w - \frac{2\pi k}{T}\right),\quad \text{for } -\frac{\pi}{\Delta} < w < \frac{\pi}{\Delta} \qquad (2.4)$$

Where δ is Dirac's delta function.

3. The Spectral Window

Using periodic condition, R_r can be written as

$$R_r = \frac{1}{N}\sum_{s=0}^{N-1-r} x_s\, y_{s+r} + \frac{1}{N}\sum_{s=N-r}^{N-1} x_s\, y_{s+r}$$

$$= \frac{1}{N}\sum_{s=0}^{N-1-r} x_s\, y_{s+r} + \frac{1}{N}\sum_{s=N-r}^{N-1} x_s\, y_{s-(N-r)} \qquad (\text{by } (2.3))$$

Let an approximation of R_r be $\hat{R}_r$

$$\hat{R}_r = \frac{1}{N-r}\sum_{s=0}^{N-1-r} x_s \ y_{s+r}, \text{ for } r = 0,1,2,\cdots,(N-1)$$

Then $\hat{R}_r = [\frac{N-r}{N}]\hat{R}_r + [\frac{N-(N-r)}{N}] \hat{R}_{-(N-r)}, \ r = 0,1,2,\cdots(N-1)$

Theory 2. $\hat{R}_r$ is consistent estimates of R_r.

proof)

$$E[\hat{R}_r] = \frac{1}{N-r}\sum_{s=0}^{N-1-r} E[x'_s y_{s+r}]$$

$$= \frac{1}{N-r}\sum_{s=0}^{N-1-r} R(\tau=r\Delta)$$

$$= R(\tau=r\Delta)$$

From Theory 2, $E[R_r]=[\frac{N-r}{N}] R(\tau=r\Delta)+[\frac{r}{N}] R(\tau=-(N-r)\Delta), \ 0\le r\le N$

These two functions will repeat themselves periodically. If we now join the function $[\frac{N-(N-r)}{N}] \hat{R}_{-(N-r)}$ from the period with r negative to the function $[\frac{(N-r)}{N}] \hat{R}_r$ from the next period with r positive, we can obtain the function represented by the absolute value $|r|$, which we will call u_r, can be written as

$$R_c(\tau) = u(\tau) + u(\tau-T) + u(\tau-2T) + u(\tau-3T) + \cdots$$

$$= \sum_{m=-\infty}^{\infty} u(\tau-mT)$$

Taking Fouries transforms of the both sides, we obtain

$$S_c(w) = \sum_{m=-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} u(\tau-mT) \ e^{-iw\tau} \ d\tau$$

Let $\tau' = \tau - mT$,

$$S_c(w) = \sum_{m=-\infty}^{\infty} e^{-iwmT}\frac{1}{2\pi}\int_{-\infty}^{\infty} u(\tau') \ e^{-iw\tau'} \ d\tau'$$

$$= \sum_{m=-\infty}^{\infty} e^{-iwmT} \ U(w), \text{ where } U(w) : \text{Four'er transform of } u(w)$$

$$= \frac{2\pi}{T} U(w) \sum_{k=-\infty}^{\infty} \delta(w - \frac{2\pi k}{T})$$

by (2.4) and $w_k = \frac{2\pi k}{T}$, $\frac{2\pi}{T} U(w_k) = S_k$

$$E[S_k] = \frac{2\pi}{T} E[U(w_k)]$$

$$= \frac{2\pi}{T} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} E[U(\tau)]\ e^{-iw_k\tau}\ d\tau\right]$$

$$= \frac{2\pi}{T} \left[\frac{1}{2\pi}\int_{-T}^{T} \frac{T-|\tau|}{T}\ R(\tau)\ e^{-iw_k\tau}\ d\tau\right]$$

The weighted spectral density $\tilde{S}(w)$ is defined by the term in brackets,

$$\tilde{S}(w_k) = \frac{1}{2\pi}\int_{-T}^{T} \frac{T-|\tau|}{T}\ R(\tau)\ e^{-iw_k\tau}\ d\tau \tag{3.1}$$

The weighting arises on account the factor $\frac{T-|\tau|}{T}$, which is described as the basic lag window.

Theory 3. $w(\tau) = \begin{cases} 1-|\tau|/T, & \text{for } 0 \le |\tau| \le T \\ 0 & \text{, elsewhere} \end{cases}$

then (1) $w(\tau) = w(-\tau)$

(2) $w(\tau=0) = 1$

(3) $\int_{-\infty}^{\infty} |\ w(\tau)\ |\ d\tau < \infty$

proof) omitted.

By Theory 3,

$$\tilde{S}(w_k) = \frac{1}{2\pi}\int_{-T}^{T} w(\tau)\ R(\tau)\ e^{-iw\tau}\ d\tau \tag{3.2}$$

The basic spectral window $W(w)$ of $w(\tau)$ is given by

$$W(w) = \frac{1}{2\pi}\int_{-T}^{T} \frac{T-|\tau|}{T}\ e^{-iw\tau}\ d\tau$$

$$= \frac{1}{2\pi}\int_{-T}^{T} \frac{T-|\tau|}{T}\ (\cos w\tau + i \sin w\tau)\ d\tau$$

$$= \frac{1}{2\pi}\int_{0}^{T} \frac{T-|\tau|}{T}\ \cos w\tau\ d\tau$$

$$= \frac{T}{2\pi}\left[\frac{\sin(wT/2)}{wT/2}\right]^2 \tag{3.3}$$

For considering the aliasing in (3.2), let replace W(w) the spectral window of $w(\tau)$ and put w' = w to avoid confusing the two w.

$$\widetilde{S}(w_k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} W(w')\ e^{-iw'\tau}\ dw'\right] R(\tau)\ e^{-iw\tau}\ d\tau$$

$$= \int_{-\infty}^{\infty} dw'\ W(w')\ \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau\ R(\tau)\ e^{-i(w-w')\tau}\right]$$

$$= \int_{-\infty}^{\infty} dw'\ W(w')\ S(w-w')$$

$$= \int_{-\infty}^{\infty} (-d\Omega)\ W(w-\Omega)\ S(\Omega)$$

where $\Omega = w - w'$

or $$\widetilde{S}(w) = \int_{-\infty}^{\infty} W(w-\Omega)\ S(\Omega)\ d\Omega$$

The expectation value of S_k is

$$E[S_k] = \frac{2\pi}{T}\int_{-\infty}^{\infty} W_a(w-w_k)\ S(w)\ dw$$

The alised spectral window function $W_a(w)$ is given by

$$W_a(w) = \sum_{j=-\infty}^{\infty} W\left(w - j\frac{2\pi}{\Delta}\right)$$

since $$\sum_{j=-\infty}^{\infty}\frac{1}{(\theta-j)^2} = \frac{\pi^2}{\sin^2\pi\theta}$$

so $$W_a(w) = \frac{T}{2\pi N^2}\left[\frac{\sin wT/2}{\sin wT/2N}\right]^2$$

Now we can compare the aliased spectral window $W_a(w)$ and the basic spectral window W(w) from (3.3). From the following figures (fig.1~fig.5), we can estimate the bandwidth of spectral windows. Although the basic spectral window (fig.1) can always be sharpened and made to decay towards zero faster by increasing the record length T, the graph of the alised spectral window shows that the bandwidth is decreased significantly as the data increase.

List of Symbols

T = record length

w = angular frequency

W = lag window function

N = number of sample values

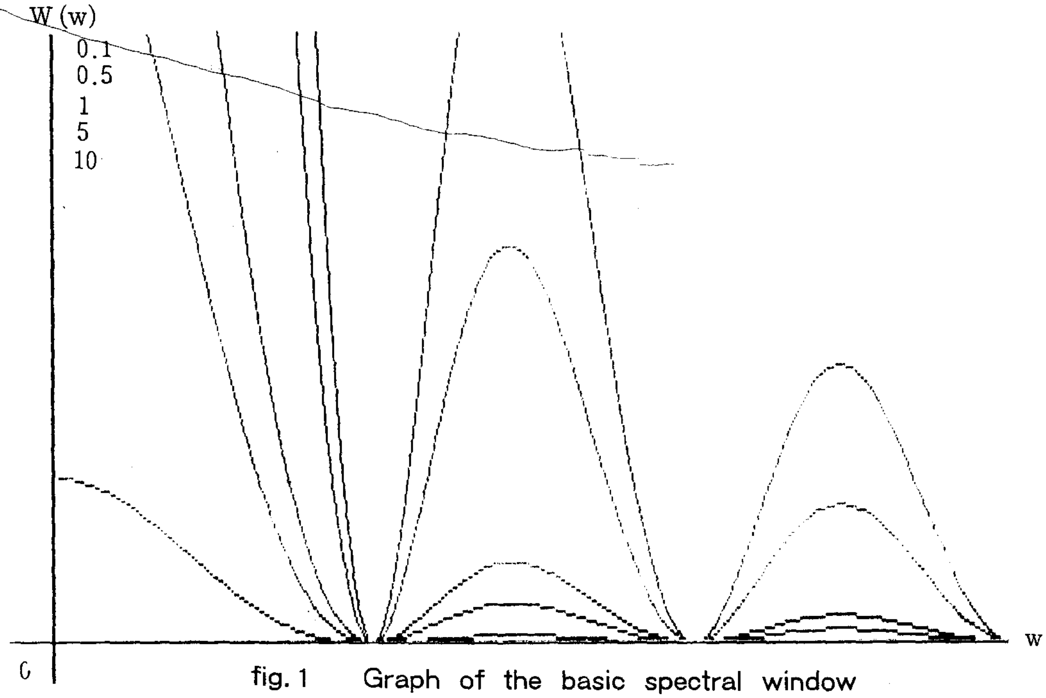

fig. 1 Graph of the basic spectral window

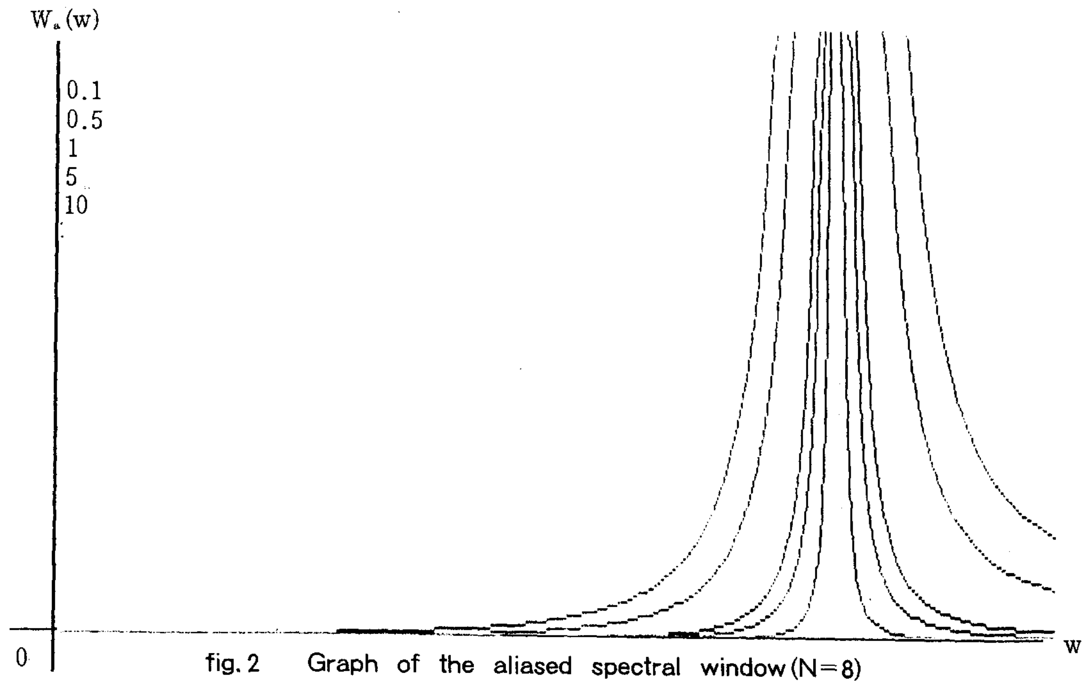

fig. 2 Graph of the aliased spectral window (N=8)

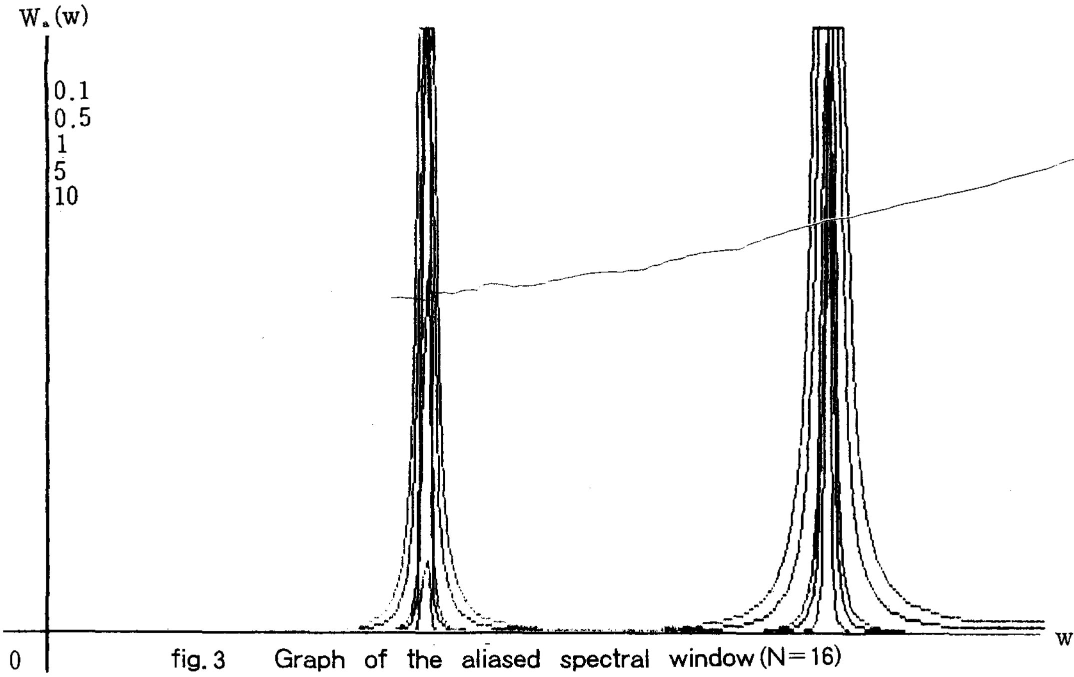

fig. 3 Graph of the aliased spectral window (N=16)

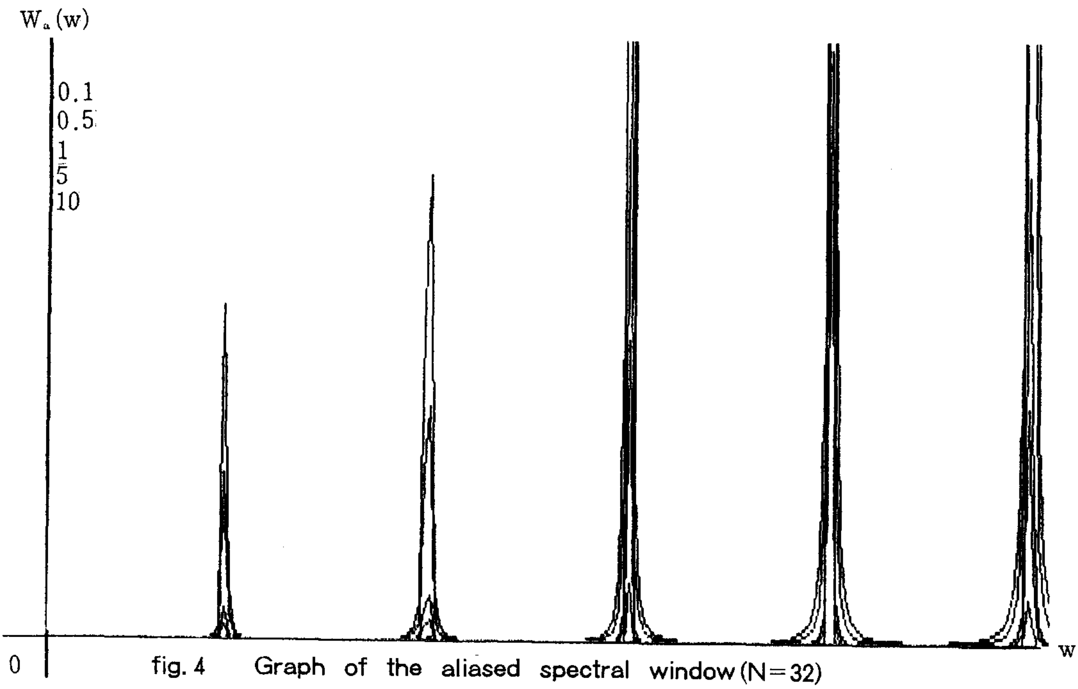

fig. 4 Graph of the aliased spectral window (N=32)

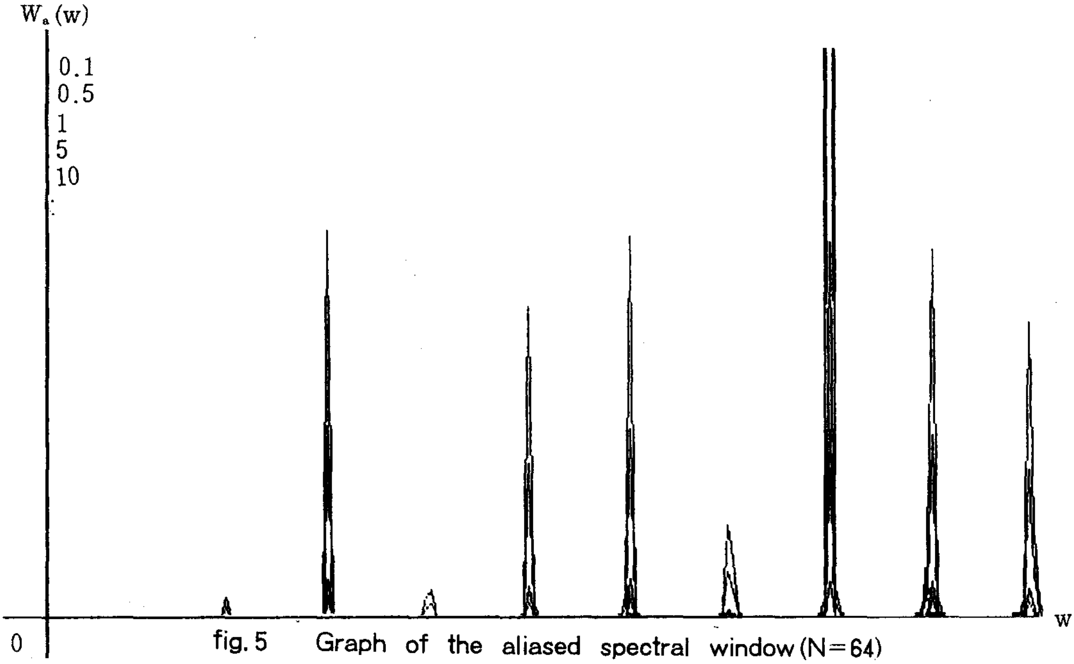

fig. 5 Graph of the aliased spectral window (N=64)

Reference

1. 김연형외, 「스펙트럴 분석 입문」, 자유아카데미, 1988.
2. 金鍾雨, *A study on the Spectral Estimation Errors by a Correlation Function,* 漢陽大 大學院, 1981.
3. 金鍾浩, "定常時系列의 스펙트럴 解析에 관한 硏究", 東國大 大學院, 1979.
4. COOLEY,J.W. and TUKEY,J.W., *'An Algorithm for the Machine Calculation of Complex Fourier Series',* Math. of Computation, vol.19, 1965.
5. D.E.NEWLAND, *Introduction to Random Vibrations and Spectral analysis,* Longman, 1975.
6. JOLLEY,L.B.W., *Summation of series, 2nd,* Dover, NY, 1961.
7. LIGHTHILL,M.J., *Introduction to Fourier Analysis and Generalised Functions,* Cambridge University Press, 1962.
8. PIET DE JONG, *The Fast Fourier Transform Spectral Estimator,* J.R.Statist. Soc.B, 1977.