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ON EXISTENCE OF SOLUTIONS FOR DIFFERENTIAL EQUATIONS ON PRODUCT WIENER SPACE

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ABSTRACT. We consider a differential equation on Wiener space. The known solutions are considered as a transformations on Wiener space. We show that the solutions can be constructed as the same limit on a product Wiener space. The solutions are also the quasi-sure limits of finite dimensional solutions on the product Wiener space.

1. INTRODUCTION

Let (X, H, μ) be an abstract Wiener space and A be a vector field on X which is, by definition, a mapping from X into H smooth in the sense of Malliavin (cf. [5]). We proved that

Theorem 1.1. (Yun [8], Theorem 5.5) If A is a vector field on X satisfying the followings
(i) $A \in W_\infty^\infty(X; H)$ and $\forall\lambda > 0$, $\int_X \exp(\lambda||A(x)||)d\mu(x) < +\infty$,
(ii) $\forall\lambda > 0$, $\forall n = 1, 2, \cdots$, $\int_X \exp(\lambda||\nabla^n A(x)||)d\mu(x) < +\infty$,
(iii) $\forall\lambda > 0$, $\int_X \exp(\lambda|\delta A(x)|)d\mu(x) < +\infty$,
then $V_t(x)$ exists for all $t \in \mathbf{R}$, q.e.x satisfying the following differential equation

$$(1.1) \qquad \begin{cases} (dV_t/dt)(x) = A(V_t(x) + x), \\ V_0(x) = 0. \end{cases}$$

The author constructed a solution $U_t(x)$ which satisfies (1.1) quasi everywhere (q.e.) (cf. [8]), i.e., for all x except in a slim set, that is, a set of (r,p)-capacity 0, for all $r \geq 0$ and $p > 1$. Here the capacities are associated with the Ornstein-Uhlenbeck operator on the Wiener space (cf. [5], [6]). By the way of its construction, we see that

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this $U_t(x)$ is a quasi continuous modification of the unique solution of Cruzeiro [1], in the sense of almost everywhere.

In the previous paper [9], we obtain further refinements of this solution. We chose a quasi continuous modification $\tilde{A}(x)$ of $A(x)$ defined everywhere on X and we constructed $U_t(x) \equiv V_t(x)+x$, $t \in \mathbf{R}, x \in X$, such that $U_t(x) \in W_\infty^\infty(X; C([-T,T] \to X))$, $X \ni x \mapsto U_t(x) \in C([-T,T] \to X)$ is quasi continuous for every $T > 0$ and satisfies

$$(1.1)' \qquad U_t(x) = x + \int_0^t \tilde{A}(U_s(x))ds \quad \text{for quasi every (q.e.) } x \in X,$$

for all $t \in \mathbf{R}$. Furthermore, we proved that we can choose its quasi continuous modification $U_t(x)$ so that the mapping $x \to U_t(x)$ preserves the class of slim sets for every t and the flow property

$$(1.2) \qquad U_t \circ U_s(x) = U_{t+s}(x),$$

holds q.e. for every t and s.

In the present paper, we prove the existence of solutions of (1.1) quasi surely as a mapping on the product Wiener space $[-M, M] \times X$. For the proof of the quasi-sure existence of solutions, we consider only the last quasi continuous modification $U_t(x)$ in all of this paper so that we can use the previous results.

2. Product Wiener space and existence of solutions

Let (X, H, μ) be an abstract Wiener space introduced by Gross. Let E be a real separable Banach space. We set

$$W_r^p(X; E) := (1 - L)^{-r/2}(L^p(X, \mu; E))$$

for the generator L (cf. [8]). Then $W_r^p(X; E)$ becomes a Banach space and we can define the Sobolev space $W_r^p(X; E)$ with the differentiability index r and the integrability index p with a norm

$$||f||_{r,p} := ||u||_{L^p} \quad \text{for} \quad f = (1 - L)^{-r/2}u, \quad u \in L^p(X, \mu; E).$$

We denote the space $\cap_r W_r^p(X; E)$ by $W_\infty^p(X; E)$ for $p \in [1, \infty)$ and $W_\infty^\infty(X; E) = \cap_p W_\infty^p(X; E)$. If E is a separable Hilbert space, we can define the gradient operator $\nabla : W_{r+1}^p(X; E) \to W_r^p(X; E \otimes H)$ and its dual, the divergence operator, $\delta : W_{r+1}^p(X; E \otimes H) \to W_r^p(E)$ as usual (cf. [1]).

Next let us recall the notion of (r,p)-capacity. The (r,p)-capacity $C_{r,p}$ is defined as follows: for an open set $O \subset X$,

$$C_{r,p}(O) = \inf\{||f||_{r,p}^p \ : \ f \in W_r^p(X; \mathbf{R}), \ f \geq 1 \text{ a.e. on } O\}$$

and for an arbitrary set $B \subset X$,

$$C_{r,p}(B) = \inf\{C_{r,p}(G) \ : \ G \text{ is open and } G \supset B\}.$$

We say that a property holds quasi-everywhere (q.e. in abbreviation) if it holds except on a set of capacity 0 for all r, p. We note that the following property holds for Sobolev spaces on an abstract Wiener space.

$$W_r^p(B) \cap C_b(X \to B) \text{ is dense in } W_r^p(B) \text{ and } 1 \in W_r^p(B).$$

Then it has been proved by I. Shigekawa that any $v \in W_r^p(B)$ admits a quasi-continuous modification and denoting it by $\tilde{v}$, it holds that

$$C_{r,p}(||\tilde{v}||_B) \leq ||v||_{r,p}^p$$

and the Chebyshev type inequality holds

$$C_{r,p}(||\tilde{v}|| \geq \lambda) \leq \frac{1}{\lambda^p}||v||_{r,p}^p.$$

Furthermore, in [9], we showed that we can modify the solution $U_t(x)$ so that it is defined for every $t \in \mathbb{R}$ and $x \in X$, satisfies (1.1)' for q.e. $x \in X$ for all $t \in \mathbb{R}$ and also has the quasi sure flow property, i.e., satisfies (1.2) for q.e. $x \in X$, for all $t, s \in \mathbb{R}$.

For the proof of Theorem 1.1, we first consider (1.1) in finite dimensional case. In [8], we proved the following theorem for finite dimensional case.

Theorem 2.1. (Yun [8], Theorem 3.5) Suppose that $B \in C^\infty$ and
(i) $\forall m = 0, 1, 2, \cdots, \forall \lambda > 0$, $\int_{\mathbb{R}^n} \exp(\lambda \cdot ||\nabla^m B(x)||) d\mu(x) < +\infty$,
(ii) $\forall \lambda > 0$, $\int_{\mathbb{R}^n} \exp(\lambda |\delta_\mu B(x)|) d\mu(x) < +\infty$.
Then a solution of the following system of differential equations

$$[L^m \nabla^n V_t(x) : m = 0, 1, \cdots, N, \quad n = 0, 1, \cdots, 2N, \quad 2m + n \leq 2N] :$$

$$\begin{aligned} \frac{d}{dt} L^m \nabla^n V_t = \nabla B \cdot L^m \nabla^n V_t & \\ + E^{m,n}(L^i \nabla^j B, L^l \nabla^r V_t : & \ i = 0, 1, \cdots, m, \\ & j = 0, 1, \cdots, n, \quad l = 0, 1, \cdots, m-1, \\ & r = 0, 1, \cdots, n, \quad 2i + j \leq 2m + n, \\ & 2l + r \leq 2(m-1) + n), \\ L^m \nabla^n V_0(x) = 0, \quad & m = 0, 1, \cdots, k, \quad n = 0, 1, \cdots, 2k, \\ & 2m + n \leq 2k, \quad k = 2, 3, \cdots, N, \end{aligned}$$

exists also for all $t \in \mathbf{R}$ starting for all $x \in \mathbf{R}^n$, where $E^{m,n}$ is some polynomial which can be obtained successively (cf. (5.9) below, [8]).

We show that for any $k \in \mathbf{N}$, $(L^k V_t)$ exists for all $t \in \mathbf{R}$, μ-a.e.x and thereby (V_t) admits a quasi-continuous modification as a $C([0,T] \to X)$-valued function for any $T > 0$. In finite dimensional case, one point has a positive (r,p)-capacity for sufficiently large r and p. Therefore we can show that a solution to (1.1) exists for every initial value $x \in X$. To deal with (LV_t), for example, we have to consider some system of differential equation ([8]). To proceed to infinite dimensional case, we adopt a finite dimensional approximation. To be precise, we take a sequence $\{A_n\}$ converging to A such that A_n depends only on finite number of coordinates and takes values in finite dimensional subspace of H. Denoting a solution for A_n by $(V_t^{(n)})$, we show that $(V_t^{(n)})$ converges quasi-everywhere and the limit satisfies (1.1).

For the existence of solutions, consider another Sobolev space as followings. For fixed $M > 0$, define the norm $||(t,x)||^2 = |t|^2 + ||x||^2$ and measure $dt/2M \otimes d\mu$ on the space $[-M,M] \times X$. For a Banach space E, the Sobolev space W_1^p on $[-M,M] \times X$ is defined by

$$W_1^p = \Big\{\phi : [-M,M]\times X \to E \mid \phi \in L^p(dt/2M \otimes d\mu) \quad \text{and} \\ \int_{-M}^{M}\int_X \Big(\Big|\Big|\frac{\partial \phi}{\partial t}\Big|\Big|_E^2 + ||\nabla\phi||_{\mathcal{L}(H,E)}\Big)^{p/2} d\mu \frac{dt}{2M} < +\infty\Big\}$$

with the norm

$$||\phi||_{1,p} \equiv ||\phi||_{W_1^p} = ||\phi||_{L^p(dt/2M\otimes d\mu)} \\ + \Big(\int_{-M}^{M}\int_X \Big(\Big|\Big|\frac{\partial \phi}{\partial t}\Big|\Big|^2 + ||\nabla\phi||^2\Big)^{p/2} d\mu \frac{dt}{2M}\Big)^{1/p}.$$

We can define the capacities on $([-M,M] \times X, dt/2M \otimes d\mu)$ in the same way. Then we have

$$C_{r,p}(\{||u|| > l\}) \le \frac{1}{l^p}||u||_{W_r^p}^p$$

if $u \in W_r^p(X)$ for all r ([3]). Let $\Omega_{l,U_t} = \{x \in X \ : \ |U_t(x)| \le l\}$. Then by the above inequality,

$$C_{r,p}(\Omega_{l,U_t}^c) \le \frac{1}{l^p}||U_t||_{W_r^p}^p.$$

Lemma 2.2. For all $p \geq 1$,

$$\lim_{n,m\to\infty} \int_{-M}^{M} \int_X \int_0^t |A^{(n)}(V_s^{(n)}(x)+x) - A^{(m)}(V_s^{(n)}(x)+x)|^p ds d\mu \frac{dt}{2M} = 0.$$

Proof. Note that

$$\int_{-M}^{M} \int_X \int_0^t |A^{(n)}(V_s^{(n)}(x)+x) - A^{(m)}(V_s^{(n)}(x)+x)|^p ds d\mu \frac{dt}{2M}$$
$$\leq \int_{-M}^{M} \int_X \int_0^M |A^{(n)}(V_s^{(n)}(x)+x) - A^{(m)}(V_s^{(n)}(x)+x)|^p ds d\mu \frac{dt}{2M}.$$

Since for all $M > 0$,

$$\lim_n \int_{-M}^{M} \int_X ||A^{(n)}(V_t^{(n)}(x)+x) - A^{(n)}(V_t(x)+x)|| d\mu(x) dt = 0$$

and $A^{(n)}(V_t^{(n)}(x)+x) - A(V_t(x)+x) \to 0$ in the space $L^1([-M,M]\times X; H)$ (Lemma 5.2, [8]),

$$\begin{aligned}
&\int_X \sup_{|t|\leq M} ||V_t^{(n+m)}(x) - V_t^{(n)}(x)|| d\mu(x) \\
&= \int_X \sup_{|t|\leq M} || \int_0^t (A^{(n+m)}(V_s^{(n+m)}(x)+x) - A^{(n)}(V_s^{(n)}(x)+x)) ds || d\mu(x) \\
&\leq \int_X \sup_{|t|\leq M} \int_0^t ||A^{(n+m)}(V_s^{(n+m)}(x)+x) - A^{(n)}(V_s^{(n)}(x)+x)|| ds d\mu(x) \\
&\leq \int_0^M \int_X ||A^{(n+m)}(V_s^{(n+m)}(x)+x) - A^{(n)}(V_s^{(n)}(x)+x)|| d\mu(x) ds \\
&\leq \int_0^M \int_X ||A^{(n+m)}(V_s^{(n+m)}(x)+x) - A(V_s(x)+x)|| d\mu(x) ds \\
&\quad + \int_0^M \int_X ||A^{(n)}(V_s^{(n)}(x)+x) - A(V_s(x)+x)|| d\mu(x) ds \\
&\longrightarrow 0 \quad \text{as } n,m \to \infty.
\end{aligned}$$

Thus we have

$$\lim_{n,m\to\infty} \int_{-M}^{M} \int_X \int_0^t |A^{(n)}(V_s^{(n)}(x)+x) - A^{(m)}(V_s^{(n)}(x)+x)|^p ds d\mu \frac{dt}{2M} = 0. \quad \square$$

By Theorem 2.1, the field A_n has the flow $U_t^{A_n}$ which can be defined for all $t \in \mathbb{R}$ starting for all $x \in \mathbb{R}^n$. Further we have, by the formula of change of variables,

$$(2.1) \qquad \frac{d(U_t^{A_n})_*\mu}{d\mu}(x) = \exp(\int_0^t \delta A_n(U_{-\xi}^{A_n}(x))d\xi) = k_t^n(x), \quad x \in \mathbb{R}^n.$$

This flow is a transformation of $\mathbb{R}^n$. We must modify the transformation on X and at the same time it does not change the formula (2.1).

We can set, for $x \in X$,

$$x = y + z, \quad z \in H_n.$$

If $U_t^{A_n}$ is the flow for A_n defined on $\mathbb{R}^n$, set

$$V_t^{A_n}(x) = U_t^{A_n}(z) + y - x = U_t^{A_n}(z) - z.$$

Then

$$\frac{V_t^{A_n}(x)}{dt} = A_n(V_t^{A_n}(x) + x) \quad \text{and} \quad V_0^{A_n}(x) = 0.$$

If ϕ is a function defined on H_n, then we have $\phi(V_t^{A_n}(x) + x) = \phi(U_t^{A_n}(x))$.

On the other hand, the measure μ can be decomposed as $\mu = \mu_n \otimes \nu_n$ with ν_n defined on $H_n^{\perp}$. Then we have

$$\int_X \psi(x)d(V_t^{A_n}(\cdot) + \cdot)_*\mu(x) = \int_{H_n^{\perp}} (\int_{H_n} \psi(V_t^{A_n}(y+z) + y + z)d\mu_n(z))d\nu_n(y)$$
$$= \int_X \psi(x)\exp(\int_0^t \delta A_n(U_{-\xi}^{A_n}(x))d\xi)d\mu(x).$$

Thus we have the following

$$(2.2) \qquad \frac{d(V_t^{A_n}(\cdot) + \cdot)_*\mu}{d\mu}(x) = \exp(\int_0^t \delta A_n(V_{-\xi}^{A_n}(x) + x)d\xi) = k_t^n(x).$$

We consider $V_t^{A_n}(x) = \tilde{U}_t^{A_n}(x) - x \in H$ and have to show the convergence of these functions. For this, we prepare the estimation of the norm of $(U_t^{A_n})'(x)$.

Lemma 2.3. (Yun [8], Lemma 4.6) Suppose that $A \in W^2_\infty(X;H)$ and $\int_X \exp(\lambda\|\nabla A(x)\|)d\mu(x) < +\infty$, $\forall\lambda > 0$. For all $t \in \mathbf{R}$, the matrix $(U_t^{A_n})'(x)$ is in the space $L^p(X;\mathrm{End}\ H)$ for all $1 \le p < +\infty$ and if $|t| < M$, we have

$$\int_X \|(U_t^{A_n})'(x)\|^p d\mu(x) \le C(M),$$

where $C(M)$ does not depend on n.

Using the above formula (2.2) and Lemma 2.3, we can prove the following lemma.

Lemma 2.4. For all $M > 0$, we have

$$\lim_{n,l\to\infty}[\sup_{|t|<M}\int_X \|A^{(n)}(V_t^{(n)}(x)) - A^{(n)}(V_t^{(l)}(x))\|d\mu(x)] = 0.$$

Theorem 2.5. For every $\varepsilon > 0$, there exists $F \subset [-M,M]\times X$ such that $C_{r,p}(F^c) < \varepsilon$ for every r, p and $V_t^{(n)}(x) \to V_t(x)$ uniformly on F.

Proof. By Lemma 2.2,

$$\begin{aligned}
&\|V^{(n)} - V^{(m)}\|_{L^p(dt/2M\otimes d\mu)}\\
&= \left(\int_{-M}^{M}\int_X\left|\int_0^t A^{(n)}(V_s^{(n)}(x)+x) - A^{(m)}(V_s^{(m)}(x)+x)ds\right|^p d\mu\frac{dt}{2M}\right)^{1/p}\\
&\le \left(\int_{-M}^{M}\int_X\int_0^t |A^{(n)}(V_s^{(n)}(x)+x) - A^{(m)}(V_s^{(n)}(x)+x)|^p ds d\mu\frac{dt}{2M}\right)^{1/p}\\
&\quad - \left(\int_{-M}^{M}\int_X\int_0^t |A^{(m)}(V_s^{(n)}(x)+x) - A^{(m)}(V_s^{(m)}(x)+x)|^p ds d\mu\frac{dt}{2M}\right)^{1/p}\\
&\longrightarrow 0 \quad \text{as} \quad n,m\to\infty.
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{\partial V_t^{(n)}}{\partial t} &= A^{(n)}(V_t^{(n)}(x)+x),\\
\nabla V_t^{(n)}(x) &= \int_0^t (\nabla A^{(n)}(V_s^{(n)}(x)+x)\cdot\nabla V_s^{(n)}(x) + \nabla A^{(n)}(V_s^{(n)}(x)+x)ds
\end{aligned}$$

and

$$\left(\int_{-M}^{M}\int_X\left(\left\|\frac{\partial V^{(n)}}{\partial t}\right\|^2+\|\nabla V_t^{(n)}\|^2\right)^{p/2}d\mu\frac{dt}{2M}\right)^{1/p}$$
$$\leq\left(\left(\int_{-M}^{M}\int_X\left\|\frac{\partial V^{(n)}}{\partial t}\right\|^p d\mu\frac{dt}{2M}\right)^{2/p}+\left(\int_{-M}^{M}\int_X\|\nabla V_t^{(n)}\|^p d\mu\frac{dt}{2M}\right)^{2/p}\right)^{1/2}.$$

Since, by Lemma 2.2,

$$\begin{aligned}
&\int_X\left\|\frac{\partial V_t^{(n)}}{\partial t}-\frac{\partial V_t^{(m)}}{\partial t}\right\|^p d\mu\\
&\quad= E[|A^{(n)}(V_t^{(n)}(x)+x)-A^{(m)}(V_t^{(m)}(x)+x)|^p]\\
&\quad\leq\Big(E[|A^{(n)}(V_t^{(n)}(x)+x)-A^{(m)}(V_t^{(n)}(x)+x)|^p]^{1/p}\\
&\qquad+E[|A^{(m)}(V_t^{(n)}(x)+x)-A^{(m)}(V_t^{(m)}(x)+x)|^p]^{1/p}\Big)^p\\
&\quad\longrightarrow 0 \quad\text{as}\quad n,m\to\infty \quad\text{uniformly}
\end{aligned}$$

and

$$\begin{aligned}
&\int_X\|\nabla V_t^{(n)}-\nabla V_t^{(m)}\|^p d\mu\\
&=\int_X\int_0^t|\nabla A^{(n)}(V_s^{(n)}(x)+x)\cdot\nabla V_s^{(n)}(x)+\nabla A^{(n)}(V_s^{(n)}(x)+x)\\
&\qquad-\nabla A^{(m)}(V_s^{(m)}(x)+x)\cdot\nabla V_s^{(n)}(x)-\nabla A^{(m)}(V_s^{(m)}(x)+x)|^p ds d\mu\\
&\longrightarrow 0 \quad\text{as}\quad n,m\to\infty \quad\text{uniformly (by Lemma 2.4),}
\end{aligned}$$

we have

$$\|V^{(n)}-V^{(m)}\|_{W_1^p}\to 0 \quad\text{as}\quad n,m\to\infty.$$

For the proof of

$$\|V^{(n)}-V^{(m)}\|_{W_r^p}\to 0 \quad\text{as}\quad n,m\to\infty,$$

we have to consider the following system of differential equations

$$\begin{aligned}\frac{d}{dt}L^m\nabla^n V_t^{(n)} = \nabla A^{(n)}\cdot L^m\nabla^n V_t^{(n)} \\ &+ E^{m,n}(L^l A^{(n)}, LA^{(n)}, L^i(\nabla^j A^{(n)}), \nabla^{2l}A^{(n)}, \\ &\nabla^2 A^{(n)}, \nabla A^{(n)}, L^{l-1}(V_t^{(n)}), \nabla^{2l-1}A^{(n)}, \\ &L^{i-1}(\nabla^{j-1}V_t^{(n)}), \nabla^{2l-2}V_t^{(n)}, \nabla^{2l-3}V_t^{(n)} : \\ &i = 1,\cdots,m-1, \quad j = 1,\cdots,2n-2, \\ &l = 2,\cdots,m, \quad 2i+j \le 2(m-1)+n),\end{aligned}$$

for some polynomial $E^{m,n}$ which can be calculated successively (see (5.9) below, [8]), where $\frac{m}{2} + n > \frac{[r]}{2} + 1$.
Adopting a finite dimensional approximation used in the proof of Theorem 5.5 ([8]), we can prove the existence of F. □

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