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A Zadeh′s extended composition operator for two 2-dimensional quadratic fuzzy numbers

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A Zadeh′s extended composition operator for two 2-dimensional quadratic fuzzy numbers

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⟨**Abstract**⟩

A Zadeh's extended composition operator for two 2-dimensional quadratic fuzzy numbers

There are many results for Zadeh's max-min composition operator based on Zadeh's expansion principle. We calculated Zadeh's max-min composite operator for two triangular fuzzy numbers and quadratic fuzzy numbers defined in $\mathbb{R}$ for various cases different from the conventional case. Then, calculations were performed to determine the relationship between the 1-dimensional quadratic fuzzy number and the 2-dimensional quadratic fuzzy number. More specifically, we cut two 2-dimensional quadratic fuzzy numbers into planes perpendicular to the xy-plane and the x-axis, and passing through the vertices. We calculated Zadeh's max-min composite operator using the membership functions of that cross section. Extended operations were performed on two different 2-dimensional quadratic fuzzy numbers. The resulting new 2-dimensional fuzzy number was cut into a plane perpendicular to the xy-plane and to the x-axis, passing through the the vertices. The resulting cut surface was compared with the results above, and it confirms that the two results are actually identical to each other.

1 Introduction

In fuzzy set theory, various types of operations between two fuzzy sets have been defined and studied. There are many results for Zadeh's max-min composition operators based on Zadeh's extension principle. We calculated max-min composition operators for two generalized triangular fuzzy sets([1]), for two generalized quadratic fuzzy sets([2]), and for two generalized trapezoidal fuzzy sets([3]). And we extended the above results to a 2-dimensional case. We proved Zadeh's extension principle for 2-dimensional triangular fuzzy numbers([4]). We calculated Zadeh's max-min composition operator for two 2-dimensional quadratic fuzzy numbers([5]), parametric operations between a 2-dimensional triangular fuzzy number and a trapezoidal fuzzy set([6]) and algebraic operations for two generalized 2-dimensional quadratic fuzzy sets([7]). The fuzzy sets in the above results([1], [2], [3]) were defined in $\mathbb{R}^+$.

In this paper, we calculated Zadeh's max-min composition operators for two triangular fuzzy numbers defined in $\mathbb{R}$. for exactly 6 positive numbers a, b, c, p, q, r, non-positive triangular fuzzy numbers $A = (-a, -b, c)$ and $B = (-p, q, r)$ and the expansion operation on quadratic fuzzy numbers in the same way. And with the result of the extended computation of a 2-dimensional quadratic fuzzy number being a 1-dimensional extension, we tried to visually confirm it using a graph. Taking the examples of two 2-dimensional quadratic fuzzy sets, we obtained the equations of the the intersections between planes perpendicular to the x-axis and passing through each vertex and two 2-dimensional quadratic fuzzy numbers. Then, the extended four operations of the two 1-dimensional quadratic fuzzy sets were calculated and graphed. Meanwhile, we computed the extended four operations of the 2-dimensional quadratic fuzzy numbers, which are the two examples above. Then we calculated the intersection between a plane perpendicular to the x-axis and passing through each vertex and the resulting 2-dimensional quadratic fuzzy number. We confirmed that the equations of the two intersections acquired in this way and the graphs are actually identical, respectively.

2 Preliminaries

Let X be a set. A classical subset A of X is often viewed as a characteristic function μ_A from X to $\{0,1\}$ such that $\mu_A(x) = 1$ if $x \in A$, and $\mu_A(x) = 0$ if $x \notin A$. $\{0,1\}$ is called a valuation set. The following definition is a generalization of this notion.

Definition 2.1. A *fuzzy set* A on X is a function from X to the interval $[0,1]$. The function is called the *membership function* of A.

Definition 2.2. The set $A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\}$ is said to be the *α-cut* of a fuzzy set A.

Definition 2.3. ([8]) A fuzzy set A on $\mathbb{R}$ is *convex* if

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)), \quad \forall x_1, x_2 \in \mathbb{R}, \quad \forall \lambda \in [0,1].$$

Definition 2.4. ([8]) A convex fuzzy set A on $\mathbb{R}$ is called a *fuzzy number* if

(1) There exists exactly one $x \in \mathbb{R}$ such that $\mu_A(x) = 1$,

(2) $\mu_A(x)$ is piecewise continuous.

Definition 2.5. ([1]) A *triangular fuzzy number* on $\mathbb{R}$ is a fuzzy number A which has a membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, \;\; a_3 \leq x, \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x < a_2, \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x < a_3. \end{cases}$$

The above triangular fuzzy number is denoted by $A = (a_1, a_2, a_3)$.

Definition 2.6. ([8]) The addition, subtraction, multiplication, and division of two fuzzy numbers are defined as

1. Addition $A(+)B$:

$$\mu_{A(+)B}(z) = \sup_{z=x+y} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B.$$

2. Subtraction $A(-)B$:

$$\mu_{A(-)B}(z) = \sup_{z=x-y} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B.$$

3. Multiplication $A(\cdot)B$:

$$\mu_{A(\cdot)B}(z) = \sup_{z=x\cdot y} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B.$$

4. Division $A(/)B$:

$$\mu_{A(/)B}(z) = \sup_{z=x/y} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B.$$

Remark 2.7. Let A and B be fuzzy sets. $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively. Then the α-cuts of $A(+)B, A(-)B$, $A(\cdot)B$ and $A(/)B$ can be calculated as the followings.

(1) $(A(+)B)_\alpha = A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}]$.

(2) $(A(-)B)_\alpha = A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}]$.

(3) $(A(\cdot)B)_\alpha = A_\alpha(\cdot)B_\alpha = [\min(a_1^{(\alpha)}b_1^{(\alpha)}, a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_1^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}),$
$\max(a_1^{(\alpha)}b_1^{(\alpha)}, a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_1^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)})]$.

(4) $(A(/)B)_\alpha = A_\alpha(/)B_\alpha = [\min(a_1^{(\alpha)}/b_1^{(\alpha)}, a_1^{(\alpha)}/b_2^{(\alpha)}, a_2^{(\alpha)}/b_1^{(\alpha)}, a_2^{(\alpha)}/b_2^{(\alpha)}),$
$\max(a_1^{(\alpha)}/b_1^{(\alpha)}, a_1^{(\alpha)}/b_2^{(\alpha)}, a_2^{(\alpha)}/b_1^{(\alpha)}, a_2^{(\alpha)}/b_2^{(\alpha)})]$.

Definition 2.8. ([8]) The *extended addition* $A(+)B$, *extended subtraction* $A(-)B$, *extended multiplication* $A(\cdot)B$ and *extended division* $A(/)B$ are fuzzy sets with membership functions defined as follows. For all $x \in A$ and $y \in B$,

$$\mu_{A(*)B}(z) = \sup_{z=x*y} \min\{\mu_A(x), \mu_B(y)\}, \quad (* = +, -, \cdot, /)$$

Definition 2.9. ([10]) Let A and B be two continuous fuzzy numbers defined on $\mathbb{R}$ and A_α, B_α, $f_A(t)$, $f_B(t)$ be the α-cuts and parametric α-functions of A and B, respectively. The parametric addition, parametric subtraction, parametric multiplication and parametric division are fuzzy numbers which have their α-cuts as the followings.

(1) parametric addition $A(+)_pB$:

$$(A(+)_pB)_\alpha = \{f_A(t) + f_B(t) \mid t \in [0,1]\}.$$

(2) parametric subtraction $A(-)_pB$:

$$(A(-)_pB)_\alpha = \{f_A(t) - f_B(1-t) \mid t \in [0,1]\}.$$

(3) parametric multiplication $A(\cdot)_pB$:

$$(A(\cdot)_pB)_\alpha = \{f_A(t) \cdot f_B(t) \mid t \in [0,1]\}.$$

(4) parametric division $A(/)_pB$:

$$(A(/)_pB)_\alpha = \{f_A(t)/f_B(1-t) \mid t \in [0,1]\}.$$

Theorem 2.10. ([10]) Let A and B be two continuous fuzzy numbers defined on $\mathbb{R}$. Then we have $A(+)_pB = A(+)B, A(-)_pB = A(-)B, A(\cdot)_pB = A(\cdot)B$ and $A(/)_pB = A(/)B$.

Definition 2.11. ([9]) A *quadratic fuzzy number* is a fuzzy number A having membership function

$$\mu_A(x) = \begin{cases} 0, & x < \alpha, \ \ \beta \le x, \\ -a(x-\alpha)(x-\beta) = -a(x-k)^2 + 1, & \alpha \le x < \beta. \end{cases}$$

where $a > 0$.

The above quadratic fuzzy number is denoted by $A = [\alpha, k, \beta]$.

Theorem 2.12. ([4]) Let A be a continuous convex fuzzy number defined on $\mathbb{R}^2$ and $A^\alpha = \{(x,y) \in \mathbb{R}^2 \mid \mu_A(x,y) = \alpha\}$ be the α-set of A. Then for all $\alpha \in (0,1)$, there exist continuous functions $f_1^\alpha(t)$ and $f_2^\alpha(t)$ defined on $[0, 2\pi]$ such that

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}.$$

Definition 2.13. ([4]) Let A and B be convex fuzzy numbers defined on $\mathbb{R}^2$ and

$$A^\alpha = \{(x,y) \in \mathbb{R}^2 \mid \mu_A(x,y) = \alpha\} = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\},$$

$$B^\alpha = \{(x,y) \in \mathbb{R}^2 \mid \mu_B(x,y) = \alpha\} = \{(g_1^\alpha(t), g_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}$$

be the α-sets of A and B, respectively. For $\alpha \in (0,1)$, the parametric operations defined by parametric addition, parametric subtraction, parametric multiplication and parametric division are fuzzy numbers that have their α-sets as the followings.

(1) parametric addition $A(+)_p B$:

$$(A(+)_p B)^\alpha = \{(f_1^\alpha(t) + g_1^\alpha(t), f_2^\alpha(t) + g_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}.$$

(2) parametric subtraction $A(-)_p B$:

$$(A(-)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\},$$

where

$$x_\alpha(t) = \begin{cases} f_1^\alpha(t) - g_1^\alpha(t+\pi), & \text{if } 0 \le t \le \pi, \\ f_1^\alpha(t) - g_1^\alpha(t-\pi), & \text{if } \pi \le t \le 2\pi, \end{cases}$$

and

$$y_\alpha(t) = \begin{cases} f_2^\alpha(t) - g_2^\alpha(t+\pi), & \text{if } 0 \le t \le \pi, \\ f_2^\alpha(t) - g_2^\alpha(t-\pi), & \text{if } \pi \le t \le 2\pi. \end{cases}$$

(3) parametric multiplication $A(\cdot)_p B$:

$$(A(\cdot)_p B)^\alpha = \{(f_1^\alpha(t) \cdot g_1^\alpha(t), f_2^\alpha(t) \cdot g_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}.$$

(4) parametric division $A(/)_p B$:

$$(A(/)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\},$$

where

$$x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t+\pi)} \quad (0 \le t \le \pi), \quad x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t-\pi)} \quad (\pi \le t \le 2\pi)$$

and

$$y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t+\pi)} \quad (0 \le t \le \pi), \quad y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t-\pi)} \quad (\pi \le t \le 2\pi).$$

For $\alpha = 0$ and $\alpha = 1$, define

$$(A(*)_p B)^0 = \lim_{\alpha \to 0^+} (A(*)_p B)^\alpha \quad \text{and} \quad (A(*)_p B)^1 = \lim_{\alpha \to 1^-} (A(*)_p B)^\alpha,$$

where $* = +, -, \cdot, /$.

Definition 2.14. ([5]) A fuzzy set A with a membership function

$$\mu_{A^2}(x,y) = \begin{cases} 1 - \left(\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2}\right), & b^2(x-x_1)^2 + a^2(y-y_1)^2 \le a^2b^2, \\ 0, & \text{otherwise}, \end{cases}$$

where $a, b > 0$ is called the *2-dimensional quadratic fuzzy number* and denoted by $[a,\ x_1,\ b,\ y_1]^2$.

Note that $\mu_A(x,y)$ is a cone. The intersections of $\mu_A(x,y)$ and the horizontal planes $z = \alpha \ \ (0 < \alpha < 1)$ are ellipses. The intersections of $\mu_A(x,y)$ and the vertical planes $y - y_1 = k(x - x_1) \ \ (k \in \mathbb{R})$ are symmetric quadratic fuzzy numbers in those planes. If $a = b$, ellipses become circles. The α-cut A_α of a 2-dimensional quadratic fuzzy number $A = [a, x_1, b, y_1]^2$ is an interior of ellipse in an xy-plane including the boundary

$$\begin{aligned} A_\alpha &= \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, b^2(x-x_1)^2 + a^2(y-y_1)^2 \le a^2b^2(1-\alpha)\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, \frac{(x-x_1)^2}{a^2(1-\alpha)} + \frac{(y-y_1)^2}{b^2(1-\alpha)} \le 1\right\}. \end{aligned}$$

Theorem 2.15. ([5]) Let $A = [a_1,\ x_1,\ b_1,\ y_1]^2$ and $B = [a_2,\ x_2,\ b_2,\ y_2]^2$ be two 2-dimensional quadratic fuzzy numbers. Then we have the following.

(1) $A(+)_pB = \Big[a_1 + a_2,\ x_1 + x_2,\ b_1 + b_2,\ y_1 + y_2\Big]^2.$

(2) $A(-)_pB = \Big[a_1 + a_2,\ x_1 - x_2,\ b_1 + b_2,\ y_1 - y_2\Big]^2.$

(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = x_1x_2 + (x_1a_2 + x_2a_1)\sqrt{1-\alpha}\cos t + a_1a_2(1-\alpha)\cos^2 t$$

and

$$y_\alpha(t) = y_1y_2 + (y_1b_2 + y_2b_1)\sqrt{1-\alpha}\sin t + b_1b_2(1-\alpha)\sin^2 t.$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = \frac{x_1 + a_1\sqrt{1-\alpha}\cos t}{x_2 - a_2\sqrt{1-\alpha}\cos t} \quad \text{and} \quad y_\alpha(t) = \frac{y_1 + b_1\sqrt{1-\alpha}\sin t}{y_2 - b_2\sqrt{1-\alpha}\sin t}.$$

Thus $A(+)_pB$ and $A(-)_pB$ become 2-dimensional quadratic fuzzy numbers, but $A(\cdot)_pB$ and $A(/)_pB$ are not 2-dimensional quadratic fuzzy numbers.

3 1-dimensional triangular fuzzy set

In this section,we calculate the extended composition operator of triangular fuzzy numbers for various cases different from the previously known results. for six positive real numbers a, b, c, p, q, r, we consider two triangular fuzzy numbers $A = (-a, -b, c)$ and $B = (-p, q, r)$. The other cases can be calculated similarly.

Theorem 3.1. ([11]) Let $-a < -p$, $c < r$, $\mu_A(0) = \alpha_1$ and $\mu_B(0) = \alpha_2$. If $\alpha_1 > \alpha_2$, then $A(+)B$ and $A(-)B$ are triangular fuzzy numbers, and $A(\cdot)B$ is a general fuzzy number. And $A(/)B$ has values in $(\alpha_2, 1]$ on $\mathbb{R}$.

Proof. Note that

$$\mu_A(x) = \begin{cases} 0, & x < -a, \;\; c < x, \\ \frac{1}{a-b}(x+a), & -a \le x < -b, \\ \frac{1}{c+b}(c-x), & -b \le x < c. \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < -p, \;\; r < x, \\ \frac{1}{q+p}(x+p), & -p \le x < q, \\ \frac{1}{r-q}(r-x), & q \le x < r. \end{cases}$$

Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively. Since $\alpha = \frac{a_1^{(\alpha)}+a}{a-b}$ and $\alpha = \frac{c-a_2^{(\alpha)}}{c+b}$, we have

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha(a-b) - a, -\alpha(c+b) + c].$$

Similarly,

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [\alpha(q+p) - p, -\alpha(r-q) + r].$$

1. Addition : By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [\alpha(a-b) - a + \alpha(q+p) - p, -\alpha(c+b) + c - \alpha(r-q) + r]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[-a-p, c+r]^c$ and $\mu_{A(+)B}(q-b) = 1$. Therefore,

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < c+r, \ \ -a-p \le x, \\ \frac{x+a+p}{a-b+q+p}, & -a-p \le x < q-b, \\ \frac{-x+c+r}{c+b+r-q}, & q-b \le x < c+r. \end{cases}$$

Hence $A(+)B$ is a triangular fuzzy number.

2. Subtraction : By the above facts, $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [\alpha(a - b) - a + \alpha(r-q) - r, -\alpha(c+b) + c - \alpha(q+p) + p]$. Thus $\mu_{A(-)B}(x) = 0$ on the interval $[-a-r, c+p]^c$ and $\mu_{A(-)B}(-b-q) = 1$. Therefore,

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < c+p, \ \ -a-r \le x, \\ \frac{x+a+r}{a-b+r-q}, & -a-r \le x < -b-q, \\ \frac{-x+c+p}{c+b+q+p}, & -b-q \le x < c+p. \end{cases}$$

Hence $A(-)B$ is a triangular fuzzy number.

3. Multiplication : (1) $\alpha_1 < \alpha \le 1$

By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b) - a) \cdot (-\alpha(r-q) + r), (-\alpha(c+b) + c) \cdot (\alpha(q+p) - p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b) - a) \cdot (-\alpha_1(r-q) + r)$ and $x = (-\alpha_1(c+b) + c) \cdot$

$(\alpha_1(q+p)-p)$ and $\mu_{A(\cdot)B}(-bq)=1$. Therefore,

$$\mu_{A(\cdot)B}(x)=\begin{cases}\frac{aq-2ar+rb-\sqrt{(-aq-2ar-rb)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)\le x<-bq, \\ \frac{bp+cp+bq+cq-\sqrt{(-bp-cp-cq-pc)^2-4(bp+cp+bq+cq)(cp+x)}}{2(bp+cp+bq+cq)}, \\ \qquad -bq\le x<(-\alpha_1(c+b)+c)\cdot(\alpha_1(q+p)-p).\end{cases}$$

(2) $\alpha_2<\alpha\le\alpha_1$

By the above facts,

$$\begin{aligned}A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)}\cdot b_2^{(\alpha)}, a_2^{(\alpha)}\cdot b_2^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (-\alpha(c+b)+c)\cdot(-\alpha(r-q)+r)].\end{aligned}$$

Thus $\mu_{A(\cdot)B}(x)=\alpha_2$ at $x=(\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r)$ and $x=(-\alpha_2(c+b)+c)\cdot(-\alpha_2(r-q)+r)$ and $\mu_{A(\cdot)B}(x)=\alpha_1$ at $x=(\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x=(-\alpha_1(c+b)+c)\cdot(-\alpha_1(r-q)+r)$. Therefore,

$$\mu_{A(\cdot)B}(x)=\begin{cases}\frac{aq-2ar+rb-\sqrt{(-aq-2ar-rb)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r)\le x \\ \qquad\qquad <(\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r), \\ \frac{-cq+rb+rc+\sqrt{(-cq+cr-rb-rc)^2-4(-bq-cq+br+cr)(-cr+x)}}{2(-bq-cq+br+cr)}, \\ \qquad (-\alpha_1(c+b)+c)\cdot(\alpha_1(q+p)-p)\le x \\ \qquad\qquad <(-\alpha_2(c+b)+c)\cdot(-\alpha_2(r-q)+r).\end{cases}$$

(3) $0<\alpha\le\alpha_2$

There are two cases (i) and (ii).

(i) By the above facts,

$$A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}]$$

$$= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (-\alpha(c+b)+c)\cdot(-\alpha(r-q)+r)].$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, cr]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r)$ and $x = (-\alpha_2(c+b)+c)\cdot(-\alpha_2(r-q)+r)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-rb)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \quad -ar \le x < (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r), \\ \frac{-cq+rb+rc-\sqrt{(-cq+cr-rb-rc)^2-4(-bq-cq+br+cr)(-cr+x)}}{2(-bq-cq+br+cr)}, \\ \quad (-\alpha_2(c+b)+c)\cdot(\alpha_2(q+p)-p) \le x < cr. \end{cases}$$

(ii) By the above facts,

$$A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_1^{(\alpha)} \cdot b_1^{(\alpha)}]$$

$$= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (\alpha(a-b)-a)\cdot(\alpha(q+p)-p)].$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, ap]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r)$ and $(\alpha_2(a-b)-a)\cdot(\alpha_2(q+p)-p)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-rb)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \quad -ar \le x < (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r), \\ \frac{2ap+aq-bp-\sqrt{(-2ap-aq+bp)^2-4(ap-bp+aq-bq)(ad-x)}}{2(ap-bp+aq-bq)}, \\ \quad (\alpha_2(a-b)-a)\cdot(\alpha_2(q+p)-p) \le x < ap. \end{cases}$$

Hence $A(\cdot)B$ is a fuzzy number.

4. Division : (1) $\alpha_1 < \alpha \le 1$

By the above facts, $A_\alpha(/)B_\alpha = \left[\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}\right] = \left[\frac{\alpha(a-b)-a}{\alpha(q+p)-p}, \frac{-\alpha(c+b)+c}{-\alpha(r-q)+r}\right].$

Thus $\mu_{A(/)B}(x) = \alpha_1$ at $x = \frac{\alpha_1(a-b)-a}{\alpha_1(q+p)-p}$ and $x = \frac{-\alpha_1(c+b)+c}{-\alpha_1(r-q)+r}$, $\mu_{A(/)B}(\frac{-b}{q}) = 1$. Therefore,

$$\mu_{A(/)B}(x) = \begin{cases} \frac{px-a}{(q+p)x-(a-b)}, & \frac{\alpha_1(a-b)-a}{\alpha_1(q+p)-p} \le x \le \frac{-b}{q}, \\ \frac{-rx+c}{(q-r)x+c+b}, & \frac{-b}{q} \le x \le \frac{-\alpha_1(c+b)+c}{-\alpha_1(r-q)+r}. \end{cases}$$

(2) $\alpha_2 < \alpha \le \alpha_1$

By the above facts, $A_\alpha(/)B_\alpha = \Big(\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}\Big] = \Big(\frac{\alpha(a-b)-a}{\alpha(q+p)-p}, \frac{-\alpha(c+b)+c}{\alpha(q+p)-p}\Big]$.

$\mu_{A(/)B}(x) = \alpha_1$ at $x = \frac{\alpha_1(a-b)-a}{\alpha_1(q+p)-p}$ and $\frac{-\alpha_1(c+b)+c}{\alpha_1(q+p)-p}$. Therefore,

$$\mu_{A(/)B}(x) = \begin{cases} \frac{px-a}{(q+p)x-(a-b)}, & -\infty < x \le \frac{(\alpha_1(a-b)-a)}{(\alpha_1(q+p)-p)}, \\ \frac{px+c}{(q+p)x+c+b}, & \frac{(-\alpha_1(c+b)+c)}{(\alpha_1(q+p)-p)} \le x < \infty. \end{cases}$$

Hence $A(/)B$ has values in $(\alpha_2, 1]$ on $\mathbb{R}$. □

Theorem 3.2. ([11]) Let $-a < -p$, $c < r$, $\mu_A(0) = \alpha_1$ and $\mu_B(0) = \alpha_2$. If $\alpha_1 = \alpha_2$, then $A(+)B$ and $A(-)B$ are triangular fuzzy numbers, and $A(\cdot)B$ is a general fuzzy number. And $A(/)B$ has values in $(\alpha_2, 1]$ on $\mathbb{R}$.

Proof. Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively. Since $\alpha = \frac{a_1^{(\alpha)}+a}{a-b}$ and $\alpha = \frac{c-a_2^{(\alpha)}}{c+b}$, we have

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha(a-b)-a, -\alpha(c+b)+c].$$

Similarly,

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [\alpha(q+p)-p, -\alpha(r-q)+r].$$

1. Addition : By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)}+b_1^{(\alpha)}, a_2^{(\alpha)}+b_2^{(\alpha)}] = [\alpha(a-b)-a+\alpha(q+p)-p, -\alpha(c+b)+c-\alpha(r-q)+r]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[-a-p, c+r]^c$

and $\mu_{A(+)B}(q-b)=1$. Therefore,

$$\mu_{A(+)B}(x)=\begin{cases} 0, & x<c+r,\ \ -a-p\le x, \\ \frac{x+a+p}{a-b+q+p}, & -a-p\le x<q-b, \\ \frac{-x+c+r}{c+b+r-q}, & q-b\le x<c+r. \end{cases}$$

Hence $A(+)B$ is a triangular fuzzy number.

2. Subtraction : By the above facts, $A_\alpha(-)B_\alpha=[a_1^{(\alpha)}-b_2^{(\alpha)},a_2^{(\alpha)}-b_1^{(\alpha)}]=[\alpha(a-b)-a+\alpha(r-q)-r,-\alpha(c+b)+c-\alpha(q+p)+p]$. Thus $\mu_{A(-)B}(x)=0$ on the interval $[-a-r,c+p]^c$ and $\mu_{A(-)B}(-b-q)=1$. Therefore,

$$\mu_{A(-)B}(x)=\begin{cases} 0, & x<c+p,\ \ -a-r\le x, \\ \frac{x+a+r}{a-b+r-q}, & -a-r\le x<-b-q, \\ \frac{-x+c+p}{c+b+q+p}, & -b-q\le x<c+p. \end{cases}$$

Hence $A(-)B$ is a triangular fuzzy number.

3. Multiplication : (1) $\alpha_1<\alpha\le 1$

By the above fact,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)}\cdot b_2^{(\alpha)},a_2^{(\alpha)}\cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r),(-\alpha(c+b)+c)\cdot(\alpha(q+p)-p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x)=\alpha_1$ at $x=(\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x=(-\alpha_1(c+b)+c)\cdot(\alpha_1(q+p)-p)$, $\mu_{A(\cdot)B}(-bp)=1$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r) \le x < -bq, \\ \frac{bp+cp+be+ce-\sqrt{(-bp-cp-cq-pc)^2-4(bp+cp+be+ce)(cp+x)}}{2(bp+cp+bq+cq)}, \\ \qquad -bq \le x < (-\alpha_1(c+b)+c)\cdot(\alpha_1(q+p)-p). \end{cases}$$

(2) $0 < \alpha \le \alpha_1$

There are two cases (i) and (ii).

(i) By the above fact,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)}\cdot b_2^{(\alpha)}, a_2^{(\alpha)}\cdot b_2^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (-\alpha(c+b)+c)\cdot(-\alpha(r-q)+r)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, cr]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x = (-\alpha_1(c+b)+c)\cdot(-\alpha_1(r-q)+r)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad -ar \le x < (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r), \\ \frac{cq-cr+rb+rc-\sqrt{(-cq+cr-rb-rc)^2-4(-bq-cq+bf+cr)(-cr+x)}}{2(-bq-cq+bf+cr)}, \\ \qquad (-\alpha_1(c+b)+c)\cdot(\alpha_1(q+p)-p) \le x < cr. \end{cases}$$

(ii) By the above fact,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)}\cdot b_2^{(\alpha)}, a_1^{(\alpha)}\cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (\alpha(a-b)-a)\cdot(\alpha(q+p)-p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, ap]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b)-a)\cdot$

$(-\alpha_1(r-q)+r)$ and $x = (\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad -ar \le x < (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r), \\ \frac{2ap+aq-bp-\sqrt{(-2ap-ae+bp)^2-4(ap-bp+aq-bq)(ap-x)}}{2(ap-bp+aq-bq)}, \\ \qquad (\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p) \le x < ap. \end{cases}$$

Hence $A(\cdot)B$ is a fuzzy number.

4. Division : If $\alpha_1 < \alpha \le 1$, by the above fact,

$A_\alpha(/)B_\alpha = \left(\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}\right] = \left(\frac{\alpha(a-b)-a}{\alpha(q+p)-p}, \frac{-\alpha(c+b)+c}{-\alpha(r-q)+r}\right]$. $\mu_{A(/)B}(\frac{-b}{q}) = 1$. Therefore,

$$\mu_{A(/)B}(x) = \begin{cases} \frac{px-a}{(q+p)x-(a-b)}, & -\infty < x \le \frac{-b}{q}, \\ \frac{-rx+c}{(q-r)x+c+b}, & \frac{-b}{q} \le x \le 0. \end{cases}$$

Hence $A(/)B$ has values in $(\alpha_1, 1]$ on $\mathbb{R}$. □

Theorem 3.3. ([11]) Let $-a < -p$, $c < r$, $\mu_A(0) = \alpha_1$ and $\mu_B(0) = \alpha_2$. If $\alpha_1 < \alpha_2$, then $A(+)B$ and $A(-)B$ are triangular fuzzy numbers, and $A(\cdot)B$ is a general fuzzy number. And $A(/)B$ has values in $(\alpha_1, 1]$ on $\mathbb{R}$.

Proof. Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively, Since $\alpha = \frac{a_1^{(\alpha)}+a}{a-b}$ and $\alpha = \frac{c-a_2^{(\alpha)}}{c+b}$, we have

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha(a-b)-a, -\alpha(c+b)+c].$$

Similarly,

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [\alpha(q+p)-p, -\alpha(r-q)+r].$$

1. Addition : By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [\alpha(a-b) - a + \alpha(q+p) - p, -\alpha(c+b) + c - \alpha(r-q) + r]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[-a-p, c+r]^c$ and $\mu_{A(+)B}(q-b) = 1$. Therefore,

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < c+r, \;\; -a-p \le x, \\ \frac{x+a+p}{a-b+q+p}, & -a-p \le x < q-b, \\ \frac{-x+c+r}{c+b+r-q}, & q-b \le x < c+r. \end{cases}$$

Hence $A(+)B$ is a triangular fuzzy number.

2. Subtraction : By the above facts, $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [\alpha(a-b) - a + \alpha(r-q) - r, -\alpha(c+b) + c - \alpha(q+p) + p]$. Thus $\mu_{A(-)B}(x) = 0$ on the interval $[-a-r, c+p]^c$ and $\mu_{A(-)B}(-b-q) = 1$. Therefore,

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < c+p, \;\; -a-r \le x, \\ \frac{x+a+r}{a-b+r-q}, & -a-r \le x < -b-q, \\ \frac{-x+c+p}{c+b+q+p}, & -b-q \le x < c+p. \end{cases}$$

Hence $A(-)B$ is a triangular fuzzy number.

3. Multiplication : (1) $\alpha_2 < \alpha \le 1$

By the above fact,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b)-a) \cdot (-\alpha(r-q)+r), (-\alpha(c+b)+c) \cdot (\alpha(q+p)-p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = (\alpha_2(a-b)-a) \cdot (-\alpha_2(r-q)+r)$ and $x = (-\alpha_2(c+b)+c) \cdot (\alpha_2(q+p)-p)$, $\mu_{A(\cdot)B}(-bp) = 1$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r) \le x < -bq, \\ \frac{bp+cp+be+ce+\sqrt{(-bp-cp-cq-pc)^2-4(bp+cp+be+ce)(cp+x)}}{2(bp+cp+be+ce)}, \\ \qquad -bq \le x < (-\alpha_2(c+b)+c)\cdot(\alpha_2(q+p)-p). \end{cases}$$

(2) $\alpha_1 < \alpha \le \alpha_2$

By the above facts,

$$A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_1^{(\alpha)} \cdot b_1^{(\alpha)}]$$

$$= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (\alpha(a-b)-a)\cdot(\alpha(q+p)-p)].$$

Thus $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x = (\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p)$ and $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r)$ and $x = (-\alpha_2(c+b)+c)\cdot(\alpha_2(q+p)-p)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r) \le x \\ \qquad\qquad < (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r), \\ \frac{2ap+aq-bp+\sqrt{(-2ap-ae+bp)^2-4(ap-bp+aq-bq)(ap-x)}}{2(ap-bp+aq-bq)}, \\ \qquad (\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p) \le x \\ \qquad\qquad < (-\alpha_2(c+b)+c)\cdot(\alpha_2(q+p)-p). \end{cases}$$

(3) $0 < \alpha \le \alpha_1$

There are two cases (i) and (ii).

(i) By the above facts,

$$A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}]$$

$$= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (-\alpha(c+b)+c)\cdot(-\alpha(r-q)+r)].$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, cr]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x = (-\alpha_1(c+b)+c)\cdot(-\alpha_1(r-q)+r)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \quad -ar \le x < (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r), \\ \frac{cq-cr+rb+rc-\sqrt{(-cq+cr-rb-rc)^2-4(-bq-cq+bf+cr)(-cr+x)}}{2(-bq-cq+bf+cr)}, \\ \quad (-\alpha_1(c+b)+c)\cdot(\alpha_1(q+p)-p) \le x < cr. \end{cases}$$

(ii) By the above facts,

$$A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_1^{(\alpha)} \cdot b_1^{(\alpha)}]$$

$$= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (\alpha(a-b)-a)\cdot(\alpha(q+p)-p)].$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, ap]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x = (\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \quad -ar \le x < (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r), \\ \frac{2ap+aq-bp-\sqrt{(-2ap-aq+bp)^2-4(ap-bp+aq-bq)(ad-x)}}{2(ap-bp+aq-bq)}, \\ \quad (\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p) \le x < ap. \end{cases}$$

Hence $A(\cdot)B$ is a fuzzy number.

4. Division : If $\alpha_1 < \alpha \le 1$, by the above fact,

$A_\alpha(/)B_\alpha = \left(\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}\right] = \left(\frac{\alpha(a-b)-a}{\alpha(q+p)-p}, \frac{-\alpha(c+b)+c}{-\alpha(r-q)+r}\right]$. $\mu_{A(/)B}(\frac{-b}{q}) = 1$. Therefore,

$$\mu_{A(/)B}(x) = \begin{cases} \frac{px-a}{(q+p)x-(a-b)}, & -\infty < x \le \frac{-b}{q}, \\ \frac{-rx+c}{(q-r)x+c+b}, & \frac{-b}{q} \le x \le 0. \end{cases}$$

Hence $A(/)B$ has values in $(\alpha_1, 1]$ on $\mathbb{R}$. □

We have computed Zadeh′s max-min composition operator for two triangular fuzzy numbers $A = (-a, -b, c)$ and $B = (-p, q, r)$. In the case of $B_0 \subset A_0$, we got three kinds of conclusions according to the three magnitude relationship between $\mu_A(0)$ and $\mu_B(0)$, i.e., $\mu_A(0) > \mu_B(0)$, $\mu_A(0) = \mu_B(0)$ and $\mu_A(0) < \mu_B(0)$. For each case, $A(+)B$ and $A(-)B$ were triangular fuzzy numbers, and $A(\cdot)B$ was a slightly distorted triangular fuzzy number, but $A(/)B$ was a different type of fuzzy number. In conclusion, $A(+)B$, $A(-)B$, $A(\cdot)B$ can be applied where the shape of the triangular fuzzy number comes out, and $A(/)B$ can be applied where appropriate.

Remark 3.4. We calculated Zadeh′s max-min composition operator for two non-positive triangular fuzzy numbers $A = (-a, -b, c)$ and $B = (-q, p, r)$ for six positive numbers a, b, c, p, q, r. Our results were obtained for the case of $-a < -p$ and $r < c$. Similar results can be obtained when $-a < -p$ and $c < r$, although the calculation is complex.

4 1-dimensional quadratic fuzzy set

In this section, an extended composition operator of quadratic fuzzy numbers was calculated for various cases using the same method as in the previous section. For six positive real numbers x_1, x_2, x_3, x_4, m and n, we considered two quadratic fuzzy numbers $A = [-x_1, -m, x_2]$ and $B = [-x_3, n, x_4]$. The other cases can be calculated similarly.

Theorem 4.1. Let $-x_1 < -x_3$, $x_2 < x_4$, $\mu_A(0) = \alpha_1$ and $\mu_B(0) = \alpha_2$. If $\alpha_1 > \alpha_2$, then $A(+)B$ and $A(-)B$ are quadratic fuzzy numbers, and $A(\cdot)B$ is a general fuzzy number. And $A(/)B$ has values in $[\alpha_1, 1]$ on $\mathbb{R}$.

Note that

$$\mu_A(x) = \begin{cases} \frac{-1}{a^2}(x+x_1)(x-x_2) = \frac{-1}{a^2}(x+m)^2 + 1, & -x_1 \le x \le x_2, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\mu_B(x) = \begin{cases} \frac{-1}{b^2}(x+x_3)(x-x_4) = \frac{-1}{b^2}(x-n)^2 + 1, & -x_3 \le x \le x_4, \\ 0, & \text{otherwise.} \end{cases}$$

Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively.

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = \left[-m - a\sqrt{1-\alpha}, -m + a\sqrt{1-\alpha}\right].$$

Similarly,

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = \left[n - b\sqrt{1-\alpha}, n + b\sqrt{1-\alpha}\right].$$

1. Addition :

$$A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}]$$
$$= \left[-m + n - (a+b)\sqrt{1-\alpha}, -m + n + (a+b)\sqrt{1-\alpha}\right].$$
$$\mu_{A(+)B}(x) = \begin{cases} \frac{a^2+2ab+b^2-(m-n+x)^2}{(a+b)^2}, & -a-b-m+n \le x \le a+b-m+n, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $A\,(+)\,B$ is a quadratic fuzzy number.

2. Subtraction :

$$A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}]$$
$$= \left[-m - n - (a+b)\sqrt{1-\alpha}, -m - n + (a+b)\sqrt{1-\alpha}\right].$$
$$\mu_{A(-)B}(x) = \begin{cases} \frac{a^2+2ab+b^2-(m+n+x)^2}{(a+b)^2}, & -a-b-m-n \le x \le a+b-m-n, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $A\,(-)\,B$ is a quadratic fuzzy number.

3. Multiplication :

(1) $\alpha_1 < \alpha \le 1$

$$A_\alpha\,(\cdot)\,B_\alpha = \left[a_1^{(\alpha)} b_2^{(\alpha)}, a_2^{(\alpha)} b_1^{(\alpha)}\right]$$
$$= \left[\left(-m - a\sqrt{1-\alpha}\right)\left(n + b\sqrt{1-\alpha}\right), \left(-m + a\sqrt{1-\alpha}\right)\left(n - b\sqrt{1-\alpha}\right)\right].$$

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{2a^2b^2-b^2m^2-a^2n^2+2abx+(bm+an)\sqrt{b^2m^2+a^2n^2-2ab(mn+2x)}}{2a^2b^2}, \\ \quad ab(-1+\alpha_1) - mn - \sqrt{-(-1+\alpha_1)(bm+an)^2} \le x \\ \qquad \le ab(-1+\alpha_1) - mn + \sqrt{-(-1+\alpha_1)(bm+an)^2}, \\ 0, \quad \text{otherwise.} \end{cases}$$

(2) $\alpha_2 < \alpha \le \alpha_1$

$$A_\alpha(\cdot)B_\alpha = \left[a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}\right]$$
$$= \left[\left(-m - a\sqrt{1-\alpha}\right)\left(n + b\sqrt{1-\alpha}\right), \left(-m + a\sqrt{1-\alpha}\right)\left(n + b\sqrt{1-\alpha}\right)\right].$$

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{2a^2b^2 - b^2m^2 - a^2n^2 + 2abx + (bm+an)\sqrt{b^2m^2+a^2n^2-2ab(mn+2x)}}{2a^2b^2}, \\ \quad ab(-1+\alpha_2) - mn - \sqrt{-(-1+\alpha_2)(bm+an)^2} \le x \\ \quad\quad \le ab(-1+\alpha_1) - mn - \sqrt{-(-1+\alpha_1)(bm+an)^2}, \\ \frac{2a^2b^2 - b^2m^2 - a^2n^2 - 2abx + (-bm+an)\sqrt{b^2m^2+a^2n^2+2ab(mn+2x)}}{2a^2b^2}, \\ \quad ab(1-\alpha_1) - mn + \sqrt{(1-\alpha_1)(bm-an)^2} \le x \\ \quad\quad \le ab(1-\alpha_2) - mn + \sqrt{(1-\alpha_2)(bm-an)^2}, \\ 0, \quad \text{otherwise.} \end{cases}$$

(3) $0 < \alpha \le \alpha_2$

$$A_\alpha(\cdot)B_\alpha = \left[a_1^{(\alpha)}b_2^{(\alpha)}, a_1^{(\alpha)}b_1^{(\alpha)}\right] or \left[a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}\right].$$

If

$$A_\alpha(\cdot)B_\alpha = \left[a_1^{(\alpha)}b_2^{(\alpha)}, a_1^{(\alpha)}b_1^{(\alpha)}\right]$$
$$= \left[\left(-m - a\sqrt{1-\alpha}\right)\left(n + b\sqrt{1-\alpha}\right), \left(-m + a\sqrt{1-\alpha}\right)\left(n + b\sqrt{1-\alpha}\right)\right].$$

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{2a^2b^2-b^2m^2-a^2n^2+2abx+(bm+an)\sqrt{b^2m^2+a^2n^2-2ab(mn+2x)}}{2a^2b^2}, \\ \quad -(a+m)(b+n) \le x \\ \qquad \le ab(-1+\alpha_1) - mn - \sqrt{-(-1+\alpha_1)(bm+an)^2}, \\ \frac{2a^2b^2-b^2m^2-a^2n^2-2abx+(-bm+an)\sqrt{b^2m^2+a^2n^2+2ab(mn+2x)}}{2a^2b^2}, \\ \quad ab(1-\alpha_1) - mn + \sqrt{(1-\alpha_1)(bm-an)^2} \le x \\ \qquad \le (a+m)(b-n), \\ 0, \quad \text{otherwise.} \end{cases}$$

If

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= \left[a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}\right] \\ &= \left[\left(-m - a\sqrt{1-\alpha}\right)\left(n + b\sqrt{1-\alpha}\right), \left(-m - a\sqrt{1-\alpha}\right)\left(n - b\sqrt{1-\alpha}\right)\right]. \end{aligned}$$

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{2a^2b^2-b^2m^2-a^2n^2+2abx+(bm+an)\sqrt{b^2m^2+a^2n^2-2ab(mn+2x)}}{2a^2b^2}, \\ \quad -(a+m)(b+n) \le x \\ \qquad \le ab(-1+\alpha_1) - mn - \sqrt{-(-1+\alpha_1)(bm+an)^2}, \\ \frac{2a^2b^2-b^2m^2-a^2n^2-2abx-(-bm+an)\sqrt{b^2m^2+a^2n^2+2ab(mn+2x)}}{2a^2b^2}, \\ \quad ab(1-\alpha_1) - mn + \sqrt{(1-\alpha_1)(bm-an)^2} \le x \\ \qquad \le (a-m)(b+n), \\ 0, \quad \text{otherwise.} \end{cases}$$

Hence $A(\cdot)B$ is a fuzzy number.

4. Division :

(1) $\alpha_1 \le \alpha \le 1$

$$A_\alpha\,(/)\,B_\alpha = \left[\frac{a_1{}^{(\alpha)}}{b_1{}^{(\alpha)}}, \frac{a_2{}^{(\alpha)}}{b_2{}^{(\alpha)}}\right] = \left[\frac{-m - a\sqrt{1-\alpha}}{n - b\sqrt{1-\alpha}}, \frac{-m + a\sqrt{1-\alpha}}{n + b\sqrt{1-\alpha}}\right].$$

$$\mu_{A(/)B}(x) = \begin{cases} \frac{a^2-m^2-2abx-2mnx+b^2x^2-n^2x^2}{(a-bx)^2}, \\ \qquad \frac{ab(1-\alpha_1)+mn+\sqrt{(1-\alpha_1)(bm+an)^2}}{(-1+\alpha_1)b^2+n^2} \le x \\ \qquad\qquad \le \frac{ab(-1+\alpha_1)-mn+\sqrt{(1-\alpha_1)(bm+an)^2}}{(-1+\alpha_1)b^2+n^2}, \\ 0, \quad \text{otherwise.} \end{cases}$$

Hence $A(/)B$ has values in $[\alpha_1, 1]$ on $\mathbb{R}$.

Example 4.2. Let $-x_1 < -x_3$, $x_2 < x_4$, $\mu_A(0) = \alpha_1$ and $\mu_B(0) = \alpha_2$. If $\alpha_1 > \alpha_2$, then $A(+)B$ and $A(-)B$ are quadratic fuzzy numbers, and $A(\cdot)B$ is a general fuzzy number. And $A(/)B$ has values in $[\alpha_1, 1]$ on $\mathbb{R}$.

Note that

$$\mu_A(x) = \begin{cases} \frac{-1}{16}(x+6)(x-2) = \frac{-1}{16}(x+2)^2 + 1, & -6 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\mu_B(x) = \begin{cases} \frac{-1}{9}(x+1)(x-5) = \frac{-1}{9}(x-2)^2 + 1, & -1 \le x \le 5, \\ 0, & \text{otherwise.} \end{cases}$$

Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively.

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = \left[-2 - 4\sqrt{1-\alpha}, -2 + 4\sqrt{1-\alpha}\right].$$

Similarly,

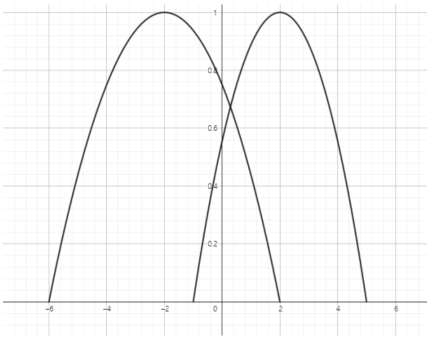

Figure 1: $\mu_A(x), \mu_B(x)$

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = \left[2 - 3\sqrt{1-\alpha}, 2 + 3\sqrt{1-\alpha}\right].$$

1. Addition :

$$A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = \left[-7\sqrt{1-\alpha}, 7\sqrt{1-\alpha}\right].$$

$$\mu_{A(+)B}(x) = \begin{cases} 1 - \frac{x^2}{49}, & -7 \le x \le 7, \\ 0, & \text{otherwise.} \end{cases}$$

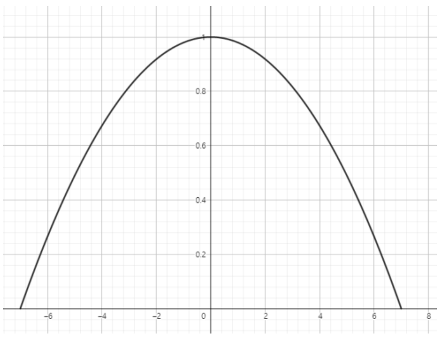

Figure 2: $\mu_{A(+)B}(x)$

Hence $A\,(+)\,B$ is a quadratic fuzzy number.

2. Subtraction :

$$A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = \left[-4 - 7\sqrt{1-\alpha}, -4 + 7\sqrt{1-\alpha}\right].$$

$$\mu_{A(-)B}(x) = \begin{cases} 1 - \frac{(x+4)^2}{49}, & -11 \le x \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

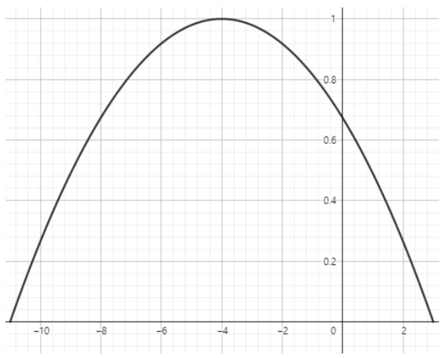

Figure 3: $\mu_{A(-)B}(x)$

Hence $A\,(-)\,B$ is a quadratic fuzzy number.

3. Multiplication :

(1) $\frac{3}{4} < \alpha \le 1$

$$\begin{aligned} A_\alpha\,(\cdot)\,B_\alpha &= \left[a_1^{(\alpha)} b_2^{(\alpha)}, a_2^{(\alpha)} b_1^{(\alpha)}\right] \\ &= \left[\left(-2 - 4\sqrt{1-\alpha}\right)\left(2 + 3\sqrt{1-\alpha}\right), \left(-2 + 4\sqrt{1-\alpha}\right)\left(2 - 3\sqrt{1-\alpha}\right)\right]. \end{aligned}$$

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{1}{72}\left(47 + 7\sqrt{1-12x} + 6x\right), & -14 \le x \le 0, \\ 0, & \text{otherwise.} \end{cases}$$

(2) $\frac{5}{9} < \alpha \leq \frac{3}{4}$

$$A_\alpha(\cdot)B_\alpha = \left[a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}\right]$$
$$= \left[\left(-2-4\sqrt{1-\alpha}\right)\left(2+3\sqrt{1-\alpha}\right), \left(-2+4\sqrt{1-\alpha}\right)\left(2+3\sqrt{1-\alpha}\right)\right].$$

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{1}{72}\left(47+7\sqrt{1-12x}+6x\right), & -\frac{56}{3} \leq x \leq -14, \\ \frac{1}{72}\left(47+\sqrt{49+12x}-6x\right), & 0 \leq x \leq \frac{8}{3}, \\ 0, & \text{otherwise.} \end{cases}$$

(3) $0 < \alpha \leq \frac{5}{9}$

$$A_\alpha(\cdot)B_\alpha = \left[a_1^{(\alpha)}b_2^{(\alpha)}, a_1^{(\alpha)}b_1^{(\alpha)}\right] or \left[a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}\right],$$

but in this case

$$A_\alpha(\cdot)B_\alpha = \left[a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}\right]$$
$$= \left[\left(-2-4\sqrt{1-\alpha}\right)\left(2+3\sqrt{1-\alpha}\right), \left(-2+4\sqrt{1-\alpha}\right)\left(2+3\sqrt{1-\alpha}\right)\right].$$

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{1}{72}\left(47+7\sqrt{1-12x}+6x\right), & -30 \leq x \leq -\frac{56}{3}, \\ \frac{1}{72}\left(47+\sqrt{49+12x}-6x\right), & \frac{8}{3} \leq x \leq 10, \\ 0, & \text{otherwise.} \end{cases}$$

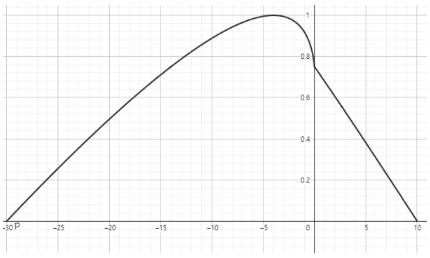

Figure 4: $\mu_{A(\cdot)B}(x)$

Hence $A\,(\cdot)\,B$ is a fuzzy number.

4. Division :

(1) $\frac{3}{4} \le \alpha \le 1$

$$A_\alpha\,(/)\,B_\alpha = \left[\frac{a_1{}^{(\alpha)}}{b_1{}^{(\alpha)}}, \frac{a_2{}^{(\alpha)}}{b_2{}^{(\alpha)}}\right] = \left[\frac{-2-4\sqrt{1-\alpha}}{2-3\sqrt{1-\alpha}}, \frac{-2+4\sqrt{1-\alpha}}{2+3\sqrt{1-\alpha}}\right].$$

$$\mu_{A(/)B}(x) = \begin{cases} \frac{5x^2-32x+12}{(3x-4)^2}, & -8 \le x \le 0, \\ 0, & \text{otherwise.} \end{cases}$$

Figure 5: $\mu_{A(/)B}(x)$

Hence $A(/)B$ has values in $[\alpha_1, 1]$ on $\mathbb{R}$.

Theorem 4.3. Let $-x_1 < -x_3$, $x_2 < x_4$, $\mu_A(0) = \alpha_1$ and $\mu_B(0) = \alpha_2$. If $\alpha_1 = \alpha_2$, then $A(+)B$ and $A(-)B$ are quadratic fuzzy numbers, and $A(\cdot)B$ is a general fuzzy number. And $A(/)B$ has values in $(\alpha_1, 1]$ on $\mathbb{R}$.

Note that

$$\mu_A(x) = \begin{cases} \frac{-1}{a^2}(x+x_1)(x-x_2) = \frac{-1}{a^2}(x+m)^2+1, & -x_1 \le x \le x_2, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\mu_B(x) = \begin{cases} \frac{-m^2}{n^2a^2}(x+x_3)(x-x_4) = \frac{-m^2}{n^2a^2}(x-n)^2+1, & -x_3 \le x \le x_4, \\ 0, & \text{otherwise.} \end{cases}$$

Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively.

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = \left[-m - a\sqrt{1-\alpha}, -m + a\sqrt{1-\alpha}\right].$$

Similarly,

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = \left[n - \frac{an}{m}\sqrt{1-\alpha}, n + \frac{an}{m}\sqrt{1-\alpha}\right].$$

1. Addition :

$$\begin{aligned} A_\alpha(+)B_\alpha &= [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] \\ &= \left[-m + n - (a + \frac{an}{m})\sqrt{1-\alpha}, -m + n + (a + \frac{an}{m})\sqrt{1-\alpha}\right]. \end{aligned}$$

$$\mu_{A(+)B}(x) = \begin{cases} \frac{a^2+2a\left(\frac{an}{m}\right)+\left(\frac{an}{m}\right)^2-(m-n+x)^2}{\left(a+\left(\frac{an}{m}\right)\right)^2}, \\ \qquad -a-\left(\frac{an}{m}\right)-m+n \le x \le a+\left(\frac{an}{m}\right)-m+n, \\ 0, \quad \text{otherwise.} \end{cases}$$

Hence $A(+)B$ is a quadratic fuzzy number.

2. Subtraction :

$$A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}]$$

$$= \left[-m - n - (a + \left(\frac{an}{m}\right))\sqrt{1-\alpha}, -m - n + (a + \left(\frac{an}{m}\right))\sqrt{1-\alpha}\right].$$

$$\mu_{A(-)B}(x) = \begin{cases} \frac{a^2+2a\left(\frac{an}{m}\right)+\left(\frac{an}{m}\right)^2-(m+n+x)^2}{\left(a+\left(\frac{an}{m}\right)\right)^2}, \\ \qquad -a - \left(\frac{an}{m}\right) - m - n \le x \le a + \left(\frac{an}{m}\right) - m - n, \\ 0, \quad \text{otherwise.} \end{cases}$$

Hence $A\,(-)\,B$ is a quadratic fuzzy number.

3. Multiplication :

(1) $\alpha_1 < \alpha \le 1$

$$A_\alpha\,(\cdot)\,B_\alpha = \left[a_1^{(\alpha)} b_2^{(\alpha)}, a_2^{(\alpha)} b_1^{(\alpha)}\right]$$

$$= \left[\left(-m - a\sqrt{1-\alpha}\right)\left(n + \left(\frac{an}{m}\right)\sqrt{1-\alpha}\right), \left(-m + a\sqrt{1-\alpha}\right)\left(n - \left(\frac{an}{m}\right)\sqrt{1-\alpha}\right)\right].$$

$\mu_{A(\cdot)B}(x) =$

$$\begin{cases} \frac{2a^2\left(\frac{an}{m}\right)^2-\left(\frac{an}{m}\right)^2m^2-a^2n^2+2a\left(\frac{an}{m}\right)x+\left(\left(\frac{an}{m}\right)m+an\right)\sqrt{\left(\frac{an}{m}\right)^2m^2+a^2n^2-2a\left(\frac{an}{m}\right)(mn+2x)}}{2a^2\left(\frac{an}{m}\right)^2}, \\ \qquad a\left(\frac{an}{m}\right)(-1+\alpha_1) - mn - \sqrt{-(-1+\alpha_1)\left(\left(\frac{an}{m}\right)m + an\right)^2} \le x \\ \qquad\qquad \le a\left(\frac{an}{m}\right)(-1+\alpha_1) - mn + \sqrt{-(-1+\alpha_1)\left(\left(\frac{an}{m}\right)m + an\right)^2}, \\ 0, \quad \text{otherwise.} \end{cases}$$

(2) $0 < \alpha \le \alpha_2$

$$A_\alpha\,(\cdot)\,B_\alpha = \left[a_1^{(\alpha)} b_2^{(\alpha)}, a_1^{(\alpha)} b_1^{(\alpha)}\right] or \left[a_1^{(\alpha)} b_2^{(\alpha)}, a_2^{(\alpha)} b_2^{(\alpha)}\right].$$

If

$$A_\alpha\,(\cdot)\,B_\alpha$$

$$= \left[a_1^{(\alpha)} b_2^{(\alpha)}, a_1^{(\alpha)} b_1^{(\alpha)}\right]$$

$$= \left[\left(-m - a\sqrt{1-\alpha}\right)\left(n + \left(\frac{an}{m}\right)\sqrt{1-\alpha}\right), \left(-m + a\sqrt{1-\alpha}\right)\left(n + \left(\frac{an}{m}\right)\sqrt{1-\alpha}\right)\right].$$

$\mu_{A(\cdot)B}\,(x) =$

$$\begin{cases} \frac{2a^2\left(\frac{an}{m}\right)^2 - \left(\frac{an}{m}\right)^2 m^2 - a^2n^2 + 2a\left(\frac{an}{m}\right)x + \left(\left(\frac{an}{m}\right)m + an\right)\sqrt{\left(\frac{an}{m}\right)^2 m^2 + a^2n^2 - 2a\left(\frac{an}{m}\right)(mn+2x)}}{2a^2\left(\frac{an}{m}\right)^2}, \\ \quad -\,(a+m)\left(\left(\frac{an}{m}\right)+n\right) \leq x \\ \qquad \leq a\left(\frac{an}{m}\right)(-1+\alpha_1) - mn - \sqrt{-(-1+\alpha_1)\left(\left(\frac{an}{m}\right)m + an\right)^2}, \\ \frac{2a^2\left(\frac{an}{m}\right)^2 - \left(\frac{an}{m}\right)^2 m^2 - a^2n^2 - 2a\left(\frac{an}{m}\right)x + \left(-\left(\frac{an}{m}\right)m + an\right)\sqrt{\left(\frac{an}{m}\right)^2 m^2 + a^2n^2 + 2a\left(\frac{an}{m}\right)(mn+2x)}}{2a^2\left(\frac{an}{m}\right)^2}, \\ \quad a\left(\frac{an}{m}\right)(1-\alpha_1) - mn + \sqrt{(1-\alpha_1)\left(\left(\frac{an}{m}\right)m - an\right)^2} \leq x \\ \qquad \leq (a+m)\left(\left(\frac{an}{m}\right)-n\right), \\ 0, \quad \text{otherwise.} \end{cases}$$

If

$$A_\alpha\,(\cdot)\,B_\alpha$$

$$= \left[a_1^{(\alpha)} b_2^{(\alpha)}, a_2^{(\alpha)} b_2^{(\alpha)}\right],$$

$$= \left[\left(-m - a\sqrt{1-\alpha}\right)\left(n + \left(\frac{an}{m}\right)\sqrt{1-\alpha}\right), \left(-m - a\sqrt{1-\alpha}\right)\left(n - \left(\frac{an}{m}\right)\sqrt{1-\alpha}\right)\right].$$

$\mu_{A(\cdot)B}\,(x) =$

$$
\begin{cases}
\dfrac{2a^2\left(\frac{an}{m}\right)^2-\left(\frac{an}{m}\right)^2m^2-a^2n^2+2a\left(\frac{an}{m}\right)x+\left(\left(\frac{an}{m}\right)m+an\right)\sqrt{\left(\frac{an}{m}\right)^2m^2+a^2n^2-2a\left(\frac{an}{m}\right)(mn+2x)}}{2a^2\left(\frac{an}{m}\right)^2}, \\
\qquad -(a+m)\left(\left(\frac{an}{m}\right)+n\right) \leq x \\
\qquad\qquad \leq a\left(\frac{an}{m}\right)(-1+\alpha_1)-mn-\sqrt{-(-1+\alpha_1)\left(\left(\frac{an}{m}\right)m+an\right)^2}, \\
\dfrac{2a^2\left(\frac{an}{m}\right)^2-\left(\frac{an}{m}\right)^2m^2-a^2n^2-2a\left(\frac{an}{m}\right)x-\left(-\left(\frac{an}{m}\right)m+an\right)\sqrt{\left(\frac{an}{m}\right)^2m^2+a^2n^2+2a\left(\frac{an}{m}\right)(mn+2x)}}{2a^2\left(\frac{an}{m}\right)^2}, \\
\qquad a\left(\frac{an}{m}\right)(1-\alpha_1)-mn+\sqrt{(1-\alpha_1)\left(\left(\frac{an}{m}\right)m-an\right)^2} \leq x \\
\qquad\qquad \leq (a-m)\left(\left(\frac{an}{m}\right)+n\right), \\
0, \quad \text{otherwise.}
\end{cases}
$$

Hence $A(\cdot)B$ is a fuzzy number.

4. Division :

(1) $\alpha_1 < \alpha \leq 1$

$$
A_\alpha(/)B_\alpha = \left[\frac{a_1{}^{(\alpha)}}{b_1{}^{(\alpha)}}, \frac{a_2{}^{(\alpha)}}{b_2{}^{(\alpha)}}\right] = \left[\frac{-m-a\sqrt{1-\alpha}}{n-\left(\frac{an}{m}\right)\sqrt{1-\alpha}}, \frac{-m+a\sqrt{1-\alpha}}{n+\left(\frac{an}{m}\right)\sqrt{1-\alpha}}\right].
$$

$$
\mu_{A(/)B}(x) = \begin{cases}
\dfrac{a^2-m^2-2a\left(\frac{an}{m}\right)x-2mnx+\left(\frac{an}{m}\right)^2x^2-n^2x^2}{\left(a-\left(\frac{an}{m}\right)x\right)^2}, \\
\qquad \dfrac{a\left(\frac{an}{m}\right)(1-\alpha_1)+mn+\sqrt{(1-\alpha_1)\left(\left(\frac{an}{m}\right)m+an\right)^2}}{(-1+\alpha_1)\left(\frac{an}{m}\right)^2+n^2} \leq x \\
\qquad\qquad \leq \dfrac{a\left(\frac{an}{m}\right)(-1+\alpha_1)-mn+\sqrt{(1-\alpha_1)\left(\left(\frac{an}{m}\right)m+an\right)^2}}{(-1+\alpha_1)\left(\frac{an}{m}\right)^2+n^2}, \\
0, \quad \text{otherwise.}
\end{cases}
$$

Hence $A(/)B$ has values in $(\alpha_1, 1]$ on $\mathbb{R}$.

Example 4.4. Let $-x_1 < -x_3$, $x_2 < x_4$, $\mu_A(0) = \alpha_1$ and $\mu_B(0) = \alpha_2$. If $\alpha_1 > \alpha_2$, then $A(+)B$ and $A(-)B$ are quadratic fuzzy numbers, and $A(\cdot)B$ is a general fuzzy number. And $A(/)B$ has values in $(\alpha_1, 1]$ on $\mathbb{R}$.

Note that

$$\mu_A(x) = \begin{cases} \frac{-1}{16}(x+6)(x-2) = \frac{-1}{16}(x+2)^2 + 1, & -6 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\mu_B(x) = \begin{cases} \frac{-1}{16}(x+2)(x-6) = \frac{-1}{16}(x-2)^2 + 1, & -2 \le x \le 6, \\ 0, & \text{otherwise.} \end{cases}$$

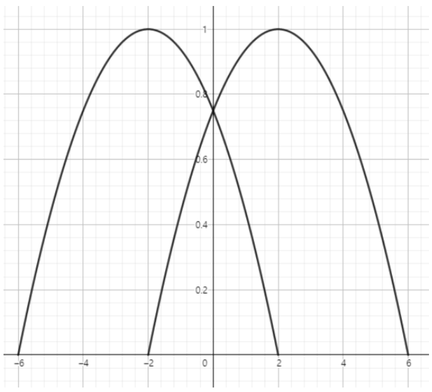

Figure 6: $\mu_A(x), \mu_B(x)$

Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively.

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = \left[-2 - 4\sqrt{1-\alpha}, -2 + 4\sqrt{1-\alpha}\right].$$

Similarly,

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = \left[2 - 4\sqrt{1-\alpha}, 2 + 4\sqrt{1-\alpha}\right].$$

1. Addition :

$A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = \left[-8\sqrt{1-\alpha}, 8\sqrt{1-\alpha}\right].$

$$\mu_{A(+)B}(x) = \begin{cases} 1 - \frac{x^2}{64}, & -8 \le x \le 8, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $A\,(+)\,B$ is a quadratic fuzzy number.

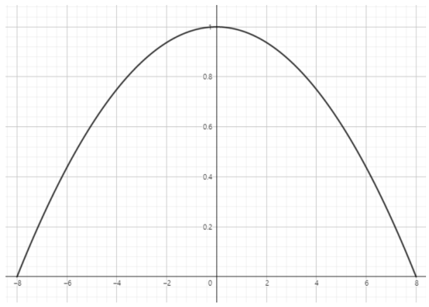

Figure 7: $\mu_{A(+)B}(x)$

2. Subtraction :

$A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = \left[-4 - 8\sqrt{1-\alpha}, -4 + 8\sqrt{1-\alpha}\right].$

$$\mu_{A(-)B}(x) = \begin{cases} \frac{1}{64}\left(48 - 8x - x^2\right), & -12 \le x \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $A\,(-)\,B$ is a quadratic fuzzy number.

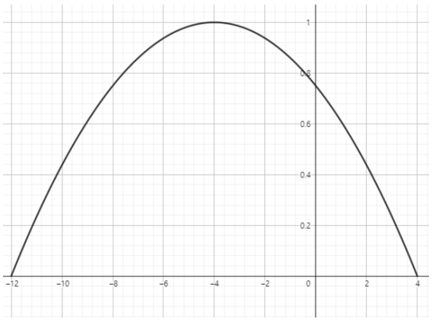

Figure 8: $\mu_{A(-)B}(x)$

3. Multiplication :

(1) $\frac{3}{4} < \alpha \leq 1$

$$A_\alpha(\cdot)B_\alpha = \left[a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_1^{(\alpha)}\right]$$
$$= \left[\left(-2-4\sqrt{1-\alpha}\right)\left(2+4\sqrt{1-\alpha}\right), \left(-2+4\sqrt{1-\alpha}\right)\left(2-4\sqrt{1-\alpha}\right)\right].$$

$$\mu_{A(\cdot)B}(x)\begin{cases} \frac{1}{16}\left(12+4\sqrt{-x}+x\right), & -16 \leq x \leq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(2) $0 < \alpha \leq \frac{3}{4}$

$$A_\alpha(\cdot)B_\alpha = \left[a_1^{(\alpha)}b_2^{(\alpha)}, a_1^{(\alpha)}b_1^{(\alpha)}\right] or \left[a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}\right],$$

but in this case

$$A_\alpha(\cdot)B_\alpha = \left[a_1^{(\alpha)}b_2^{(\alpha)}, a_1^{(\alpha)}b_1^{(\alpha)}\right]$$
$$= \left[\left(-2-4\sqrt{1-\alpha}\right)\left(2+4\sqrt{1-\alpha}\right), \left(-2+4\sqrt{1-\alpha}\right)\left(2+4\sqrt{1-\alpha}\right)\right].$$

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{1}{16}\left(12+4\sqrt{-x}+x\right), & -36 \leq x \leq -16, \\ \frac{12-x}{16}, & 0 \leq x \leq 12, \\ 0, & \text{otherwise.} \end{cases}$$

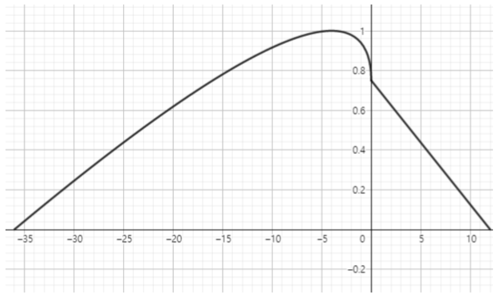

Figure 9: $\mu_{A(\cdot)B}(x)$

Hence $A\,(\cdot)\,B$ is a fuzzy number.

4. Division :

(1) $\frac{3}{4} < \alpha \le 1$

$$A_\alpha\,(/)\,B_\alpha = \left[\frac{a_1{}^{(\alpha)}}{b_1{}^{(\alpha)}}, \frac{a_2{}^{(\alpha)}}{b_2{}^{(\alpha)}}\right] = \left[\frac{-2-4\sqrt{1-\alpha}}{2-4\sqrt{1-\alpha}}, \frac{-2+4\sqrt{1-\alpha}}{2+4\sqrt{1-\alpha}}\right].$$

$$\mu_{A(/)B}(x) = \begin{cases} \frac{3x^2-10x+3}{4(x-1)^2}, & x \le 0, \\ 0, & \text{otherwise.} \end{cases}$$

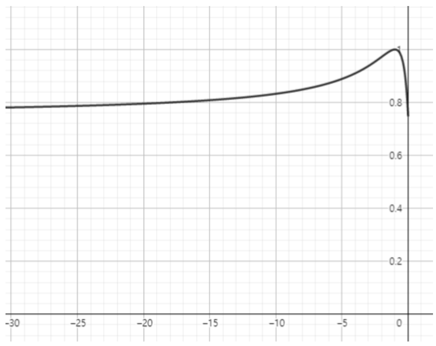

Figure 10: $\mu_{A(/)B}(x)$

Hence $A(/)B$ has values in $(\alpha_1, 1]$ on $\mathbb{R}$.

Theorem 4.5. Let $-x_1 < -x_3$, $x_2 < x_4$, $\mu_A(0) = \alpha_1$ and $\mu_B(0) = \alpha_2$. If $\alpha_1 < \alpha_2$, then $A(+)B$ and $A(-)B$ are quadratic fuzzy numbers, and $A(\cdot)B$ is a general fuzzy number. And $A(/)B$ has values in $(\alpha_1, 1]$ on $\mathbb{R}$.

Note that

$$\mu_A(x) = \begin{cases} \frac{-1}{a^2}(x+x_1)(x-x_2) = \frac{-1}{a^2}(x+m)^2 + 1, & -x_1 \le x \le x_2, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\mu_B(x) = \begin{cases} \frac{-1}{b^2}(x+x_3)(x-x_4) = \frac{-1}{b^2}(x-n)^2 + 1, & -x_3 \le x \le x_4, \\ 0, & \text{otherwise.} \end{cases}$$

Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively.

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = \left[-m - a\sqrt{1-\alpha}, -m + a\sqrt{1-\alpha}\right].$$

Similarly,

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = \left[n - b\sqrt{1-\alpha}, n + b\sqrt{1-\alpha}\right].$$

1. Addition :

$$\begin{aligned} A_\alpha(+)B_\alpha &= [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] \\ &= \left[-m + n - (a+b)\sqrt{1-\alpha}, -m + n + (a+b)\sqrt{1-\alpha}\right]. \end{aligned}$$

$$\mu_{A(+)B}(x) = \begin{cases} \frac{a^2+2ab+b^2-(m-n+x)^2}{(a+b)^2}, & -a-b-m+n \le x \le a+b-m+n, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $A(+)B$ is a quadratic fuzzy number.

2. Subtraction :

$$A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}]$$

$$= \left[-m-n-(a+b)\sqrt{1-\alpha}, -m-n+(a+b)\sqrt{1-\alpha}\right].$$

$$\mu_{A(-)B}(x) = \begin{cases} \frac{a^2+2ab+b^2-(m+n+x)^2}{(a+b)^2}, & -a-b-m-n \le x \le a+b-m-n, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $A(-)B$ is a quadratic fuzzy number.

3. Multiplication :

(1) $\alpha_2 < \alpha \le 1$

$$A_\alpha(\cdot)B_\alpha = \left[a_1^{(\alpha)} b_2^{(\alpha)}, a_2^{(\alpha)} b_1^{(\alpha)}\right]$$

$$= \left[\left(-m-a\sqrt{1-\alpha}\right)\left(n+b\sqrt{1-\alpha}\right), \left(-m+a\sqrt{1-\alpha}\right)\left(n-b\sqrt{1-\alpha}\right)\right].$$

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{2a^2b^2-b^2m^2-a^2n^2+2abx+(bm+an)\sqrt{b^2m^2+a^2n^2-2ab(mn+2x)}}{2a^2b^2}, \\ \quad ab(-1+\alpha_1)-mn-\sqrt{-(-1+\alpha_1)(bm+an)^2} \le x \\ \quad \le ab(-1+\alpha_1)-mn+\sqrt{-(-1+\alpha_1)(bm+an)^2}, \\ 0, \quad \text{otherwise.} \end{cases}$$

(2) $\alpha_1 < \alpha \le \alpha_2$

$$A_\alpha(\cdot)B_\alpha = \left[a_1^{(\alpha)} b_1^{(\alpha)}, a_1^{(\alpha)} b_2^{(\alpha)}\right]$$

$$= \left[\left(-m-a\sqrt{1-\alpha}\right)\left(n-b\sqrt{1-\alpha}\right), \left(-m-a\sqrt{1-\alpha}\right)\left(n+b\sqrt{1-\alpha}\right)\right].$$

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{2a^2b^2 - b^2m^2 - a^2n^2 - 2abx - (-bm+an)\sqrt{b^2m^2+a^2n^2+2ab(mn+2x)}}{2a^2b^2}, \\ \qquad ab(1-\alpha_1) - mn - \sqrt{(1-\alpha_1)(bm-an)^2} \le x \\ \qquad\qquad \le (a-m)(b+n), \\ \frac{2a^2b^2 - b^2m^2 - a^2n^2 + 2abx + (bm+an)\sqrt{b^2m^2+a^2n^2-2ab(mn+2x)}}{2a^2b^2}, \\ \qquad ab(-1+\alpha_1) - mn - \sqrt{-(-1+\alpha_1)(bm+an)^2} \le x \\ \qquad\qquad \le ab(-1+\alpha_1) - mn + \sqrt{-(-1+\alpha_1)(bm+an)^2}, \\ 0, \quad \text{otherwise.} \end{cases}$$

(3) $0 < \alpha \le \alpha_1$

$$A_\alpha(\cdot)B_\alpha = \left[a_1^{(\alpha)}b_2^{(\alpha)}, a_1^{(\alpha)}b_1^{(\alpha)}\right] or \left[a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}\right].$$

If

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= \left[a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}\right] \\ &= \left[\left(-m - a\sqrt{1-\alpha}\right)\left(n + b\sqrt{1-\alpha}\right), \left(-m + a\sqrt{1-\alpha}\right)\left(n + b\sqrt{1-\alpha}\right)\right]. \end{aligned}$$

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{2a^2b^2 - b^2m^2 - a^2n^2 + 2abx + (bm+an)\sqrt{b^2m^2+a^2n^2-2ab(mn+2x)}}{2a^2b^2}, \\ \qquad -(a+m)(b+n) \le x \\ \qquad\qquad \le ab(-1+\alpha_1) - mn - \sqrt{-(-1+\alpha_1)(bm+an)^2}, \\ \frac{2a^2b^2 - b^2m^2 - a^2n^2 - 2abx + (-bm+an)\sqrt{b^2m^2+a^2n^2+2ab(mn+2x)}}{2a^2b^2}, \\ \qquad ab(1-\alpha_1) - mn + \sqrt{(1-\alpha_1)(bm-an)^2} \le x \\ \qquad\qquad \le (a+m)(b-n), \\ 0, \quad \text{otherwise.} \end{cases}$$

If

$$A_\alpha (\cdot) B_\alpha = \left[a_1^{(\alpha)} b_2^{(\alpha)}, a_1^{(\alpha)} b_1^{(\alpha)}\right]$$
$$= \left[\left(-m - a\sqrt{1-\alpha}\right)\left(n + b\sqrt{1-\alpha}\right), \left(-m - a\sqrt{1-\alpha}\right)\left(n - b\sqrt{1-\alpha}\right)\right].$$

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{2a^2b^2 - b^2m^2 - a^2n^2 + 2abx + (bm+an)\sqrt{b^2m^2+a^2n^2-2ab(mn+2x)}}{2a^2b^2}, \\ \quad -(a+m)(b+n) \le x \\ \qquad \le ab(-1+\alpha_1) - mn - \sqrt{-(-1+\alpha_1)(bm+an)^2}, \\ \frac{2a^2b^2 - b^2m^2 - a^2n^2 - 2abx - (-bm+an)\sqrt{b^2m^2+a^2n^2+2ab(mn+2x)}}{2a^2b^2}, \\ \quad ab(1-\alpha_1) - mn + \sqrt{(1-\alpha_1)(bm-an)^2} \le x \\ \qquad \le (a-m)(b+n), \\ 0, \quad \text{otherwise.} \end{cases}$$

Hence $A(\cdot)B$ is a fuzzy number.

4. Division :

(1) $\alpha_1 < \alpha \le 1$

$$A_\alpha (/) B_\alpha = \left[\frac{a_1{}^{(\alpha)}}{b_1{}^{(\alpha)}}, \frac{a_2{}^{(\alpha)}}{b_2{}^{(\alpha)}}\right] = \left[\frac{-m - a\sqrt{1-\alpha}}{n - b\sqrt{1-\alpha}}, \frac{-m + a\sqrt{1-\alpha}}{n + b\sqrt{1-\alpha}}\right].$$

$$\mu_{A(/)B}(x) = \begin{cases} \frac{a^2 - m^2 - 2abx - 2mnx + b^2x^2 - n^2x^2}{(a-bx)^2}, \\ \quad \frac{ab(1-\alpha_1) + mn + \sqrt{(1-\alpha_1)(bm+an)^2}}{(-1+\alpha_1)b^2 + n^2} \le x \\ \qquad \le \frac{ab(-1+\alpha_1) - mn + \sqrt{(1-\alpha_1)(bm+an)^2}}{(-1+\alpha_1)b^2 + n^2}, \\ 0, \quad \text{otherwise.} \end{cases}$$

Hence $A(/)B$ has values in $(\alpha_1, 1]$ on $\mathbb{R}$.

Example 4.6. Let $-x_1 < -x_3$, $x_2 < x_4$, $\mu_A(0) = \alpha_1$ and $\mu_B(0) = \alpha_2$. If $\alpha_1 < \alpha_2$, then $A(+)B$ and $A(-)B$ are quadratic fuzzy numbers, and $A(\cdot)B$ is a general fuzzy number. And $A(/)B$ has values in $(\alpha_1, 1]$ on $\mathbb{R}$

Note that

$$\mu_A(x) = \begin{cases} \frac{-1}{16}(x+6)(x-2) = \frac{-1}{16}(x+2)^2 + 1, & -6 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\mu_B(x) = \begin{cases} \frac{-1}{16}(x+3)(x-5) = \frac{-1}{16}(x-1)^2 + 1, & -3 \le x \le 5, \\ 0, & \text{otherwise.} \end{cases}$$

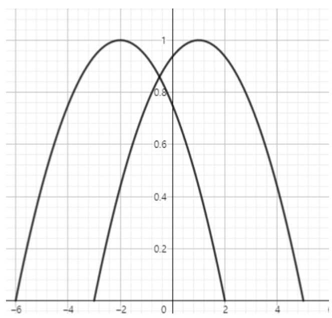

Figure 11: $\mu_A(x), \mu_B(x)$

Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively.

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = \left[-2 - 4\sqrt{1-\alpha}, -2 + 4\sqrt{1-\alpha}\right].$$

Similarly,

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = \left[1 - 4\sqrt{1-\alpha}, 1 + 4\sqrt{1-\alpha}\right].$$

1. Addition :

$$A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = \left[-1 - 8\sqrt{1-\alpha}, -1 + 8\sqrt{1-\alpha}\right].$$

$$\mu_{A(+)B}(x) = \begin{cases} \frac{-1}{64}\left(x^2 + 2x - 63\right), & -9 \le x \le 7, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $A\,(+)\,B$ is a quadratic fuzzy number.

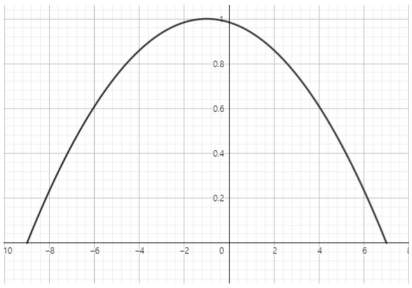

Figure 12: $\mu_{A(+)B}(x)$

2. Subtraction :

$$A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = \left[-3 - 8\sqrt{1-\alpha}, -3 + 8\sqrt{1-\alpha}\right].$$

$$\mu_{A(-)B}(x) = \begin{cases} \frac{-1}{64}\left(x^2 + 6x - 55\right), & -11 \le x \le 5, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $A\,(-)\,B$ is a quadratic fuzzy number.

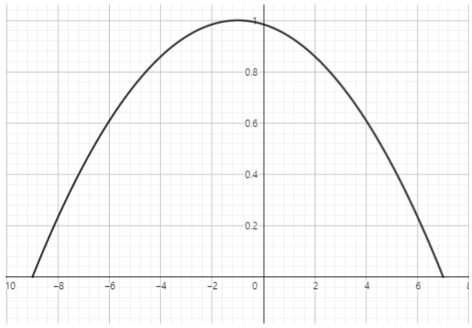

Figure 13: $\mu_{A(-)B}(x)$

3. Multiplication :

(1) $\frac{15}{16} < \alpha \leq 1$

$$A_\alpha\,(\cdot)\,B_\alpha = \left[a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_1^{(\alpha)}\right]$$

$$= \left[\left(-2-4\sqrt{1-\alpha}\right)\left(1+4\sqrt{1-\alpha}\right), \left(-2+4\sqrt{1-\alpha}\right)\left(1-4\sqrt{1-\alpha}\right)\right].$$

$$\mu_{A(\cdot)B}\,(x) \begin{cases} \frac{1}{32}\left(27+3\sqrt{1-4x}+2x\right), & -6 \leq x \leq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(2) $\frac{3}{4} < \alpha \leq \frac{15}{16}$

$$A_\alpha\,(\cdot)\,B_\alpha = \left[a_1^{(\alpha)}b_1^{(\alpha)}, a_1^{(\alpha)}b_2^{(\alpha)}\right]$$

$$= \left[\left(-2-4\sqrt{1-\alpha}\right)\left(1-4\sqrt{1-\alpha}\right), \left(-2-4\sqrt{1-\alpha}\right)\left(1+4\sqrt{1-\alpha}\right)\right].$$

$$\mu_{A(\cdot)B}\,(x) = \begin{cases} \frac{1}{32}\left(27+3\sqrt{1-4x}+2x\right), & -12 \leq x \leq -6, \\ \frac{1}{32}\left(27+\sqrt{9+4x}-2x\right), & 0 \leq x \leq 4, \\ 0, & \text{otherwise.} \end{cases}$$

(3) $0 < \alpha \leq \frac{3}{4}$

$$A_\alpha\,(\cdot)\,B_\alpha = \left[a_1^{(\alpha)}b_2^{(\alpha)}, a_1^{(\alpha)}b_1^{(\alpha)}\right] or \left[a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}\right],$$

but in this case

$$A_\alpha\,(\cdot)\,B_\alpha = \left[a_1^{(\alpha)} b_2^{(\alpha)}, a_1^{(\alpha)} b_1^{(\alpha)}\right]$$
$$= \left[\left(-2-4\sqrt{1-\alpha}\right)\left(1+4\sqrt{1-\alpha}\right), \left(-2-4\sqrt{1-\alpha}\right)\left(1-4\sqrt{1-\alpha}\right)\right].$$

$$\mu_{A(\cdot)B}\,(x) = \begin{cases} \frac{1}{32}\left(27+3\sqrt{1-4x}+2x\right), & -30 \le x \le -12, \\ \frac{1}{32}\left(27+\sqrt{9+4x}-2x\right), & 4 \le x \le 18, \\ 0, & \text{otherwise.} \end{cases}$$

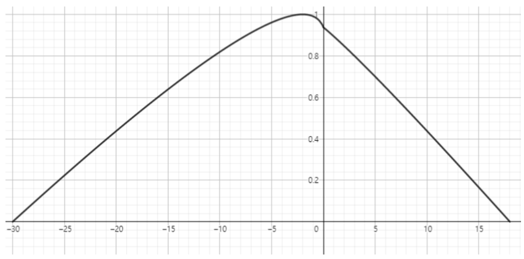

Figure 14: $\mu_{A(\cdot)B}(x)$

Hence $A\,(\cdot)\,B$ is a fuzzy number.

4. Division :

(1) $\frac{15}{16} < \alpha \le 1$

$$A_\alpha\,(/)\,B_\alpha = \left[\frac{a_1{}^{(\alpha)}}{b_1{}^{(\alpha)}}, \frac{a_2{}^{(\alpha)}}{b_2{}^{(\alpha)}}\right] = \left[\frac{-2-4\sqrt{1-\alpha}}{1-4\sqrt{1-\alpha}}, \frac{-2+4\sqrt{1-\alpha}}{1+4\sqrt{1-\alpha}}\right].$$

$$\mu_{A(/)B}(x) = \begin{cases} \frac{15x^2-36x+12}{16(x-1)^2}, & x \le \frac{-1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $A(/)B$ has values in $(\alpha_1, 1]$ on $\mathbb{R}$.

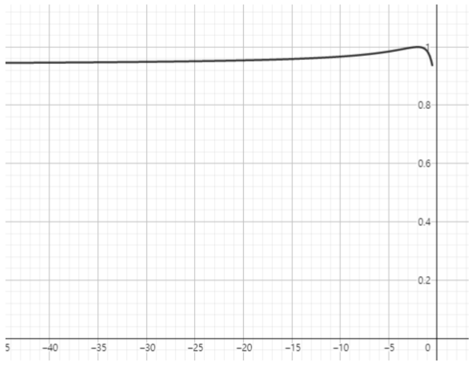

Figure 15: $\mu_{A(/)B}(x)$

We have computed Zadeh's max-min composition operator for two quadratic fuzzy numbers $A = [-x_1, -m, x_2]$ and $B = [-x_3, n, x_4]$. We got three kinds of conclusions according to the three magnitude relationship between $\mu_A(0)$ and $\mu_B(0)$, i.e., $\mu_A(0) > \mu_B(0)$, $\mu_A(0) = \mu_B(0)$ and $\mu_A(0) < \mu_B(0)$. For each case, $A(+)B$ and $A(-)B$ were quadratic fuzzy numbers, and $A(\cdot)B$ was a fuzzy numbers, but $A(/)B$ was a different type of fuzzy number. In conclusion, $A(+)B$, $A(-)B$, $A(\cdot)B$ can be applied where the shape of the quadratic fuzzy number comes out, and $A(/)B$ can be applied where appropriate.

5 2-dimensional quadratic fuzzy set

In this section, taking the examples of two 2-dimensional quadratic fuzzy sets, we obtain the equations of the intersections between planes perpendicular to the x-axis and passing through each vertex and two 2-dimensional quadratic fuzzy numbers. Then, the extended four operations of the two 1-dimensional quadratic fuzzy sets are calculated and graphed. Meanwhile, we computed the extended four operations of the 2-dimensional quadratic fuzzy numbers, which are the two examples above. Then we calculated the intersection between a plane perpendicular to the x-axis and passing through each vertex and the resulting 2-dimensional quadratic fuzzy number. We confirmed that the equations of the two intersections acquired in this way and the graphs are actually identical, respectively.

Let $A^2 = [a,\ b,\ c,\ d]^2$ and $B^2 = [p,\ q,\ r,\ s]^2$. The intersections of A^2, B^2 and planes perpendicular to the x-axis and passing through each vertex are called A and B, respectively. The membership functions of A and B are $\mu_A = \frac{-1}{c^2}(y-d)^2 + 1$ and $\mu_B = \frac{-1}{r^2}(y-s)^2 + 1$. We calculate exactly the above four operations using α-cuts. Let $A_\alpha = \left[{a_1}^{(\alpha)}, {a_2}^{(\alpha)}\right]$ and $B_\alpha = \left[{b_1}^{(\alpha)}, {b_2}^{(\alpha)}\right]$ be the α-cuts of A and B, respectively. Then we have

$$A_\alpha = \left[d - c\sqrt{1-\alpha}, d + c\sqrt{1-\alpha}\right], B_\alpha = \left[s - r\sqrt{1-\alpha}, s + r\sqrt{1-\alpha}\right].$$

(1) Addition: Since

$$A_\alpha\,(+)\,B_\alpha = \left[(d+s) - (c+r)\sqrt{1-\alpha}, (d+s) + (c+r)\sqrt{1-\alpha}\right],$$

we have

$$\mu_{A(+)B}(y) = \begin{cases} 1 - \frac{(y-(d+s))^2}{(c+r)^2}, & (d+s)-(c+r) \le y \le (d+s)+(c+r), \\ 0, & \text{otherwise.} \end{cases}$$

By Theorem 2.15([5]), $A(+)_p B = [a+p, b+q, c+r, d+s]^2$. Thus

$$\mu_{A(+)B}(x,y) = \begin{cases} 1 - \left(\frac{(x-(b+q))^2}{(a+p)^2} + \frac{(y-(d+s))^2}{(c+r)^2} \right), \\ \qquad (c+r)^2(x-(b+q))^2 + (a+p)^2(y-(d+s))^2 \\ \qquad \le (a+p)^2(c+r)^2, \\ 0, \qquad \text{otherwise.} \end{cases}$$

Substituting $x = b+q$ into $\mu_{A(+)B}(x,y)$,

$$\mu_{A(+)B}(b+q,y) = \begin{cases} 1 - \left(\frac{(y-(d+s))^2}{(c+r)^2} \right), & (y-(d+s))^2 \le (c+r)^2, \\ 0, & \text{otherwise.} \end{cases}$$

This result indicates that $\mu_{A(+)B}(y)$ and $\mu_{A(+)B}(b+q,y)$ match.

(2) Subtraction: Since

$A_\alpha(-)B_\alpha = \left[(d-s)-(c+r)\sqrt{1-\alpha}, (d-s)+(c+r)\sqrt{1-\alpha}\right]$, we have

$$\mu_{A(-)B}(y) = \begin{cases} 1 - \frac{(y-(d-s))^2}{(c+r)^2}, & (d-s)-(c+r) \le y \le (d-s)+(c+r), \\ 0, & \text{otherwise.} \end{cases}$$

By Theorem 2.15([5]), $A(-)_p B = [a+p, b-q, c+r, d-s]^2$. Thus

$$\mu_{A(-)B}(x,y) = \begin{cases} 1 - \left(\frac{(x-(b-q))^2}{(a+p)^2} + \frac{(y-(d-s))^2}{(c+r)^2} \right), & (c+r)^2 (x-(b-q))^2 + (a+p)^2 (y-(d-s))^2 \\ & \le (a+p)^2 (c+r)^2, \\ 0, & \text{otherwise.} \end{cases}$$

Substituting $x = b - q$ into $\mu_{A(-)B}(x,y)$,

$$\mu_{A(-)B}(b-q,y) = \begin{cases} 1 - \left(\frac{(y-(d-s))^2}{(c+r)^2} \right), & (y-(d-s))^2 \le (c+r)^2, \\ 0, & \text{otherwise.} \end{cases}$$

This result indicates that $\mu_{A(-)B}(y)$ and $\mu_{A(-)B}(b-q,y)$ match.

(3) Multiplication: Since

$$A_\alpha (\cdot) B_\alpha = \left[\left(d - c\sqrt{1-\alpha}\right)\left(s - r\sqrt{1-\alpha}\right), \left(d + c\sqrt{1-\alpha}\right)\left(s + r\sqrt{1-\alpha}\right) \right]$$

$$\mu_{(A\times B)}(y) = \begin{cases} \frac{2c^2r^2 - d^2r^2 - c^2s^2 - 2cry + (dr+cs)\sqrt{d^2r^2 - 2cdrs + c^2s^2 + 4cry}}{2c^2r^2}, & \\ & (d-c)(s-r) \le y \le (d+c)(s+r), \\ 0, & \text{otherwise.} \end{cases}$$

By Theorem 2.15([5]), $\left(A(\cdot)_p B\right)^\alpha = \{(x_a(t), y_a(t)) \mid 0 \le t \le 2\pi\}$, where

$$\begin{cases} x_\alpha(t) = bq + (bp+qa)\sqrt{1-\alpha}\cos t + ap(1-\alpha)\cos^2 t, \\ y_\alpha(t) = ds + (dr+sc)\sqrt{1-\alpha}\sin t + cr(1-\alpha)\sin^2 t. \end{cases}$$

Since $x_\alpha\left(\frac{3\pi}{2}\right) = bq$ and $y_\alpha\left(\frac{3\pi}{2}\right) = ds + (dr+sc)\sqrt{1-\alpha} + cr(1-\alpha)$, we have

$$\alpha = \frac{2c^2r^2 - d^2r^2 - c^2s^2 - 2cry_\alpha\left(\frac{3\pi}{2}\right) + (dr+cs)\sqrt{d^2r^2 - 2cdrs + c^2s^2 + 4cry_\alpha\left(\frac{3\pi}{2}\right)}}{2c^2r^2}$$

This result indicates that $\mu_{A(\cdot)B}(y)$ and $\mu_{A(\cdot)B}(bq, y)$ match.

(4) Division: Since

$A_\alpha (/) B_\alpha = \left[\frac{d-c\sqrt{1-\alpha}}{s+r\sqrt{1-\alpha}}, \frac{d+c\sqrt{1-\alpha}}{s-r\sqrt{1-\alpha}}\right]$, we have

$$\mu_{A(/)B}(y) = \begin{cases} \frac{c^2-d^2+2cry+2dsy+r^2y^2-s^2y^2}{(c+ry)^2}, & \frac{d-c}{s+r} \le y \le \frac{d+c}{s-r}, \\ 0, & \text{otherwise.} \end{cases}$$

By Theorem 2.15([5]), $\left(A(/)_p B\right)^\alpha = \{(x_a(t), y_a(t)) \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = \frac{b + a\sqrt{1-\alpha}\cos t}{q - p\sqrt{1-\alpha}\cos t}, \quad y_\alpha(t) = \frac{d + c\sqrt{1-\alpha}\sin t}{s - r\sqrt{1-\alpha}\sin t}.$$

Since $x_\alpha\left(\frac{3\pi}{2}\right) = \frac{b}{q}$ and $y_\alpha\left(\frac{3\pi}{2}\right) = \frac{d-c\sqrt{1-\alpha}}{s+r\sqrt{1-\alpha}}$, we have

$$\alpha = \frac{c^2 - d^2 + 2cr\left(y_\alpha\left(\frac{3\pi}{2}\right)\right) + 2ds\left(y_\alpha\left(\frac{3\pi}{2}\right)\right) + r^2\left(y_\alpha\left(\frac{3\pi}{2}\right)\right)^2 - s^2\left(y_\alpha\left(\frac{3\pi}{2}\right)\right)^2}{\left(c + r\left(y_\alpha\left(\frac{3\pi}{2}\right)\right)\right)^2}.$$

This result indicates that $\mu_{A(/)B}(y)$ and $\mu_{A(/)B}\left(\frac{b}{q}, y\right)$ match.

In general, calculation and comparison are very complicated, so for convenience of calculation, only special cases that were perpendicular to the xy-plane and the x-axis, and pass through the vertices were calculated and compared. As a result, it was confirmed through some cases that a 1-dimensional quadratic fuzzy number can be extended to a 2-dimensional quadratic fuzzy number and well defined.

Example 5.1. ([12]) Let $A^2 = [6, 10, 8, 16]^2$ and $B^2 = [4, 8, 5, 12]^2$. The graphs of μ_{A^2} and μ_{B^2} are as follows:

The intersections of A^2, B^2 and planes perpendicular to the x-axis and passing through each vertex are called A and B, respectively. The membership functions of A and B are $\mu_A = \frac{-1}{64}(y-16)^2 + 1$ and $\mu_B = \frac{-1}{25}(y-12)^2 + 1$.

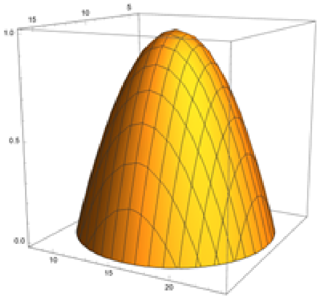

Figure 16: $\mu_{A^2}(x,y)$

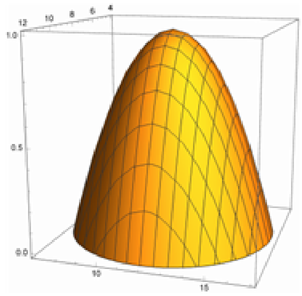

Figure 17: $\mu_{B^2}(x,y)$

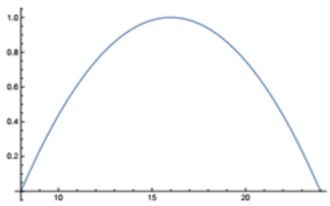

Figure 18: $\mu_A(y)$

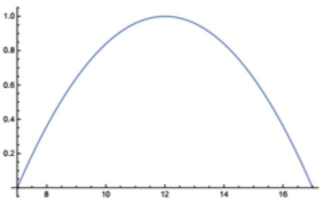

Figure 19: $\mu_B(y)$

We calculate exactly the above four operations using α-cuts. Let $A_\alpha = \left[a_1{}^{(\alpha)}, a_2{}^{(\alpha)}\right]$ and $B_\alpha = \left[b_1{}^{(\alpha)}, b_2{}^{(\alpha)}\right]$ be the α-cuts of A and B, respectively. Then we have

$$A_\alpha = \left[16 - 8\sqrt{1-\alpha}, 16 + 8\sqrt{1-\alpha}\right], B_\alpha = \left[12 - 5\sqrt{1-\alpha}, 12 + 5\sqrt{1-\alpha}\right].$$

(1) Addition: Since $A_\alpha\,(+)\,B_\alpha = \left[28 - 13\sqrt{1-\alpha}, 28 + 13\sqrt{1-\alpha}\right]$, we have

$$\mu_{A(+)B}\,(y) = \begin{cases} \frac{1}{169}\left(-615 + 56y - y^2\right), & 15 \le y \le 41, \\ 0, & \text{otherwise.} \end{cases}$$

By Theorem 2.15([5]), $A(+)_p B = [10, 18, 13, 28]^2$. Thus

$$\mu_{A(+)B}\,(x,y) = \begin{cases} 1 - \left(\frac{(x-18)^2}{10^2} + \frac{(y-28)^2}{13^2}\right), & 13^2(x-18)^2 + 10^2(y-28)^2 \le 10^2 13^2, \\ 0, & \text{otherwise.} \end{cases}$$

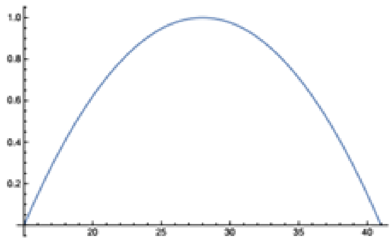

Figure 20: $\mu_{A(+)B}(y)$

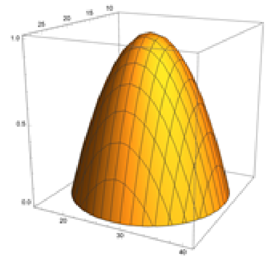

Figure 21: $\mu_{A(+)B}(x, y)$

Substituting $x = 18$ into $\mu_{A(+)B}(x, y)$,

$$\mu_{A(+)B}(18, y) = \begin{cases} 1 - \left(\frac{(y-28)^2}{13^2}\right), & (y-28)^2 \leq 13^2, \\ 0, & \text{otherwise.} \end{cases}$$

This result indicates that $\mu_{A(+)B}(y)$ and $\mu_{A(+)B}(18, y)$ match. The section cut perpendicular to the x-axis at the vertex of Figure 21 is shown in Figure 22, and this section is shown in Figure 20.

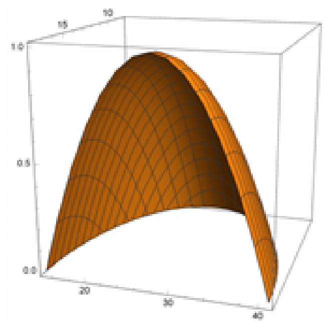

Figure 22: $\mu_{A(+)B}(18, y)$

(2) Subtraction: Since $A_\alpha (-) B_\alpha = \left[4 - 13\sqrt{1-\alpha}, 4 + 13\sqrt{1-\alpha}\right]$, we have

$$\mu_{A(-)B}(y) = \begin{cases} \frac{1}{169}\left(153 + 8y - y^2\right), & -9 \leq y \leq 17, \\ 0, & \text{otherwise.} \end{cases}$$

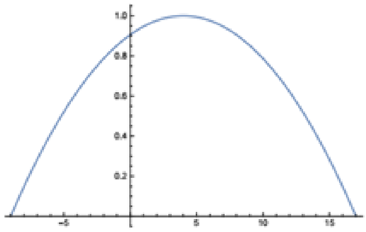

Figure 23: $\mu_{A(-)B}(y)$

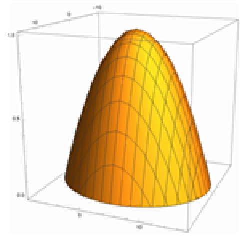

Figure 24: $\mu_{A(-)B}(x,y)$

By Theorem 2.15([5]), $A(-)_p B = [10, 2, 13, 4]^2$. Thus

$$\mu_{A(-)B}(x,y) = \begin{cases} 1 - \left(\frac{(x-2)^2}{10^2} + \frac{(y-4)^2}{13^2}\right), & 13^2(x-2)^2 + 10^2(y-4)^2 \le 10^2 13^2, \\ 0, & \text{otherwise.} \end{cases}$$

Substituting $x = 2$ into $\mu_{A(-)B}(x,y)$,

$$\mu_{A(-)B}(2,y) = \begin{cases} 1 - \left(\frac{(y-4)^2}{13^2}\right), & (y-4)^2 \le 13^2, \\ 0, & \text{otherwise.} \end{cases}$$

This result indicates that $\mu_{A(-)B}(y)$ and $\mu_{A(-)B}(2,y)$ match. The section cut perpendicular to the x-axis at the vertex of Figure 24 is shown in Figure 25, and this section is shown in Figure 23.

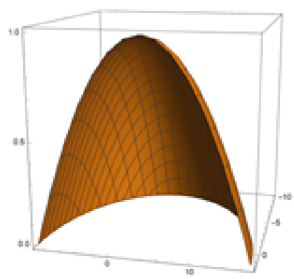

Figure 25: $\mu_{A(-)B}(2,y)$

(3) Multiplication: Since

$$A_\alpha (\cdot) B_\alpha = [(16 - 8\sqrt{1-\alpha})\left(12 - 5\sqrt{1-\alpha}\right), \left(16 + 8\sqrt{1-\alpha}\right)\left(12 + 5\sqrt{1-\alpha}\right)],$$

we have

$$\mu_{A(\cdot)B}(y) = \begin{cases} \frac{1}{200}\left(-776 - 5y + 44\sqrt{2}\sqrt{8+5y}\right), & 56 \le y \le 408, \\ 0, & \text{otherwise.} \end{cases}$$

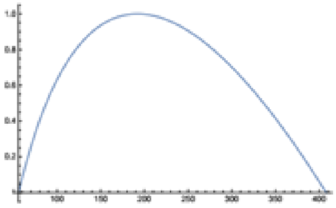

Figure 26: $\mu_{A(\times)B}(y)$

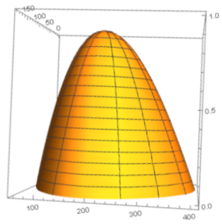

Figure 27: $\mu_{A(\times)B}(x, y)$

By Theorem 2.15([5]), $\left(A\left(\cdot\right)_p B\right)^{\alpha} = \{(x_a(t), y_a(t)) \mid 0 \le t \le 2\pi\}$, where

$$\begin{cases} x_{\alpha}(t) = 80 + 88\sqrt{1-\alpha}\cos t + 24(1-\alpha)\cos^2 t, \\ y_{\alpha}(t) = 192 + 176\sqrt{1-\alpha}\sin t + 40(1-\alpha)\sin^2 t. \end{cases}$$

Since $x_{\alpha}\left(\frac{3\pi}{2}\right) = 80$ and $y_{\alpha}\left(\frac{3\pi}{2}\right) = 192 - 176\sqrt{1-\alpha} + 40(1-\alpha)$, we have

$$\alpha = \frac{1}{200}\left(-776 - 5\left(y_{\alpha}\left(\frac{3\pi}{2}\right)\right) + 44\sqrt{2}\sqrt{8 + 5\left(y_{\alpha}\left(\frac{3\pi}{2}\right)\right)}\right).$$

This result indicates that $\mu_{A(\cdot)B}(y)$ and $\mu_{A(\cdot)B}(80, y)$ match. The section cut perpendicular to the x-axis at the vertex of Figure 27 is shown in Figure 28, and this section is shown in Figure 26.

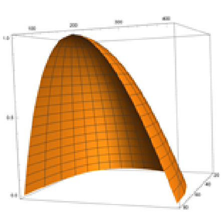

Figure 28: $\mu_{A(\cdot)B}(80, y)$

(4) Division: Since $A_\alpha\,(/)\,B_\alpha = \left[\frac{16-8\sqrt{1-\alpha}}{12+5\sqrt{1-\alpha}}, \frac{16+8\sqrt{1-\alpha}}{12-5\sqrt{1-\alpha}}\right]$, we have

$$\mu_{A(/)B}\,(y) = \begin{cases} \frac{-192+464y-119y^2}{(8+5y)^2}, & \frac{8}{17} \leq y \leq \frac{24}{7}, \\ 0, & \text{otherwise.} \end{cases}$$

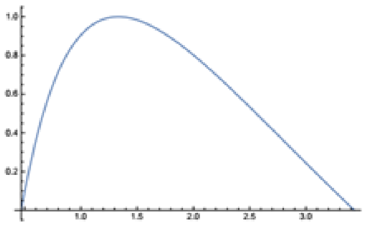

Figure 29: $\mu_{A(/)B}(y)$

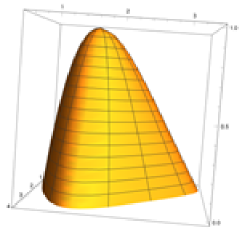

Figure 30: $\mu_{A(/)B}(x,y)$

By Theorem 2.15([5]), $\left(A\,(/)_p\,B\right)^\alpha = \{(x_a\,(t)\,, y_a\,(t))\mid 0 \leq t \leq 2\pi\}$, where

$$x_\alpha\,(t) = \frac{10+6\sqrt{1-\alpha}\cos t}{8-4\sqrt{1-\alpha}\cos t}, \quad y_\alpha\,(t) = \frac{16+8\sqrt{1-\alpha}\sin t}{12-5\sqrt{1-\alpha}\sin t}.$$

Since $x_\alpha\left(\frac{3\pi}{2}\right) = \frac{10}{8}$ and $y_\alpha\left(\frac{3\pi}{2}\right) = \frac{16-8\sqrt{1-\alpha}}{12+5\sqrt{1-\alpha}}$, we have

$$\alpha = \frac{-192+464\left(y_\alpha\left(\frac{3\pi}{2}\right)\right)-119\left(y_\alpha\left(\frac{3\pi}{2}\right)\right)^2}{\left(8+5\left(y_\alpha\left(\frac{3\pi}{2}\right)\right)\right)^2}.$$

This result indicates that $\mu_{A(/)B}\,(y)$ and $\mu_{A(/)B}\left(\frac{10}{8}, y\right)$ match. The section cut perpendicular to the x-axis at the vertex of Figure 30 is shown in Figure 31, and this section is shown in Figure 29.

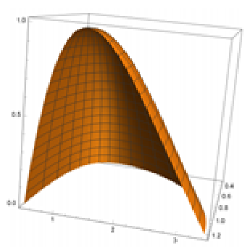

Figure 31: $\mu_{A(/)B}(\frac{10}{8}, y)$

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〈국문 초록〉

Zadeh의 확장 원리에 기반을 둔 Zadeh의 최대-최소 합성 연산자에 대한 많은 결과가 있다. $\mathbb{R}$에서 정의된 두 개의 삼각 퍼지수와 이차 퍼지수에 대해 기존의 경우와 다른 다양한 경우에 대해 Zadeh의 최대-최소 합성 연산자를 계산하였다. 그리고, 1차원 2차 퍼지수와 2차원 2차 퍼지수 사이의 관계를 설명하기 위한 계산을 수행하였다. 구체적으로, 두 개의 2차원 2차 퍼지수를 꼭짓점을 통과하면서 xy-평면과 x-축에 수직인 평면으로 자르면, 그로 인해 생기는 그 단면의 소속도 함수를 사용하여 Zadeh의 최대-최소 합성 연산자를 계산하였다. 그리고 두 개의 서로 다른 2차원 2차 퍼지 숫자에 대해 확장된 연산을 수행하였다. 그 결과 생성된 새로운 2차원 2차 퍼지수를 꼭짓점을 통과하면서 xy-평면과 x-축에 수직인 평면으로 자르면, 그로 인해 생기는 단면을 위와 비교한 결과 실제로 두 결과가 같은 것을 확인하였다.