ON SOME SPACES OF NONLINEAR MAPS

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1. Introduction.

In [3] Furi and Vignoli introduced a class of all quasi-bounded (nonlinear) maps on a Banach space X and defined a spectrum for this class. They gave some of the basic properties for such spectrum, and extended surjectivity results previously obtained by Granas and Kranosel'skij.

Canavati defined a numerical range for a broader class of all numerically bounded maps on a Banach space X and studied it in a more systematic way [2]. Kim and Yang defined a new class of all numerically bounded maps on a Hilbert C^*-module and studied their properties [5].

In this paper, we shall define some classes of n-tuples of continuous maps on a Banach space X and show that these are Banach spaces. For reasons that are going to be apparent in later sections, we found more convenient to deal with maps of the form $F : X \times X^* \to X$, instead of maps $f : X \to X$, the later being a particular case of the former. Here X^* denotes the dual space of X. In particular if $n = 1$, our spaces coincide with those of Canavati. That is, our concepts generalize those of Canavati.

Thoughout this paper, let X be a Banach space over $\mathbf{K}(\mathbf{R}$ or $\mathbf{C}), X^*$ its dual space, and denote by $< x, x^* >$ $(x \in X, x^* \in X^*)$ the duality map between X and X^*. If $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{K}^n$, we set $|\lambda| = \left(\sum_{i=1}^{n} |\lambda_i|^2\right)^{\frac{1}{2}}$. For an n-tuple $\mathbf{F} = (F_1, \cdots, F_n)$ of maps and $x \in X$, $\mathbf{F}(x)$ means $\mathbf{F}(x) = (F_1(x), \cdots, F_n(x))$, and $< \mathbf{F}(x), x^* >$ denotes $(< F_1(x), x^* >, \cdots, < F_n(x), x^* >)$.

2. Some Spaces of n-tuples of Nonlinear Maps.

DEFINITION 2.1. *Let X be a Banach space over the field $\mathbf{K}$.* (a) $\mathbf{B}^n(X)$ *is the vector space of all n-tuples* $\mathbf{f} = (f_1, \cdots, f_n)$ *of continuous maps* $f_i : X \to X$ *such that*

$$\|\mathbf{f}(x)\| = \left(\sum_{j=1}^{n} \|f_j(x)\|^2 \right)^{\frac{1}{2}} \leq M\|x\|$$

for some $M \geq 0$ *and all* $x \in X$. *We define the joint norm* $\|\mathbf{f}\|$ *of* $\mathbf{f} = (f_1, \cdots, f_n)$ *as the smallest* $M \geq 0$ *such that this inequality holds for all* $x \in X$. *An element of* $\mathbf{B}^n(X)$ *is called a jointly bounded n-tuple on* X. (b) $\mathbf{Q}^n(X)$ *is the vector space of all jointly quasibounded n-tuples on* X. *That is, the space of all n-tuples* $\mathbf{f} = (f_1, \cdots, f_n)$ *of continuous maps* $f_i : X \to X$ *such that there exist* $A, B \geq 0$ *satisfying*

$$\|\mathbf{f}(x)\| = \left(\sum_{j=1}^{n} \|f_j(x)\|^2 \right)^{\frac{1}{2}} \leq A + B\|x\|, \quad x \in X. \tag{1}$$

Denote $|\mathbf{f}|$ the joint quasinorm of $\mathbf{f} = (f_1, \cdots, f_n)$ to be the infimum of all $B \geq 0$ for which (1) holds for some $A \geq 0$, i.e.,

$$|\mathbf{f}| = \limsup_{\|x\| \to \infty} \frac{\|\mathbf{f}(x)\|}{\|x\|}$$

In particular, if $n = 1$, then $\mathbf{B}^n(X)$ is the vector space of all bounded maps on X and $\mathbf{Q}^n(X)$ is the vector space of all quasibounded maps on X. Notice that $\|\cdot\|$ is a norm on $\mathbf{B}^n(X)$ and $|\cdot|$ is a semi-norm on $\mathbf{Q}^n(X)$.

The norm $\times$ weak* topology in $X \times X^*$, is the topology in $X \times X^*$ given by the norm topology on X and the weak* topology on X^*[1,2].

We define the following subsets of $X \times X^*$,

$$\Pi_r = \{ (x, x^*) \in X \times X^* : \|x\| = \|x^*\| \geq r, \ |x\|^2 = < x, x^* > \}$$

for $r > 0$, and

$$\Pi_0 = \bigcup_{r>0} \Pi_r.$$

LEMMA 2.2[2]. *Let π denote the natural projection of $X \times X^*$ onto X, and let E be a subset of Π_r that is relatively closed in Π_r with respect to the norm $\times$ weak* topology. Then $\pi(E)$ is a (norm) closed subset of X.*

LEMMA 2.3[2]. *Each $\Pi_r (r > 0)$ and Π_0 are connected subsets of $X \times X^*$ with the norm $\times$ weak* topology, unless X has dimension one over $\mathbf{R}$.*

From now on we shall assume that Π_0 has the norm $\times$ weak* topology induced as a subset of $X \times X^*$. Also we shall assume that X does not have dimension one over $\mathbf{R}$.

DEFINITION 2.4. *Let $\mathbf{F} = (F_1, \cdots, F_n)$ be an n-tuple of continuous maps from Π_0 into X. We say that $\mathbf{F}$ is jointly $*$-bounded if .*

$$\|\mathbf{F}\|_* = \sup_{\Pi_o} \frac{\|\mathbf{F}(x, x^*)\|}{\|x\|} < \infty$$

We denote by $\mathbf{B}^n_*(X)$ the vector space of all jointly $*$-bounded n-tuples.

Notice that $\|\cdot\|_*$ is a norm on $\mathbf{B}^n_*(X)$. We can consider the vector space $\mathbf{B}^n(X)$ of all n-tuples of bounded maps as a vector subspace of $\mathbf{B}^n_*(X)$ in a natural way, namely ; if $\mathbf{f} = (f_1, \cdots, f_n) \in \mathbf{B}^n(X)$, the mapping $\mathbf{F}(x, x^*) = \mathbf{f}(x) = (f_1(x), \cdots, f_n(x))$ belongs to $\mathbf{B}^n_*(X)$ and $\|\mathbf{f}\| = \|\mathbf{F}\|_*$.

THEOREM 2.5. *$\mathbf{B}^n_*(X)$ is a Banach space.*

Proof. This is a standard argument, and so it will be omitted.

DEFINITION 2.6. *Let $\mathbf{F} = (F_1, \cdots, F_n)$ be an n-tuple of continuous maps from Π_0 into X. We say that $\mathbf{F}$ is jointly $*$-quasibounded if*

$$|\mathbf{F}|_* = \lim_{r \to \infty} \sup_{\Pi_r} \frac{\|\mathbf{F}(x, x^*)\|}{\|x\|} < +\infty.$$

We denote by $\mathbf{Q}^n_*(X)$, the vector space of all jointly $*$-quasibounded n-tuples.

Notice that $|\cdot|_*$ is a seminorm on $\mathbf{Q}^n_*(X)$. Obiously one has $\mathbf{B}^n_*(X) \subseteq \mathbf{Q}^n_*(X)$ and

$$|\mathbf{F}|_* \leq \|\mathbf{F}\|_*.$$

We can consider the vector space $\mathbf{Q}^n(X)$ as a vector space $\mathbf{Q}^n_*(X)$ in a natural way, namely ; if $\mathbf{f} \in \mathbf{Q}^n(X)$, then the mapping $\mathbf{F}(x, x^*) = \mathbf{f}(x)$ belongs to $\mathbf{Q}^n_*(X)$ and $|\mathbf{f}| = |\mathbf{F}|_*$.

LEMMA 2.7. *For any* $\mathbf{F} \in \mathbf{Q}_*^n(X)$, *there exists a sequence* $< \mathbf{F}_m >$ *in* $\mathbf{B}_*^n(X)$ *such that* $|\mathbf{F}_m - \mathbf{F}|_* = 0 \ \ (m = 1, 2, 3, \cdots)$ *and*

$$\|\mathbf{F}_m\|_* \longrightarrow |\mathbf{F}|_* \quad \text{as} \quad m \to \infty.$$

Proof. Let $\rho^2 = \|x\|^2 + \|x^*\|^2$, and define

$$\mathbf{F}_m(x, x^*) = \begin{cases} \mathbf{F}(x, x^*) & \text{if } \rho \geq m, \\ \frac{\rho}{m}\mathbf{F}(\frac{m}{\rho}x, \frac{m}{\rho}x^*) & \text{if } 0 < \rho < m. \end{cases}$$

We have

$$\|\mathbf{F}_m\|_* = \sup_{\Pi_o} \frac{\|\mathbf{F}_m(x, x^*)\|}{\|x\|} = \sup_{\Pi_{m/\sqrt{2}}} \frac{\|\mathbf{F}(x, x^*)\|}{\|x\|}.$$

Therefore $\mathbf{F}_m \in \mathbf{B}_*^n(X)$ for all m large enough and

$$\|\mathbf{F}_m\|_* \to |\mathbf{F}|_* \quad \text{as} \quad m \to \infty.$$

DEFINITION 2.8. *(a) Let* $\mathbf{F}, \mathbf{G} \in \mathbf{Q}_*^n(X)$. *The n-tuple* $\mathbf{F}$ *is said to be jointly* $*$*-asymptotically equivalent to* $\mathbf{G}$ *(j.*- a.e) if* $|\mathbf{F} - \mathbf{G}|_* = 0$. *It is easy to see that this is an equivalence relation. (b)* $\widetilde{\mathbf{Q}}_*^n(X)$ *is the normed space of all equivalence class of jointly* $*$*-quasibounded n-tuples, i.e.* $\widetilde{\mathbf{Q}}_*^n(X) = \mathbf{Q}_*^n(X)/N^n(|\cdot|_*)$, *where* $\mathbf{F} \in N^n(|\cdot|_*)$ *iff* $|\mathbf{F}|_* = 0$. *The norm on* $\widetilde{\mathbf{Q}}_*^n(X)$ *is the one induced by* $|\cdot|_*$ *and will be denoted in the same way.*

From Lemma 2.7, we see that the mapping $\mathbf{B}_*^n(X) \to \widetilde{\mathbf{Q}}_*^n(X), \ \mathbf{F} \to \widetilde{\mathbf{F}}$ is onto.

Furthermore we have:

THEOREM 2.9. $\widetilde{\mathbf{Q}}_*^n(X)$ *is a Banach space.*

Proof. Let $\{\widetilde{\mathbf{F}}_m =< \widetilde{F}_m^{(1)}, \cdots, \widetilde{F}_m^{(n)} >\}$ be any sequence in $\widetilde{\mathbf{Q}}_*^n(X)$ such that $\sum |\widetilde{\mathbf{F}}_m|_*$ converges. We have to show that $\sum \widetilde{\mathbf{F}}_m = (\sum \widetilde{F}_m^{(1)}, \cdots, \sum \widetilde{F}_m^{(n)})$ converges. i.e. $\sum \widetilde{F}_m^{(j)}$ converges for each $j = 1, \cdots, n$. By Lemma 2.7, for any positive integer m, we can choose $\mathbf{G}_m \in \mathbf{B}_*^n(X)$ such that

$$\widetilde{\mathbf{G}}_m = \widetilde{\mathbf{F}}_m \text{ and } \|\mathbf{G}_m\|_* \leq |\mathbf{F}_m|_* + 2^{-m}.$$

Since $\mathbf{B}_*^n(X)$ is a Banach space, $\sum \mathbf{G}_m$ converges to an element $\mathbf{G} \in \mathbf{B}_*^n(X)$. From the continuity of the linear projection $B_*^n(X) \to \widetilde{\mathbf{Q}}_*^n(X)$, we obtain $\sum \widetilde{\mathbf{G}}_m = \sum \widetilde{\mathbf{F}}_m = \widetilde{\mathbf{G}}$.

DEFINITION 2.10. *Let $\mathbf{F} = (F_1, \cdots, F_n)$ be an n-tuple of continuous maps from Π_0 into X. We say that $\mathbf{F}$ is jointly $*$-numerically bounded if*

$$\omega_*(\mathbf{F}) = \lim_{r\to\infty} \sup_{\Pi_r} \frac{|< \mathbf{F}(x,x^*), x^* >|}{\|x\|\|x^*\|} < +\infty.$$

We denote by $\mathbf{W}^n_*(X)$, the vector space of all jointly $*$-numerically bounded n-tuples.

Notice that w_* is a seminorm on $\mathbf{W}^n_*(X)$. If $\mathbf{F} \in \mathbf{W}^n_*(X)$, then we let

$$\alpha_*(\mathbf{F}) = \lim_{r\to\infty} \inf_{\Pi_r} \frac{|< \mathbf{F}(x,x^*), x^* >|}{\|x\|\|x^*\|}.$$

Obviously one has $\mathbf{Q}^n_*(X) \subseteq \mathbf{W}^n_*(X)$ and $w_*(\mathbf{F}) \leq |\mathbf{F}|_*$.

DEFINITION 2.11. *Let $\mathbf{F} = (F_1, \cdots, F_n) \in W^n_*(X)$ and for $j = 1, \cdots, n$, consider the maps*

$$F_j^\nu : \Pi_0 \to X \quad \text{and} \quad F_j^\tau : \Pi_0 \to X$$

given by

$$F_j^\nu(x,x^*) = \frac{< F_j(x,x^*), x^* >}{\|x\|\|x^*\|} x$$

and

$$F_j^\tau(x,x^*) = F_j(x,x^*) - F_j^\nu(x,x^*).$$

Then $\mathbf{F} = \mathbf{F}^\nu + \mathbf{F}^\tau$ (i.e., $F_j = F_j^\nu + F_j^\tau$ for $j = 1, \cdots, n$.) The n-tuples $\mathbf{F}^\nu = (F_1^\nu, \cdots, F_n^\nu)$ and $\mathbf{F}^\tau = (F_1^\tau, \cdots, F_n^\tau)$ are called the jointly normal and jointly tangent components of $\mathbf{F}$ respectively.

The following Lemma follows immediately from the definitions.

LEMMA 2.12. *Let $\mathbf{F} = (F_1, \cdots, F_n) \in \mathbf{W}^n_*(X)$. Then*
(a) $< \mathbf{F}^\nu(x,x^*), x^* > = < \mathbf{F}(x,x^*), x^* >, \quad (x,x^*) \in \Pi_0$.
(b) $< \mathbf{F}^\tau(x,x^*), x^* > = 0, \quad (x,x^*) \in \Pi_0$.
(c) $\mathbf{F}^\nu \in \mathbf{Q}^n_*(X)$ and $|\mathbf{F}^\nu|_* = \omega_*(\mathbf{F})$.

The following result is also obvious.

THEOREM 2.13. *Let $\mathbf{F} = (F_1, \cdots, F_n)$ be an n-tuple of continuous maps from Π_0 into X. Then $\mathbf{F} \in \mathbf{W}^n_*(X)$ if and only if there exists n-tuples $\mathbf{G}, \mathbf{H}$ of continuous maps from Π_0 into X with $\mathbf{G} \in \mathbf{Q}^n_*(X)$ and $\mathbf{H}$ satisfying $< \mathbf{H}(x, x^*), x^* >= 0$ $((x, x^*) \in \Pi_0)$, such that $\mathbf{F} = \mathbf{G} + \mathbf{H}$. Such an n-tuple $\mathbf{H}$ is said to be a jointly $*$-orthogonal n-tuple.*

DEFINITION 2.14. *(a) Let $\mathbf{F}, \mathbf{G} \in \mathbf{W}^n_*(X)$. The n-tuple $\mathbf{F}$ is said to be jointly $*$-asymptotically numerically equivalent (i.e.j. $*$-a.n.e) to $\mathbf{G}$ if $\omega_*(\mathbf{F} - \mathbf{G}) = 0$.*

It is easy to see that there is an equivalence relation.

(b) $\widehat{\mathbf{W}}^n_*(X)$ is the normed space of all equivalence classes of jointly $*$-numerically bounded n-tuples, i.e., $\widehat{\mathbf{W}}^n_*(X) = \mathbf{W}^n_*(X)/\mathbf{N}^n(\omega_*)$, where $\mathbf{F} \in N^n(\omega_*)$ iff $\omega_*(\mathbf{F}) = 0$. The norm on $\widehat{\mathbf{W}}^n_*(X)$ is the one induced by ω_*, and it will be denoted in the same way.

Now let $\sim$: $\mathbf{Q}^n_*(X) \to \widetilde{\mathbf{Q}}_*(X)$ and $\wedge : \mathbf{W}^n_*(X) \to \widehat{\mathbf{W}}^n_*(X)$ be natural linear projections. Then we have the following commutative diagram of continous linear maps

$$\begin{array}{ccc}
\mathbf{W}^n_*(X) & \xrightarrow{\ \wedge\ } & \widehat{\mathbf{W}}_*(X) \\
\uparrow j & \overset{q}{\nearrow} & \uparrow r \\
\mathbf{Q}^n_*(X) & \xrightarrow{\ \sim\ } & \widetilde{\mathbf{Q}}^n_*(X)
\end{array}$$

where j is the inclusion map of $\mathbf{Q}^n_*(X)$ into $\mathbf{W}^n_*(X)$, $q(\mathbf{F}) = \widehat{\mathbf{F}}$ and $r(\widetilde{\mathbf{F}}) = \widehat{\mathbf{F}}$.

Note that the map r is well-defined, because if $\mathbf{F}, \mathbf{G} \in \mathbf{Q}^n_*(X)$ are such that $\widetilde{\mathbf{F}} = \widetilde{\mathbf{G}}$, then $\omega_*(\mathbf{F} - \mathbf{G}) \leq |\mathbf{F} - \mathbf{G}|_* = 0$, and hence $\widehat{\mathbf{F}} = \widehat{\mathbf{G}}$.

THEOREM 2.15. *$\widehat{\mathbf{W}}^n_*(X)$ is a Banach space.*

Proof. Let $\{\widehat{\mathbf{F}}_m\}$ be a sequence in $\widehat{\mathbf{W}}^n_*(X)$ such that $\sum \omega_*(\widehat{\mathbf{F}}_m) < \infty$. We have to show that $\sum \widehat{\mathbf{F}}_m = (\sum \widetilde{F}^{(1)}_m, \cdots, \widetilde{F}^{(n)}_m)$ converges.

Since $\omega_*(\widehat{\mathbf{F}}) = \omega_*(\mathbf{F}) = |\mathbf{F}^\nu|_* = |\widetilde{\mathbf{F}}^\nu|_*$ $(\mathbf{F} \in \mathbf{W}^n_*(X))$, where $\mathbf{F}^\nu \in \mathbf{Q}^n_*(X)$ (Lemma 2.12) is the jointly normal component of $\mathbf{F}$, we have

$$\sum |\widetilde{\mathbf{F}}^\nu_m|_* = \sum \omega_*(\widehat{\mathbf{F}}_m) < \infty. \tag{1}$$

But $\{\widetilde{\mathbf{F}}_m^\nu\}$ is a sequence in the Banach space $\widetilde{\mathbf{Q}}_*^n(X)$, and it follows from (1) and Theorem 2.9 that the series $\sum \widetilde{\mathbf{F}}_m^\nu$ converges to an element $\widetilde{\mathbf{F}} \in \widetilde{\mathbf{Q}}_*^n(X)$. Since the mapping $r : \widetilde{\mathbf{Q}}_*^n(X) \to \widehat{\mathbf{W}}_*^n(X)$ is linear and continous, we must have

$$\sum \widehat{\mathbf{F}}_m^\nu = \sum r(\widetilde{\mathbf{F}}_m^\nu) = r(\widetilde{\mathbf{F}}) = \widehat{\mathbf{F}}. \qquad (2)$$

But $\widehat{\mathbf{F}} = \widehat{\mathbf{F}}^\nu$ for $\mathbf{F} \in \mathbf{W}_*^n(X)$. Hence from (2) we obtain $\sum \widehat{\mathbf{F}}_m = \widehat{\mathbf{F}}$.

Acknowledgement. The author would like to thank the KOSEF for his partially financial support, 1991.

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