A NOTE ON THE SUBSPACE OF THE NONHOLONOMIC SPACE

이를 教育學 碩士學位 論文으로 提出함

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I. INTRODUCTION

In Euclidean space of three dimensions the distance ds between adjacent points whose rectangular Cartesian coordinates are (x, y, z) and $(x+dx, y+dy, z+dz)$ is given by $ds^2=dx^2+dy^2+dz^2$.

More generally, for any system of oblique curvilinear coordinates (u, v, w) we have

$$ds^2=a\,du^2+b\,dv^2+c\,dw^2+2f\,dvdw+2g\,dwdu+2h\,dudv,$$

where a,b,c,f,g,h are functions of the coordinates. Thus the square of the linear element ds is given by a quadratic form in the differentials of the coordinates.

This idea was generalized and extented to space of n dimensions by Riemann, who defined the infinitesimal distance ds between the adjacent points, whose coordinates in any system are x^i and x^i+dx^i, $(i=1, 2, \cdots, n)$ by the relation

$$ds^2 = g_{ij}\,dx^i dx^j, \quad (i, j = 1, 2, \cdots, n) \cdots\cdots (1)$$

where the coefficients g_{ij} are functions of the coordinates x^i.

The quadratic differential form in the second member of (1) is called a Remannian metric ; and a space which is characterized by such a metric is a Riemannian space.

Throughout this paper, let V_n be a n-dimensional Riemannian space referred to a real coordinate system x^λ and defined by a fundamental metric tensor $h_{\lambda\mu}$, whose determinant

(1, 1) $h \overset{\text{def}}{=} \mathrm{Det}(\!(h_{\lambda\mu})\!) \neq 0$

If $\underset{i}{e}^{\lambda}(i=1, 2,\cdots, n)$ are a set of a n linearly independent unit vectors, then there is a unique reciprocal set of n linearly independent covariant vectors $\overset{i}{e}_{\mu}$, $(i=1, 2,\cdots, n)$, satisfying

$$(1,2) \quad \underset{i}{e}^{\lambda} \overset{i}{e}_{\mu} = \delta_{\mu}^{\lambda}, \quad \underset{j}{e}^{\lambda} \overset{i}{e}_{\lambda} = \delta_{j}^{i}.$$

with the vectors $\underset{i}{e}^{\lambda}$ and $\overset{i}{e}_{\mu}$ a nonholonomic frame of V_n is defined in the following ways; if $T_{\mu\cdots}^{\lambda\cdots}$ are holonomic components of a tensor its nonholonomic components are defined by

$$(1,3) \quad {}^{*}T_{j\cdots}^{i\cdots} \overset{\text{def}}{=} T_{\mu\cdots}^{\lambda\cdots} \underset{i}{e}^{\lambda} \overset{j}{e}_{\mu} \cdots .$$

An easy inspection (1, 2) and (1, 3) shows that

$$(1, 4) \quad T_{\mu\cdots}^{\lambda\cdots} = {}^{*}T_{j\cdots}^{i\cdots} \underset{i}{e}^{\lambda} \overset{j}{e}_{\mu} \cdots .$$

In this paper, we will investigate properties of metric and lengths of the elements of arc connecting two points in subspace and nonholonomic subspace of an n-dimensional Riemannian space V_n.

In particular, we obtain that the metric for a subspace of V_n is equal to the metric for a nonholonomic subspace of ${}^{*}V_n$, and the length of elements of arc connecting the two points is the same, whether calculated with respect to nonholonomic subspace or nonholonomic space.

II. PRELIMINARY RESULTS

THEOREM. 2.1. We have

$$(2,1)\,a \quad T^{\lambda} = {}^{*}T^{i}\, \underset{i}{e}^{\lambda} \quad .$$

$$(2,2)\,b \quad T^{\lambda\mu} = {}^{*}T^{ij}\, \underset{i}{e}^{\lambda}\, \underset{j}{e}^{\mu} \, .$$

Consider a symmetric covariant tensor a whose determinant $a \overset{\text{def}}{=} ((a_{\lambda\mu})) \neq 0$

It is well-known that the quantities defined by

$$a^{\lambda\nu} \overset{\text{def}}{=} \frac{\text{cofactor of } a_{\lambda\nu} \text{ in } a}{a}$$

is a symmetric contravariant tensor satisfying

$$(2,2) \qquad a_{\lambda\mu}\, a^{\lambda\nu} = \delta_{\mu}^{\nu} \, .$$

Let $a_{\lambda\mu}$ and ${}^{*}a_{ij}$ be holonomic and nonholonomic components of the covariant tensor, and take a coordinate system y^i for which we have at a point p of V_n

$$(2,3) \qquad \frac{\partial y^{i}}{\partial x^{\lambda}} = \overset{i}{e}_{\lambda} \, , \quad \frac{\partial x^{\nu}}{\partial y^{i}} = \underset{i}{e}^{\nu} \, .$$

THEOREM. 2.2. We have

$$(2,4) \qquad {}^{*}a_{ij}\, {}^{*}a^{ik} = \delta_{j}^{k} \, .$$

III. SUBSPACE OF THE ${}^*V_n$

Let V_n be a Riemannian space of n dimensions, referred to coordinates x^λ,$(\lambda = 1, 2, \cdots, n)$ and having the metric $a_{\lambda\mu} dx^\lambda dx^\mu$.

Then we have the followings

THEOREM. 3.1. The metric in the nonholonomic frame is represented by

$$(3,1) \quad {}^*a_{ij}\, dy^i\, dy^j = a_{\lambda\mu}\, dx^\lambda dx^\mu .$$

PROOF. From (1,4) and (2,3) ,

$$a_{\lambda\mu}\, dx^\lambda dx^\mu = {}^*a_{ij}\, \overset{i}{e}_\lambda \overset{j}{e}_\mu\, dx^\lambda\, dx^\mu$$
$$= {}^*a_{ij}\, dy^i\, dy^j \quad .$$

DEFINITION 3.2. The space which is characterized by the nonholonomic frame is nonholonomic space ${}^*V_n$ with n dimension.

DEFINITION 3.3. Points of v_n whose coordinates are expressible as functions of m idependent variables $\bar{x}^\alpha$, $(\alpha = 1, 2, \cdots, n)$, $(m \langle n)$,are said to constitute a $\overline{V}_m$ immersed in V_n, and $\overline{V}_m$ is said to be a subspace of V_n.

DEFINITION 3.4. ${}^*\overline{V}_m$ whose coordinate are expressible as functions of m independent variables $\bar{x}^p$, $(p = 1, 2, \cdots, m)$, $(m \langle n)$, are subspace of ${}^*V_n$.

Let $\bar{a}_{\alpha\beta}\, d\bar{x}^\alpha d\bar{x}^\beta$ be the metric for subspace $\overline{V}_m$ of V_n if $\bar{x}^\alpha$ and $\bar{x}^\alpha + d\bar{x}^\alpha$ are adjacent points of $\overline{V}_m$, whose coordinate in the x's are

$x^\lambda + dx^\lambda$ we must have $dx^\lambda = \frac{\partial x^\lambda}{\partial \bar{x}^\alpha} d\bar{x}^\alpha$ (α takes the values 1, 2,⋯, m and λ takes the values 1, 2,⋯,n).

Let ${}^*\bar{a}_{pq} d\bar{y}^p d\bar{y}^q$ be the metric for nonholonomic subspace ${}^*\bar{V}_m$ of ${}^*V_n$, if $\bar{y}^p$ and $\bar{y}^p + d\bar{y}^p$ are adjacent points of ${}^*\bar{V}_m$, whose coordinates in the y-coordinate system are y^i and $y^i + dy^i$, we must have by the reciprocal relations,

$$(3,2)a \qquad dy^i = \frac{\partial y^i}{\partial \bar{y}^p} d\bar{y}^p,$$

$$(3,2)b \qquad d\bar{y}^p = \frac{\partial \bar{y}^p}{\partial y^i} dy^i.$$

THEOREM 3.5. The metric for subspace $\bar{V}_m$ of V_n. is equal to the metric for nonholonomic subspace ${}^*\bar{V}_m$ of ${}^*V_n$.

PROOF. Using (2,1)b and (2,3), we have the results as in the following way ;

$$(3,3) \qquad \bar{a}_{\alpha\beta} d\bar{x}^\alpha d\bar{x}^\beta = {}^*\bar{a}_{pq} \overset{p}{e}_\alpha \overset{q}{e}_\beta d\bar{x}^\alpha d\bar{x}^\beta$$

$$= {}^*\bar{a}_{pq} d\bar{y}^p d\bar{y}^q$$

COROLLARY. 3.6. The metric in the ${}^*\bar{V}_m$ is represented by holonomic covariant tensor.

PROOF. Multiply both side of (3,3) by $\frac{\partial \bar{x}^\alpha}{\partial \bar{y}^p} \cdot \frac{\partial \bar{x}^\beta}{\partial \bar{y}^q}$.

According to (2,3), we have the following results (3,4)

$${}^*\bar{a}_{pq} d\bar{x}^\alpha d\bar{x}^\beta = \bar{a}_{\alpha\beta} \underset{p}{e}^\alpha \underset{q}{e}^\beta d\bar{x}^\alpha d\bar{x}^\beta .$$

$$\text{Hence } {}^*\bar{a}_{pq} = \bar{a}_{\alpha\beta} \underset{p}{e}^\alpha \underset{q}{e}^\beta \qquad (3,4) .$$

The length ds of the elements of arc connecting the two points is the same, whether calculated with respect to V_n or $\overline{V}_m$.

$$(3,5)\quad ds^2 = a_{\lambda\mu}\, dx^\lambda\, dx^\mu = \bar{a}_{\alpha\beta}\, d\bar{x}^\alpha\, d\bar{x}^\beta = d\bar{s}^2.$$

THEOREM. 3.7. The length $^*ds$ of the elements of arc connecting the two points is the same, whether calculated with respect to $^*V_n$ or $^*\overline{V}_m$.

PROOF. By virture of (3,1),

$$(3,6)\quad {}^*ds^2 = {}^*a_{ij}\, dy^i\, dy^j = a_{\lambda\mu}\, dx^\lambda\, dx^\mu = ds^2.$$

From (3,3),

$$^*d\bar{s}^2 = {}^*\bar{a}_{pq} d\bar{y}^p d\bar{y}^q = \bar{a}_{\alpha\beta} d\bar{x}^\alpha d\bar{x}^\beta = d\bar{s}^2.$$

By means of (3,5),

$$(3,7)\quad {}^*ds = {}^*d\bar{s}.$$

COROLLARY. 3.8. The metric in the $^*\overline{V}_m$ is equal to the metric in the $^*V_n$.

PROOF. By means of (3,1) and (3,3), (3,5),

$$(3,8)\quad {}^*a_{ij}\, dy^i\, dy^j = {}^*\bar{a}_{pq}\, d\bar{y}^p\, d\bar{y}^q.$$

THEOREM. 3.9. The nonholonomic tensor of the $^*\overline{V}_m$ is determined by the nonholonomic component of $^*V_n$.

PROOF. Using (3,2)a and (3,2)b, (3,8), we have

$$\overset{*}{\bar{a}}_{pq} = \overset{*}{a}_{ij} \frac{\partial y^i}{\partial \bar{y}^p} \cdot \frac{\partial y^j}{\partial \bar{y}^q} . \tag{3, 9}$$

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〈國文抄錄〉

NONHOLONOMIC 空間의 部分空間에 관한 小考

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本 論文에서는 n 次元 Riemann 空間 V_n의 部分空間 $\bar{V}_m$(但, $m<n$), Nonholonomic Frame에 依해서 결정되는 Nonholonomic 空間 $^*V_n$와 이의 部分空間 $^*\bar{V}_m$상에서 距離와 길이의 여러가지 性質을 調査하였다. 特히, V_n의 部分空間 $\bar{V}_m$上의 距離는 Nonholonomic 部分空間上의 距離와 같고, 두 點을 잇는 弧의 길이는 Nonholonomic 空間 또는 Nonholonomic 部分空間에 關하여 計算하여도 같음을 밝혔다.