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# Lattices of Formations of Algebraic Structures

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# Lattices of Formations of Algebraic Structures

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In Memory of
*My Grandma*

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JEJU ISLAND

*"In our relations with other people, we mainly discuss and evaluate their character and behavior. That is why I have withdrawn from nearly all so-called relations. This has made my old age rather lonely. My life has been full of hard work and I am grateful. It began as toil for bread and butter and ended in a love of science."*

Smultronställlet (1957, Ingmar Bergman)

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DR. EBERHARD ISAK BORG

# Abstract

## Lattices of Formations of Algebraic Structures

In the dissertation, various lattices of formations of algebraic structures are investigated. The languages corresponding to multiply local formations are described. Let $\sigma$ be a partition of the set of all primes. It is proved that every law of the lattice of all formations is fulfilled in the lattice of all multiply $\sigma$-local formations. It is shown that the lattice of all functor-closed totally composition formations is algebraic, and that the law system of the lattice of all functor-closed formations coincides with the law system of the lattice of all functor-closed multiply partially composition formations. It is proved that the lattice of all $\mathfrak{X}$-local formations is algebraic and modular. Let $\mathfrak{M}$ be the class of all multioperator $T$-groups satisfying the minimality and maximality conditions for $T$-subgroups. It is proved that every law of the lattice of all functor-closed $\mathfrak{M}$-formations is fulfilled in the lattice of all functor-closed multiply partially foliated $\mathfrak{M}$-formations.



**Keywords:** *Monoid, Language, Group, Ring, Multioperator T-Group, Fuzzy Set, Lattice, Formation, Saturated Formation, Local Formation, Composition Formation*

**Mathematics Subject Classification (2020)** 20F17, 20D10, 20M35, 68Q70, 03E72

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# Contents

# Chapter 1

# Introduction

*"After completion of the classification of simple groups, the main problem in the theory of finite groups remains the problem of mastering mechanisms of their interaction in arbitrary groups. The most important handicap here is p-groups. These small bricks are encountered almost everywhere and, in addition, the possibilities of their interaction are displayed infinitely, like a horde of insects.*
*The theory of formations is an attempt to be engaged in the theory of groups, that is to say, modulo p-groups. For all that, separate groups are considered through these classes as if in a diminishing glass, and structured operations on groups can be treated, in a definite respect, as construction of normalizers, extensions, joins, etc."*
January 28, 1994 (Kiel, Germany [96])

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WOLFGANG GASCHÜTZ

A *formation* of finite groups is a class of finite groups closed under taking quotients and subdirect products (Gaschütz, 1962 [42]).

## 1.1 The initial idea

All considered groups are finite. A *class* $\mathfrak{X}$ is a set of groups with the property that if $G \in \mathfrak{X}$, then every group isomorphic to $G$ belongs to $\mathfrak{X}$. A *variety* of groups may be defined as a nonempty class of groups closed under taking homomorphic images and subcartesian products [75], formations extend this notion.

The theory of saturated formations introduced by Gaschütz [42] became an integrated part of group theory by now. Recall that the Frattini subgroup $\Phi(G)$ of a group $G$ is the intersection of all maximal subgroups of $G$. A formation $\mathfrak{F}$ is said to be *saturated* if $G/\Phi(G) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$.

Gaschütz–Lubeseder–Schmid theorem states that any formation is saturated iff it is local. That makes saturated formations one of the most suitable classes for a better understanding of a group structure. Further it was found various generalizations of saturated formations, such as Baer-local, $\mathfrak{X}$-local, $\sigma$-local, foliated formations, etc.

Let $M$ and $N$ be normal subgroups of a finite group $G$ such that $N \subseteq M$. Then $M/N$ is said to be a *chief factor* of $G$ if $M/N$ is a minimal normal subgroup of $G/N$. This chief factor is *complemented* if there exists a maximal subgroup $K$ of $G$ such that $G = KM$ and $K \cap M = N$. If $M \subseteq \Phi(G)$, then the chief factor $M/N$ is not complemented. The *centralizer* of a chief factor $M/N$ in $G$ is denoted by $C_G(M/N)$, i.e., it is the set of all elements of $G$ that commute with all elements $gN$ of $M/N$.

The symbol $\mathbb{P}$ denotes the set of all primes. Consider a function $f$ with domain $\mathbb{P}$ whose images are formations of groups. The class $\mathfrak{F} = LF(f)$ of all groups $G$, such that either $G = 1$ or $G \neq 1$ and $G/\mathrm{C}_G(M/N) \in f(p)$ for every complemented chief factor $M/N$ of $G$ and any prime $p$ dividing the order of $M/N$, is a formation. The notation $\mathfrak{F} = LF(f)$ originally has the implicit meaning that $\mathfrak{F}$ is a *local formation* with a formation function $f$.

If $\mathfrak{F} = LF(f)$ for some formation function $f$, then $f$ is called a *local satellite* of $\mathfrak{F}$. When values of local satellite of a formation are themselves local formations, then that leads to the following definition. Every formation is 0-multiply local. For a positive integer $n$, a formation $\mathfrak{F}$ is called *$n$-multiply local* if $\mathfrak{F} = LF(f)$, and all nonempty values of $f$ are $(n-1)$-multiply local formations. If $n = 1$, then $\mathfrak{F}$ is just a local formations. A formation is called *totally local* if it is $n$-multiply local for all positive integers $n$.

Consider some standard examples of local formations [37, IV, (3.4)]. The class of all nilpotent groups is a local formation with $f(p) = 1$ for all $p \in \mathbb{P}$. Indeed, a chief factor $M/N$ of a nilpotent group is always central, i.e., $C_G(M/N) = G$. The class of all supersolvable groups is also a local formation. A chief factor of a supersolvable group has always prime order. Then the formation of all supersolvable groups has a local satellite whose value is with the class of all abelian groups of exponent dividing $p-1$ for all primes $p$. However, a nonsaturated formation cannot be defined locally. For instance, the class of all abelian groups is a nonsaturated formation, and it is impossible to find a local definition for it.

It is well-known that the lattice of all varieties of groups is modular but is not distributive [75]. The lattice of all locally finite varieties is a sublattice of the lattice of all hereditary formations [107]. Although many results of the theory of formations are some counterparts to the corresponding results of the theory of varieties, at the same time, the methods of their proof are very different from the corresponding proofs of the theory of varieties. Moreover, unlike the lattice of all varieties of groups, it turned out that many lattices of formations are algebraic.

In 1986 Skiba [107] proved that the lattice of all saturated formations is modular. Later it was found fruitful applications of this fact, in particular law systems of the lattices of formations have been studied; see Chapters 4 and 5 in [107], Chapter 4 in [44], and [135]. In [107] it is shown that the law system of the lattice of all $\tau$-closed $m$-multiply saturated formations coincides with the law system of the lattice of all $\tau$-closed $n$-multiply saturated formations for nonnegative integers $m$ and $n$, and in [47] it is proved that for any infinite set of primes $\omega$ the law system of the lattice of all $m$-multiply $\omega$-saturated formations coincides with the law system of the lattice of all $n$-multiply $\omega$-saturated formations. The mentioned result was generalized for the lattices of functor-closed $n$-multiply $\omega$-saturated formations [100]. In [137] the analogous fact established for multiply $\omega$-composition formations.

Finally we note that in the papers [13, 14] it was proposed a new approach of formation theory application in the theory of formal languages.

## 1.2 Research contribution

The present work contains some contributions to the theory of formations of algebraic structures, which originated in 1962 after the introduction by Gaschütz the concept of local formation of finite solvable groups, and has been enriched by the contributions of Baer, Ballester-Bolinches, Dixon, Förster, Guo, Shemetkov, Skiba, Vedernikov, Vorob'ev, et al. Further studies revealed that formations are of a general algebraic nature and can be applied to the study of not necessarily solvable finite and infinite groups, Lie algebras, monoids, rings, and even of general algebraic systems such as multirings and multioperator $T$-groups.

## Structure of the dissertation

The work is organized as follows.

### Chapter 2

This chapter contains the relevant literature review details. The basic concepts of formation theory are introduced. It is shown that the lattice of all formations of finite rings is algebraic and modular. Some applications for fuzzy rings are discussed.

### Chapter 3

Ballester-Bolinches, Pin, and Soler-Escrivà developed a general method to describe the languages corresponding to saturated formations of finite groups. In the present chapter it is shown that the mentioned result is applicable to the languages corresponding to multiply local formations of finite groups. Moreover for a subgroup functor $\tau$ (in Skiba's sense), the languages corresponding to $\tau$-closed saturated formations of finite groups are described.

Let $n$ be a positive integer, and let $\sigma = \{\sigma_i \mid i \in I\}$ be a partition of the set of all primes. It is shown that every law of the lattice of all formations is fulfilled in the lattice of all $n$-multiply $\sigma$-local formations of finite groups. This immediately implies the modularity of the lattice of all $n$-multiply $\sigma$-local formations of finite groups, i.e., it is obtained as a corollary the recent result of Chi, Safonov and Skiba.

**Chapter 4**

Let $\tau$ be a subgroup functor such that all subgroups of a finite group $G$ containing in $\tau(G)$ are subnormal in $G$. It is shown that the lattice of all $\tau$-closed totally composition formations of finite groups is inductive and algebraic. Thus it was found the solution of Skiba–Shemetkov problem on algebraic lattices of composition formations of finite groups.

Let $n$ be a positive integer and $\omega$ be a nonempty set of primes. It is established that the lattice of all $\tau$-closed $n$-multiply $\omega$-composition formations is $\mathfrak{G}$-separated, where $\mathfrak{G}$ is the class of all finite groups. It is proved that every law of the lattice of all $\tau$-closed formations is fulfilled in the lattice of all $\tau$-closed $n$-multiply $\omega$-composition formations. As an application, it is shown that the lattice of $\tau$-closed

$n$-multiply $\omega$-composition formations is modular but not distributive.

It is established that the law system of the lattice of all $\tau$-closed formations of finite groups coincides with the law system of the lattice of all $\tau$-closed $n$-multiply $\omega$-composition formations of finite groups. Thus it was found the solution of Skiba–Shemetkov problem on laws in the case of infinite set of primes $\omega$.

### Chapter 5

Let $\mathfrak{X}$ be a class of simple groups with a completeness property $\pi(\mathfrak{X}) = \operatorname{char}\mathfrak{X}$. Förster introduced the concept of $\mathfrak{X}$-local formation in order to obtain a common extension of well-known theorems of Gaschütz–Lubeseder–Schmid and Baer. It is proved that the lattice of all $\mathfrak{X}$-local formations of finite groups is algebraic and modular.

### Chapter 6

Let $\mathfrak{M}$ be the class of all multioperator $T$-groups satisfying the minimality and maximality conditions for $T$-subgroups, and let $n$ be a positive integer. It is proved that every law of the lattice of all $\tau$-closed $\mathfrak{M}$-formations is fulfilled in the lattice of all $\tau$-closed $n$-multiply $\Omega_1$-foliated $\mathfrak{M}$-formations with direction $\varphi$, such that $\varphi_0 \leqslant \varphi$. The Frattini theory of $\tau$-closed $n$-multiply $\Omega_1$-foliated $\mathfrak{M}$-formations is developed.

### Chapter 7

The chapter devoted to further applications. We give a brief discussion of further possible applications of our results and future directions for research involving the methods and results of this work.

To the best author's knowledge the main results presented here, and not attributed to others or described as well-known, are new. They have a theoretical significance and may be used in the study on the theory of algebraic structures and their classes.

### Derived works

The contributions appearing in the present dissertation have been published as research papers. The corresponding bibliographical references are papers without co-authors [117, 118, 119, 120, 121, 122, 123, 124, 125]. The authors contributed equally to the papers [126, 127, 128, 129, 130, 137, 138]. The author also thanks the authors of the literature for the provision of the initial ideas for this work.

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The author declares that he has no competing interests.

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# Chapter 2

# Formations of Algebraic Structures

## 2.1 Literature review & background

In the universe of all finite groups, the definition of a variety leads to the concept of a formation, — a class closed under taking homomorphic images and finite subdirect products is called a *formation*. This concept of the 1960s appeared first in the scope of finite solvable groups. Later, several authors investigated formations of algebraic structures. Jakubík [62] proved that the collection of all formations of lattice ordered groups is a complete Brouwerian lattice, also the collection of all formations of finite monounary algebras forms a complete lattice; see [63].

In the papers [13, 14, 19], it was proposed an approach of formation theory application in the theory of formal languages. The Eilenberg theorem [38] implies that there exists a bijection between the set of all varieties of regular languages and the set of all varieties of finite groups. The Formation Theorem (Ballester-Bolinches, Pin, Soler-Escrivà [13]) states that there is a one-to-one correspondence between formations of finite groups and formations of languages. This result makes formations useful for the study of abstract machines and automata, which commonly appear in theory of computation, compiler construction, artificial intelligence, parsing, formal verification and another aspects of theoretical computer science. Moreover, formations are a useful tool to study finite rings (see [32, 125]) which find interesting applications in coding theory (see [20, 102]).

In the group theory, formations are some of the most important classes. In the books [12, 44, 45, 99, 107, 135], it was demonstrated that constructions and results of lattice theory are useful for studying groups and formations of groups. In the scope of groups, formations generalize some notions such as $\sigma$-solvability and $\sigma$-nilpotency, and help to understand better the structure of groups. The motivation to study $\sigma$-local formations rises from the result of Chi, Safonov and Skiba [29, Theorem 1.3], which deals with so-called $\Sigma_t$-closed formations. Recently these interesting classes of groups, introduced first by Kramer [68] and studied by Shemetkov [95], have found new powerful applications set out in [29, 30, 111].

Thus methods of the general formations theory find various applications for investigation of groups, rings, and modules. There is some duality in the research of formations of finite groups and formations of other algebraic structures. This reasoning gives us the motivation to consider their properties from a unified viewpoint.

Such a unified approach can be realized considering formations of so-called (multi-operator) $T$-groups. Some particular cases of $T$-groups are groups, modules, rings, and multirings (i.e., $T$-groups with the condition every $t \in T$ on $G$ is distributive with respect to addition). Lattices of formations of $T$-groups have been studied in the theory of partially foliated formations introduced by Vedernikov [131, 34].

The references contain about 140 entries. However, it is not a comprehensive bibliography of this field of study. With numerous exceptions, it may contain only items referred to in the text. To find the references for a topic, please use online databases, such as MathSciNet or Zentralblatt.

## 2.2 Formations of monoids and formal languages

### 2.2.1 Classes of monoids

Recall that a *monoid* is an algebraic structure with a single associative binary operation and an identity element. A *group* is a monoid where each element has an inverse element. The simplest example of a monoid that is not a group is the set $\{0, 1\}$ with the usual multiplication.

Monoids are commonly used theoretical foundations of computer science. Various abstract data types in computer programming may be described using monoids: because the operation takes two values of a given type and returns a new value of the same type, it can be chained indefinitely, and associativity abstracts away the details of construction. We note that a *list* (array) is a fine example of a monoid (the *identity* of a list is an *empty list*, and the associative operation is appending).

**Example 2.1** (R [54]). Following Hammill [54], we use R to show that numeric vectors with concatenation and an empty vector form a monoid. Given a set of values is all the numeric vectors. We take R's **c** function as the monoidal operation. To check that this is indeed a monoid, first we shall make an infix version of **c**.

```
"%c%" <- c
```

Does applying the monoidal operation to two elements of our set give another element of that set?

```
A <- 1:3
B <- 4:7
C <- 8:10
class(A); class(B); class(C)


## [1] "integer"
## [1] "integer"
## [1] "integer"


class(A %c% B %c% C)
```

```
## [1] "integer"
```

Thus, the operation appears to preserve the type. Let us check now if it is associative.

```
v1 <- (A %c% B) %c% C
v2 <- A %c% (B %c% C)
all.equal(v1, v2)
```

```
## [1] TRUE
```

It is. Let us check if we have an identity element.

```
O <- integer(0)
v1 <- A %c% O
v2 <- O %c% A
all.equal(v1, A)
```

```
## [1] TRUE
```

```
all.equal(v2, A)
```

```
## [1] TRUE
```

Thus, we deal with a monoid.

**Example 2.2** (C# [124]). We describe a monoid using C# code as follows.

- ```
  public interface IMonoid<T>{
  T Zero { get; }
  T Append(T a, T b); }
  ```

We implement a monoid a singleton.

- ```
  public static class Singleton<T> where T : new() {
  private static readonly T _instance = new T();
  public static T Instance { get { return _instance; } }
  }
  ```

For instance we may implement monoid $\mathrm{gcd}(a, b)$.

- ```
  public class GCDMonoid : IMonoid<int> {
  public GCDMonoid() {}
  private int gcd(int a, int b) {
  return b == 0 ? a : gcd(b, a % b); }
  public int Zero {
  get { return 0; } }
  public int Append(int a, int b) {
  return gcd(a, b);}}
  ```

**Example 2.3** (Scala [74]). Following Noll [74], we use Scala to implement a monoid using a trait as a type class.
  ```

- `trait Monoid[T] {`

  `def e: T`

  `def op(a: T, b: T): T }`

In Algebird [1], an additive monoid for the standard type `Seq` is defined as follows:

- `Seq` is a concatenation monoid`Seq` is a concatenation monoid;
- `op` (plus) is the concatenation operation;
- `e` (zero), the identity element, is the empty `Seq`.

The implementation is listed below.

- `class SeqMonoid[T] extends Monoid[Seq[T]] {`

  `override def zero = Seq[T]()`

  `override def plus(left : Seq[T], right : Seq[T])`

  `= left ++ right }`

Implicits need to be used because this is how the notion of type classes is implemented in Scala.

- `implicit def seqMonoid[T] : Monoid[Seq[T]]`

  `= new SeqMonoid[T]`

In Chapter 10 of the book [31], we may find some interesting samples of monoids implemented in Scala.

More samples of monoids in computer programming are discussed here:

- marmelab.com/blog/2018/04/18/functional-programming-2-monoid.html
- blog.axosoft.com/monoids-practical-category-theory
- fsharpforfunandprofit.com/posts/monoids-without-tears
- medium.com/@sjsyrek/five-minutes-to-monoid-fe6f364d0bba
- doc.sagemath.org/pdf/en/reference/monoids/monoids.pdf

**Definition 2.4.** ([13]) A *formation of monoids* is a class of monoids $\mathfrak{F}$ satisfying the following two conditions:

1. any quotient of a monoid of $\mathfrak{F}$ also belongs to $\mathfrak{F}$;
2. the subdirect product of any finite family of monoids of $\mathfrak{F}$ is also in $\mathfrak{F}$.

**Example 2.5** (Example 5 [10])**.** Following Ballester-Bolinches et al. [10], we consider some nontrivial examples of monoid formations.

- A monoid $M$ has a zero if there exists an element $0 \in M$ such that the equation $m0 = 0 = 0m$ holds for each element $m \in M$. Note that this 0 is unique. Finite monoids with zero constitute a formation, which is not a variety of finite monoids. Moreover, monoids with zero of a given formation of monoids constitute a formation.
- A monoid is called periodic if all its cyclic submonoids are finite. The set of all periodic monoids is a formation of monoids.

- A monoid $M$ is called aperiodic if there exists a positive integer $k$ such that $m^k = m^{k+1}$ for all $m \in M$. The class of all aperiodic monoids is a formation of monoids.

- A monoid is called relatively regular if it contains a fnite ideal. The class of all relatively regular monoids is a formation of monoids.

### 2.2.2 Formations of languages

Recall that *languages* are subsets of a certain type of monoid, the free monoid over an alphabet, and regular languages are precisely the behaviours of finite automata [10]. A language is called *regular* [10] if its syntactic monoid is a finite monoid (note that a regular language is a group language if its syntactic monoid is a finite group). Following the standard notation, we denote by $A^*$ a *free monoid* on a set $A$, i.e., the set of all words with letters from $A$. In the sequel, a *class* of regular languages $\mathcal{C}$ will associate with any finite alphabet $A$ a set $\mathcal{C}(A^*)$ of regular languages of $A^*$; see [14, Section 4].

**Definition 2.6** ([13]). A *formation of languages* is a class of regular languages $\mathcal{F}$ satisfying the following two conditions:

1. for each alphabet $A$, $\mathcal{F}(A^*)$ is closed under Boolean operations and quotients;

2. if $L$ is a language of $\mathcal{F}(B^*)$ and $\eta : B^* \to M$ denotes its syntactic morphism, then for each monoid morphism $\alpha : A^* \to B^*$ such that $\eta \circ \alpha$ is surjective,

the language $\alpha^{-1}(L)$ belongs to $\mathcal{F}(A^*)$.

Following [13], we associate with any formation of monoids $\mathfrak{M}$ the class of languages $\mathcal{F}(\mathfrak{M})$ as follows: for any alphabet $A$, we denote by $\mathcal{F}(\mathfrak{M})(A^*)$ the set of languages of $A^*$ fully recognised by some monoid of $\mathfrak{M}$ (note that this is equivalent that the syntactic monoid is in $\mathfrak{M}$). For a formation of languages $\mathcal{F}$, we denote by $\mathfrak{M}(\mathcal{F})$ the formation of monoids generated by the syntactic monoids of the languages of $\mathcal{F}$; see [13].

**Theorem 2.7** (Formation Theorem [13]). *The correspondences*

$$\mathfrak{M} \to \mathcal{F}(\mathfrak{M}) \text{ and } \mathcal{F} \to \mathfrak{M}(\mathcal{F})$$

*are two mutually inverse, order preserving, bijections between formations of monoids and formations of languages. In particular, there is a one-to-one correspondence between formations of finite groups and formations of languages.*

## 2.3 Formations of finite groups

We consider only finite groups. Closure operations on classes of groups were introduced in [95, p. 12], [37, pp. 374–375] and [12, p. 89]. Let $\mathfrak{Y}$ be a class of finite groups. Following [37], we define the closure operations as follows.

$\mathrm{Q}\mathfrak{Y} = (G : \exists H \in \mathfrak{Y}$ and an epimorphism from $H$ onto $G)$;

$\mathrm{R}_0\mathfrak{Y} = (G : \exists N_i \trianglelefteq G\,(i = 1, \dots, r)$ with $G/N_i \in \mathfrak{Y}$, $N_1 \cap \cdots \cap N_r = 1)$.

Formations are classes of groups introduced in the 1960s (see [42]).

### 2.3.1 Semigroup of all formations

**Definition 2.8** ([37]). A *formation* is a class of groups $\mathfrak{F}$ which is both $\mathrm{Q}$-closed and $\mathrm{R}_0$-closed, i.e., satisfying the following two conditions:

1. if $G \in \mathfrak{F}$, then $G/N \in \mathfrak{F}$;
2. if $G/N_1$, $G/N_2 \in \mathfrak{F}$, then $G/N_1 \cap N_2 \in \mathfrak{F}$,

for any normal subgroups $N$, $N_1$, $N_2$ of $G$. Note that, a class $\mathfrak{F}$ is a formation iff $\mathfrak{F} = \mathrm{QR}_0\mathfrak{F}$. If $\mathfrak{X}$ is a class of groups, we write $\mathrm{form}\,\mathfrak{X}$ instead of $\mathrm{QR}_0\mathfrak{X}$ to denote the formation generated by $\mathfrak{X}$. Let $G$ be a group. Then the formation

$$\mathrm{form}\, G = \mathrm{QR}_0(G)$$

is called *one-generated*.

**Example 2.9** (p. 11 [135]). The following classes of groups are formations.

- $\emptyset$ is the empty formation.
- $(1)$ is the class of all identity groups.
- $\mathfrak{G}$ is the class of all finite groups.
- $\mathfrak{A}$ is the class of all abelian groups.
- $\mathfrak{N}$ is the class of all nilpotent groups.
- $\mathfrak{S}$ is the class of all solvable groups.

- $\mathfrak{G}_p$ or $\mathfrak{N}_p$ is the class of all $p$-groups, where $p$ is a prime.

However, the class of all finite cyclic groups is not a formation.

**Definition 2.10** ([37])**.** For any group $G$ and a class of finite groups $\mathfrak{F} \supseteq (1)$, we denote by $G_{\mathfrak{F}}$ the $\mathfrak{F}$*-radical of* $G$, i.e., the product of all normal $\mathfrak{F}$-subgroups of the group $G$.

**Definition 2.11** ([37])**.** For any group $G$ and a nonempty formation $\mathfrak{F}$, we denote by $G^{\mathfrak{F}}$ the $\mathfrak{F}$*-residual of* $G$, i.e., the intersection of all normal subgroups $N$ of $G$ such that $G/N \in \mathfrak{F}$.

**Definition 2.12** ([37])**.** The formation

$$\mathfrak{M}\mathfrak{F} = \{G \mid G^{\mathfrak{F}} \in \mathfrak{M}\}$$

is the *product of formations* $\mathfrak{M}$ and $\mathfrak{F}$.

For any formations $\mathfrak{F}_1$, $\mathfrak{F}_2$, and $\mathfrak{F}_3$, we have

$$(\mathfrak{F}_1\mathfrak{F}_2)\mathfrak{F}_3 = \mathfrak{F}_1(\mathfrak{F}_2\mathfrak{F}_3);$$

see [37, p. 338]. Thus, the set of all formations with the formation product defined above is a semigroup. We denote it by $G\mathfrak{G}$.

By Corollaries 7.14 and 7.15 in [99], the sets of all local formations, all $n$-multiply local formations and all totally local formations are subsemigroups of $G\mathfrak{G}$. We will study these types of formations in the next subsections.

### 2.3.2 $\tau$-Closed formations

The concept of subgroup functor turned out to be useful in group theory; see, e.g., [107, 64, 11] .

**Definition 2.13** ([107]). In each group $G$ we select a system of subgroups $\tau(G)$ and say that $\tau$ is a *subgroup functor* if

1. $G \in \tau(G)$ for every group $G$;

2. for every epimorphism $\varphi : A \to B$ and any $H \in \tau(A)$ and $T \in \tau(B)$, we have $H^{\varphi} \in \tau(B)$ and $T^{\varphi^{-1}} \in \tau(A)$.

If $\tau(G) = \{G\}$ then the functor $\tau$ is called *trivial*.

For any set of groups $\mathfrak{Y}$, the symbol $\mathrm{s}_{\tau}$ denotes the set of groups $H$ such that $H \in \tau(G)$ for some group $G \in \mathfrak{Y}$.

A class of groups $\mathfrak{F}$ is called *$\tau$-closed* if $\mathrm{s}_{\tau}(\mathfrak{F}) = \mathfrak{F}$. For instance $\mathfrak{F}$ is called s-closed [37] (or hereditary) if it contains all the subgroups of $G \in \mathfrak{F}$ (i.e., $\tau(\mathfrak{F}) = \mathrm{s}(\mathfrak{F})$), and $\mathrm{s}_n$-closed [37] (or normally hereditary) if it contains all the normal subgroups of $G \in \mathfrak{F}$ (i.e., $\tau(\mathfrak{F}) = \mathrm{s}_n(\mathfrak{F})$). A formation $\mathfrak{F}$ is *$\tau$-closed* if $\tau(G) \subseteq \mathfrak{F}$ for every group $G$ of $\mathfrak{F}$.

Let $\Theta$ be a set of formations. By $\Theta\mathrm{form}\,\mathfrak{Y}$, we denote the intersection of all formations of $\Theta$ containing a set of groups $\mathfrak{Y}$, i.e., the classes

- $\tau\mathrm{form}\,\mathfrak{Y}$,

- $l_n^\tau \mathrm{form}\,\mathfrak{Y}$,
- $l_\infty^\tau \mathrm{form}\,\mathfrak{Y}$

are the the intersections, respectively, of all $\tau$-closed formations, all $\tau$-closed $n$-multiply local formations and all $\tau$-closed totally local formations containing a given set of groups $\mathfrak{Y}$. For trivial subgroup functor, we have $\mathrm{form}\,\mathfrak{Y}$, $l_n\mathrm{form}\,\mathfrak{Y}$, and $l_\infty\mathrm{form}\,\mathfrak{Y}$.

We say that $\tau$ is a *closed subgroup functor* if for any groups $G$ and $H \in \tau(G)$ we have $\tau(H) \subseteq \tau(G)$.

Following [107], we define a partial order $\leq$ on the set of all subgroup functors as follows: $\tau_1 \leq \tau_2$ if and only if $\tau_1(G) \subseteq \tau_2(G)$ for any group $G \in \mathfrak{X}$. By $\overline{\tau}$, we denote the intersection of all closed subgroup functors $\tau_i$ such that $\tau \leq \tau_i$. The functor $\overline{\tau}$ is called the *closure* of $\tau$; see [107].

Recall that a group class closed under taking homomorphic images is called a *semiformation* [95].

Let $\mathfrak{Y}$ be a class of groups. By [107], the intersection of all $\tau$-closed semiformations containing $\mathfrak{Y}$ is called the *$\tau$-closed semiformation generated by* $\mathfrak{Y}$.

**Lemma 2.14** (Lemma 1.2.21 [107]). *Let $\mathfrak{F}$ be a $\tau$-closed semiformation generated by $\mathfrak{Y}$. Then $\mathfrak{F} = QS_{\overline{\tau}}(\mathfrak{Y})$.*

**Lemma 2.15** (Lemma 1.2.22 [107]). *Let $\mathfrak{Y}$ be a set of groups. Then*

$$\tau\mathrm{form}\,\mathfrak{Y} = QR_0S_{\overline{\tau}}(\mathfrak{Y}).$$

Let $\mathfrak{Y}$ be a class of groups. The symbol $\tau\mathrm{form}\,\mathfrak{Y}$ denotes the *$\tau$-closed formation generated by* $\mathfrak{Y}$, i.e., the intersection of all $\tau$-closed formations containing $\mathfrak{Y}$; see [107].

**Lemma 2.16** (Corollary 1.2.24 [107]). *Let $\{\mathfrak{M}_i \mid i \in I\}$ be a set of $\tau$-closed formations. Then*

$$\tau\mathrm{form}\left(\bigcup_{i\in I}\mathfrak{M}_i\right) = \mathrm{form}\left(\bigcup_{i\in I}\mathfrak{M}_i\right).$$

The symbol $\mathrm{fin}(\mathfrak{M})$ denotes the class of all finite groups such that $G \in \mathfrak{M}$ where $\mathfrak{M}$ is a variety of groups; see [107].

**Lemma 2.17** (Lemma 3.4.3 [107]). *For every variety of groups $\mathfrak{M}$ the map* fin *of the form*

$$\mathfrak{M} \to \mathrm{fin}\,\mathfrak{M}$$

*is an embedding of the lattice and semigroup of locally finite varieties into the algebra of all formations.*

### 2.3.3 Lattices of formations

By $\Theta$, we denote a set of classes of finite groups. Any formation of finite groups in $\Theta$ will be called a $\Theta$-*formation*. In the present work, we study complete lattices of formations.

**Definition 2.18** (p. 151 [107]). When the intersection of each set of $\Theta$-formations is in $\Theta$, and we can find a $\Theta$-formation $\mathfrak{F}$ such that $\mathfrak{M} \subseteq \mathfrak{F}$ for any $\Theta$-formation

$\mathfrak{M}$, then $\Theta$ is called a *complete lattice of formations*. Note that any complete lattice of formations is a complete lattice in the ordinary sense. Let $\mathfrak{M}$ and $\mathfrak{H}$ belong to $\Theta$. Then $\mathfrak{M} \bigvee_\Theta \mathfrak{H}$ is the *least upper bound* for $\{\mathfrak{M}, \mathfrak{H}\}$ in $\Theta$, and $\mathfrak{M} \cap \mathfrak{H}$ is the *greatest lower bound* for $\{\mathfrak{M}, \mathfrak{H}\}$ in $\Theta$.

Let $\Theta$ be a complete lattice of formations of finite groups, and let formations $\mathfrak{M}$ and $\mathfrak{H}$ belong to $\Theta$. Denote by $\mathfrak{M} \bigvee_\Theta \mathfrak{H}$ the formation $\mathrm{form}_\Theta(\mathfrak{M} \cup \mathfrak{H})$. In particular, for the lattice of all formation of finite groups the following equation holds:

$$\mathfrak{M} \bigvee \mathfrak{H} = \mathrm{form}(\mathfrak{M} \cup \mathfrak{H}).$$

**Definition 2.19.** A lattice of formations $\Theta$ is called *modular* if for any $\Theta$-formations $\mathfrak{X}$, $\mathfrak{H}$, and $\mathfrak{F}$ such that $\mathfrak{X} \subseteq \mathfrak{H}$, we have $\mathfrak{H} \cap (\mathfrak{X} \bigvee_\Theta \mathfrak{F}) = \mathfrak{X} \bigvee_\Theta (\mathfrak{H} \cap \mathfrak{F})$.

**Definition 2.20.** A lattice of formations $\Theta$ is called *distributive* if for any $\Theta$-formations $\mathfrak{F}_1$, $\mathfrak{F}_2$ and $\mathfrak{F}_3$ we have $\mathfrak{F}_1 \cap (\mathfrak{F}_2 \bigvee_\Theta \mathfrak{F}_3) = (\mathfrak{F}_1 \cap \mathfrak{F}_2) \bigvee_\Theta (\mathfrak{F}_1 \cap \mathfrak{F}_3)$.

### 2.3.4 Saturated formations

The *Frattini subgroup* $\Phi(G)$ of a group $G$ is the intersection of all maximal subgroups of $G$; see [37].

**Definition 2.21** ([37])**.** A formation $\mathfrak{F}$ is said to be *saturated*

$$\text{if } G/\Phi(G) \in \mathfrak{F} \text{ implies } G \in \mathfrak{F}.$$

It is well-known that a formation is saturated iff it is local, this circumstance makes saturated formations one of the most suitable classes for studying the structure of finite groups.

**Definition 2.22** ([37, 44, 12])**.** The set of all primes is denoted by $\mathbb{P}$. For any formation function

$$f : \mathbb{P} \to \{\text{formations of groups}\}, \tag{2.1}$$

the symbol $LF(f)$ denotes the set of all groups $G$ such that either $G = 1$ or $G \neq 1$ and $G/\mathrm{C}_G(H/K) \in f(p)$ for every chief factor $H/K$ of $G$ and each $p \in \pi(H/K)$. The class $LF(f)$ is a saturated formation for any function $f$ of the form 2.1. If $\mathfrak{F} = LF(f)$ for some formation function $f$, then $f$ is called a *local satellite* of the formation $\mathfrak{F}$; see [44, p. 2] for more details.

*Remark* 2.23 ([37, 118]). The notation $\mathfrak{F} = LF(f)$ originally has the implicit meaning that $\mathfrak{F}$ is a *local formation* with a formation function $f$; see Section 3 in [37, IV]. By the Gaschütz–Lubeseder–Schmid theorem, a formation of finite groups is saturated iff it is local; see Section 4 in [37, IV].

Let $p$ be a prime. Following [107], we put for every set of groups $\mathfrak{Y}$:

$$\mathfrak{Y}^{\tau}(F_p) = \begin{cases} \tau\text{form}(G/F_p(G) \mid G \in \mathfrak{Y}) & \text{if } p \in \pi(\mathfrak{Y}); \\ \emptyset & \text{if } p \notin \pi(\mathfrak{Y}). \end{cases}$$

For trivial subgroup functor, we use the notion

$$\mathfrak{Y}(F_p) = \begin{cases} \mathrm{form}(G/F_p(G) \mid G \in \mathfrak{Y}) & \text{if } p \in \pi(\mathfrak{Y}); \\ \emptyset & \text{if } p \notin \pi(\mathfrak{Y}). \end{cases}$$

Recall that $\pi(\mathfrak{Y}) = \bigcup_{G \in \mathfrak{Y}} \pi(G)$, and $F_p(G) = O_{p',p}(G)$; for more details see [37].

We use the notion of canonical satellite of a formation $\mathfrak{F}$ to study the construction of $\mathfrak{F}$ from a simpler formation.

**Definition 2.24** ([113])**.** Let $\mathfrak{F} = LF(F)$, where $F(p) = \mathfrak{N}_p\mathfrak{F}(F_p)$ for all $p \in \mathbb{P}$. Then $F$ is called the *canonical local satellite* of the formation $\mathfrak{F}$. Note that any local (saturated) formation has the unique canonical local satellite.

### 2.3.5 Multiply local formations

When values of local satellite of a formation are themselves local formations, then that leads to the following definition.

**Definition 2.25** (p. 275 [44])**.** Every formation is 0-multiply local. For $n > 0$, a formation $\mathfrak{F}$ is called *n-multiply local* if $\mathfrak{F} = LF(f)$, and all nonempty values of $f$ are $(n-1)$-multiply local formations. If $n = 1$, then we have just local formations.

**Example 2.26** ([107])**.** Let $\mathfrak{M} = \mathfrak{N}^n\mathfrak{H}$ and $\mathfrak{F} = \mathfrak{N}_p\mathfrak{M}$, where the formation $\mathfrak{H} \neq \emptyset$ is not local. By Example 1.3.3 in [107], formations $\mathfrak{M}$ and $\mathfrak{F}$ are $n$-multiply local.

*Remark* 2.27 ([113]). Let $n$ be a positive integer, and $\mathfrak{F} = LF(F)$ be an $n$-multiply local formation. Then by Lemma 11 in [113] $F(p)$ is an $(n-1)$-multiply local formation for all primes $p$. Moreover, by Theorem 2 in [113] $\mathfrak{F}$ is an $n$-multiply local iff

$$\mathfrak{N}_p l_n \mathrm{form}\, \mathfrak{F}(F_p) \subseteq \mathfrak{F}$$

for all primes $p$. If $\mathfrak{F} = l_n \mathrm{form}\, \mathfrak{X}$, then Theorem 1.3.13 in [107] implies $\pi(\mathfrak{F}) = \pi(\mathfrak{X})$. By Remarks 1 and 2 in [113] we have

$$\mathfrak{N}_p \mathfrak{F}(F_p) = \mathfrak{N}_p l_{n-1} \mathrm{form}\, \mathfrak{F}(F_p).$$

The following lemma efficiently describes $n$-multiply local formations.

**Lemma 2.28** (Theorem 3 [113]). *Let $n$ be a positive integer, and $\mathfrak{F} = l_n \mathrm{form}\, \mathfrak{X}$ for any nonempty set of groups $\mathfrak{X}$. Then*

$$\mathfrak{F} = \mathrm{form} \bigcup_{p \in \mathbb{P}} \mathfrak{N}_p l_{n-1} \mathrm{form}\, \mathfrak{X}(F_p).$$

**Definition 2.29** ([44, 107]). A formation is called *totally local* if it is $n$-multiply local for all positive integers $n$.

Note that some well-studied formations are totally local; see [99, 107]. For instance the classes of all $\pi$-nilpotent and $\pi$-solvable groups are totally local formations for all sets $\pi$ of primes.

The following lemma efficiently describes totally local formations.

**Lemma 2.30** (Theorem 1.3.16 [107]). *Let $\mathfrak{F} = l_\infty \mathrm{form}\, \mathfrak{X}$ for any nonempty set of groups $\mathfrak{X}$. Then*

$$\mathfrak{F} = \mathrm{form} \bigcup_{p \in \mathbb{P}} \mathfrak{N}_p l_\infty \mathrm{form}\, \mathfrak{X}(F_p).$$

*Remark* 2.31 (see [107]). If there exits such integer $t$ that $\mathfrak{F}$ is a $t$-multiply local formation but it is not a $(t+1)$-local formation, then we write $ind_l(\mathfrak{F}) = t$. We have $ind_l(\mathfrak{F}) = \infty$ if $\mathfrak{F}$ is a totally local formation, and $ind_l(\mathfrak{F}) = 0$ if $\mathfrak{F}$ is not local. Let $\mathfrak{F} = LF(F)$. Then by Lemma 1.3.1 in [107], $ind_l(\mathfrak{F}) = n$ iff there exists a prime $p$ such that $ind_l(F(p)) = n-1$ and $ind_l(F(q)) \geq n-1$ for all $q \in \pi(\mathfrak{F}) \setminus \{p\}$.

**Example 2.32** (Example 1.3.2 [107]). Let $\mathfrak{F}$ be a formation of all supersolvable groups. Then $\mathfrak{F} = LF(F)$, where $F(p) = \mathbf{Ab}(p-1)$ for all primes $p$; see Example 6.3 in [14]. In particular, $F(3)$ is formation of all elementary abelian 2-groups. Then $F(3) = \mathfrak{N}_3 F(3)$ is not a local formation. Thus $ind_l(\mathfrak{F}) = 1$.

### 2.3.6 $\sigma$-Local formations

Let $\sigma = \{\sigma_i \mid i \in I\}$ be a *partition* [29] of the set of all primes $\mathbb{P}$, i.e.,

$$\sigma = \{\sigma_i \mid i \in I\}, \text{ where } \mathbb{P} = \cup_{i \in I} \sigma_i \text{ and } \sigma_i \cap \sigma_j = \emptyset \text{ for all } i \neq j.$$

**Definition 2.33.** A finite group $G$ is said to be

- *$\sigma$-primary* [108] if it is a $\sigma_i$-subgroup for some $i$;
- *$\sigma$-solvable* [108] if every its chief factor is $\sigma$-primary;
- *$\sigma$-nilpotent* [49] if it is a the direct product of some its $\sigma$-primary subgroups;
- *meta-$\sigma$-nilpotent* [109] if it is an extension of some $\sigma$-nilpotent finite group be the $\sigma$-nilpotent group.

When $\sigma = \{\{2\}, \{3\}, \dots\}$, a finite group is $\sigma$-solvable (respectively, $\sigma$-nilpotent) iff it is solvable (respectively, nilpotent).

The $\sigma$-nilpotent groups and classes of meta-$\sigma$-nilpotent groups are of a very special interest in the resent years (refer for instance to [109], [111, Introduction] and [27, 49, 48, 50, 51, 52, 67, 71, 110, 140] for an account of recent headway).

In the scope of groups, formations generalize some notions such as $\sigma$-solvability and $\sigma$-nilpotency, and help to understand better the structure of groups. The motivation to study $\sigma$-local formations rises from the result of Chi, Safonov and Skiba [29, Theorem 1.3], which deals with so-called $\Sigma_t$-closed formations. Recently these interesting classes of groups, introduced first by Kramer [68] and studied by Shemetkov [95], have found new powerful applications set out in [29, 30, 111].

**Definition 2.34** ([29])**.** Given a partition $\sigma$ of the set of all primes $\mathbb{P}$ and a function $f$ with domain $\sigma$ whose images are formations of groups, i.e., $f$ is a function of the form $f : \sigma \to \{\text{formations of groups}\}$. The class $\mathfrak{F} = LF_\sigma(f)$ of all $\sigma$-groups $G$ such that either $G = 1$ or $G \neq 1$ and $G/O_{\sigma_i',\sigma_i}(G) \in f(\sigma_i)$ for all $\sigma_i \in \sigma(G)$ is a formation. Such a formation is called the *$\sigma$-local formation* [29] defined by the formation $\sigma$-function $f$ (a *$\sigma$-local definition of* $\mathfrak{F}$) with support

$$Supp(f) = \{\sigma_i \mid f(\sigma_i) \neq \emptyset\}.$$

The $\sigma$-function $f$ is called *integrated* if $f(\sigma_i) \subseteq LF_\sigma(f)$ for all $\sigma_i$, and *full* if $f(\sigma_i) = \mathfrak{G}_{\sigma_i} f(\sigma_i)$ for all $\sigma_i$. Here $\mathfrak{G}_{\sigma_i} f(\sigma_i) = \{G \mid G^{f(\sigma_i)} \in \mathfrak{G}_{\sigma_i}\}$, and $\mathfrak{G}_{\sigma_i}$ is the class of all $\sigma_i$-groups.

If the values of $\sigma$-local definitions of some formation are themselves $\sigma$-local formations, then this leads to the definition of multiply $\sigma$-local formation.

**Definition 2.35** ([29])**.** Every formation is $0$-*multiply* $\sigma$-*local*, by definition. For $n > 0$, we say that the formation $\mathfrak{F}$ is $n$-*multiply* $\sigma$-*local* provided either $\mathfrak{F} = (1)$ is the class of all identity groups or $\mathfrak{F} = LF_\sigma(f)$, where $f(\sigma_i)$ is $(n-1)$-multiply $\sigma$-local for all $\sigma_i \in \sigma(\mathfrak{F})$.

### 2.3.7 Solvably saturated formations

Baer-local (or composition) formations build a broader than saturated formations family of classes of finite groups. Baer's theorem states that Baer-local formations of finite groups are precisely solvably saturated formations of finite groups; see p. 373 in [37].

**Definition 2.36** ([44])**.** A formation $\mathfrak{F}$ is said to be *solvably saturated* if it contains each group $G$ with $G/\Phi(N) \in \mathfrak{F}$ for some solvable normal subgroup $N$ of $G$.

*Remark* 2.37 ([44]). Each local (saturated) formation is composition (solvably saturated) formation.

Let $p$ be a prime, and $G$ be a finite group. The subgroup $C^p(G)$ of $G$ is the intersection of the centralizers of all the abelian $p$-chief factors of $G$. It is assumed that $C^p(G) = G$ if $G$ has no abelian $p$-chief factors.

Let $\mathfrak{X}$ be a set of finite groups. Following [114], we will write $\mathrm{Com}(\mathfrak{X})$ to denote the class of all groups $L$ such that $L$ is isomorphic to some abelian composition factors of some group in $\mathfrak{X}$. When $\mathfrak{X} = \{G\}$ we just use $\mathrm{Com}(G)$ instead of $\mathrm{Com}(\mathfrak{X})$. Later on, the symbol $R(G)$ means the product of all solvable normal subgroups of $G$, and $\pi(\mathfrak{X})$ denotes the set of all primes dividing the order of all groups $G \in \mathfrak{X}$.

**Definition 2.38** **([44]).** Consider a function $f$ of the form

$$f : \mathbb{P} \cup \{0\} \to \{\text{formations of groups}\}, \tag{2.2}$$

and the class of finite groups

$$CLF(f) = (G \mid G/R(G) \in f(0);\ \ G/C^p(G) \in f(p) \text{ for all}\, p \in \pi(\mathrm{Com}(G))).$$

If $\mathfrak{F}$ is a formation such that $\mathfrak{F} = CLF(f)$ for a function $f$ of the form 2.2, then $\mathfrak{F}$ is said to be *composition* formation with *composition satellite* $f$.

When values of composition satellites of some formation are themselves composition formations, we have the following definition.

**Definition 2.39** **([114]).** Every formation is 0-multiply composition, by definition. For $n > 0$, a formation $\mathfrak{F}$ is said to be *$n$-multiply composition* if $\mathfrak{F} = CLF(f)$, and all nonempty values of $f$ are $(n-1)$-multiply composition formations. If $n = 1$, then we have composition formations. A formation is *totally composition* when it is $n$-multiply composition for any $n > 0$. Note that, many formations of finite groups are totally composition. For instance, $\emptyset$ and $(1)$ are totally composition formations.

### 2.3.8 Multiply $\omega$-composition formations

The symbol $R_\omega(G)$ is the $\mathfrak{S}_\omega$-radical of a finite group $G$.

**Definition 2.40** ([114]). Let $f$ be a function of the form

$$f : \omega \cup \{\omega'\} \to \{\text{formations of groups}\}. \tag{2.3}$$

Given the class of finite groups $CF_\omega(f) =$

$$(G \mid G/R_\omega(G) \in f(\omega') \text{ and } G/C^p(G) \in f(p) \text{ for all } p \in \omega \cap \pi(\mathrm{Com}(G))).$$

If $\mathfrak{F}$ is a formation such that $\mathfrak{F} = CF_\omega(f)$ for a function $f$ of the form (2.3), then $\mathfrak{F}$ is said to be *$\omega$-composition formation* with an *$\omega$-composition satellite $f$*.

**Definition 2.41** ([114] ). Any formation is $0$-multiply $\omega$-composition. For $n > 0$, a formation $\mathfrak{F}$ is said to be *$n$-multiply $\omega$-composition* if $\mathfrak{F} = CF_\omega(f)$ and all nonempty values of $f$ are $(n-1)$-multiply $\omega$-composition formations.

Note that $n$-multiply $\omega$-composition formations of finite groups have many interesting applications in the theory of formations of finite groups; see [114, 44].

### 2.3.9 $\mathfrak{X}$-local formations

In 1985, Förster [39] introduced $\mathfrak{X}$-local formations of finite groups to obtain a common extension of Gaschütz-Lubeseder-Schmid and Baer's theorems. We note that these classes of groups have been studied later on in [7, 8, 9, 26, 12, 98]. Let $\mathfrak{X} = \mathfrak{J}$, i.e., the class of all simple groups. Then $\mathfrak{X}$-local formations saturated (local). Let

$\mathfrak{X} = \mathbb{P}$, i.e., the class of all abelian simple groups. Then $\mathfrak{X}$-local formations are solvably saturated (Baer-local); see [12, p. 125].

Let $\mathfrak{X}$ be a class of finite groups. In the sequel, $\operatorname{char}\mathfrak{X}$ is the set of orders of all simple abelian groups in $\mathfrak{X}$. By $\pi(\mathfrak{X})$, we denote the set of all primes dividing the orders of all groups $G \in \mathfrak{X}$. We fix $\mathfrak{X}$ as a nonempty class of simple groups satisfying the condition $\pi(\mathfrak{X}) = \operatorname{char}\mathfrak{X}$, and denote $\mathfrak{X}' = \mathfrak{J} \setminus \mathfrak{X}$.

**Definition 2.42** ([12])**.** Let $f$ be a function of the form

$$f : (\operatorname{char}\mathfrak{X}) \cup \mathfrak{X}' \to \{\text{formations of groups}\}, \tag{2.4}$$

where a formation $f(X)$ could be possibly empty. Then we say that $f$ is an $\mathfrak{X}$-*formation function*. The $\mathfrak{X}$-*local formation* $LF_{\mathfrak{X}}(f)$ *defined by* $f$ is the class of all groups $G$ satisfying the following two conditions:

1. if $H/K$ is an $\mathfrak{X}_p$-chief factor of $G$, then $G/\mathrm{C}_G(H/K) \in f(p)$;
2. $G/L \in f(E)$, whenever $G/L$ is a monolithic quotient of $G$ such that the composition factor of $\operatorname{Soc}(G/L)$ is isomorphic to $E$, if $E \in \mathfrak{X}'$.

(Here the symbol $\operatorname{Soc}(G)$ denotes the socle of a group $G \neq 1$, i.e., it is the product of all minimal normal subgroups of finite group $G$.) A formation $\mathfrak{F}$ of finite groups is called an $\mathfrak{X}$-*local formation* if $\mathfrak{F} = LF_{\mathfrak{X}}(f)$ for some $\mathfrak{X}$-formation function $f$, and we say that $f$ is an $\mathfrak{X}$-*local definition* of $\mathfrak{F}$ or $f$ *defines* $\mathfrak{F}$; for more details see pp. 126–127 in [12], and pp. 374–375 in [37].

*Remark* 2.43 (Example 3.1.61 [12]). Note that any formation $\mathfrak{F}$ of finite groups is $\mathfrak{X}$-local when $\mathfrak{X} = \emptyset$ because $\mathfrak{F} = LF_{\mathfrak{X}}(f)$, where $f(S) = \mathfrak{F}$ for all $S \in \mathfrak{J}$.

*Remark* 2.44 (Lemma 2.1 [9]). We note that without loss of generality in the definition of $\mathfrak{X}$-local formation, can be assumed that $\mathfrak{X}$-formation function has the same value on all $X \in \mathfrak{X}'$.

By Remark 2.44, we can modify the second condition in Definition 2.42 for $\mathfrak{X}' \neq \emptyset$ as follows: if $G/L$ is a monolithic quotient of $G$ such that $\mathrm{Soc}(G/L)$ is an $\mathfrak{X}'$-chief factor of $G$, then $G/L \in f(\mathfrak{X}')$; see e.g., p. 29 in [26], and Definition 2.1 in [9].

*Remark* 2.45 (Theorem 5.1 [98]). Every nonempty $\mathfrak{X}$-local formation has an $\mathfrak{X}$-composition satellite. Thus, we can apply the properties of partially composition formations of finite groups for studying $\mathfrak{X}$-local formations of finite groups.

*Remark* 2.46 (Definition 3.2.12, Remarks 3.2.13 [12]). We note that Frattini-like subgroup associated with a class $\mathfrak{X}$ of simple groups have been introduced in [7]. Theorem 3.2.14 in [12] states that the corresponding $\mathfrak{X}$-saturated formations of finite groups are exactly $\mathfrak{X}$-local formations of finite groups.

## 2.4 Formations of rings and some generalizations

By Theorem 4.6 in the book [37], every formation of finite groups is saturated iff it is local. In contrast to the group case, not every saturated formation of Lie and

Leibniz algebras, rings, etc. can be locally defined. However, these formations have found various applications. Consider some examples.

**Example 2.47** (Formations of monounary algebras)**.** The lattice all formations of finite monounary algebras is isomorphic to the lattice of all hereditary subsets of a certain poset [63]. The lattice of all formations of finite monounary algebras is distributive, but for the lattice of formations of at most countable monounary algebras this is not true; see [80].

**Example 2.48** (Formations of lattice ordered groups)**.** Jakubík [62] showed that the set of all formations of lattice ordered groups forms a complete Brouwerian lattice, and the set of all formations of $GMV$-algebras is isomorphic to a principal ideal of the lattice all formations of lattice ordered groups.

**Example 2.49** (Formations of solvable Lie and Leibniz algebras)**.** The theory of saturated formations of solvable Lie algebras is set out in Barnes and Gastineau-Hills [18], and Barnes [16]. Over a field of nonzero characteristic, a saturated formation of solvable Lie algebras has at most one local definition, but a locally defined saturated formation of solvable Leibniz algebras other than that of nilpotent algebras has more than one local definition [17].

**Example 2.50** (Formations of multirings)**.** Christensen [32] showed that there exist Frattini closed formations of finite rings that are not local. Shemetkov [99] introduced the concept of formations of multirings, which a special case is formations of finite rings. In the book [99], we can find not only various examples of applications of this formations, but also related problems are discussed.

**Question 2.1** (Problem 3.51 [99])**.** *Is it true that any one-generated $\mathfrak{n}$-multiply local formation of rings has only a finite set of $\mathfrak{n}$-multiply local subformations?*

**Question 2.2** (Problem 22.8 [99])**.** *How to describe finite non-one-generated formations of rings for which all proper subformations are one-generated?*

This short overview gives the motivation to study formations of finite rings.

### 2.4.1 Classes of finite rings

A *class of rings* $\mathfrak{Y}$ is a set of rings with the following property: if $R \in \mathfrak{Y}$, then any ring isomorphic to $R$ belongs to $\mathfrak{Y}$.

**Definition 2.51** ([32])**.** We refer to a class of rings as a *homomorph* whenever it contains all homomorphic images of its members and as a *formation* if in addition it is subdirect product closed; i.e., a *formation* is a class of finite rings $\mathfrak{F}$ which is both $\mathrm{Q}$-closed and $\mathrm{R}_0$-closed in the sense of [99, 37, 12, 32].

The smallest formation of finite rings containing a class of finite rings $\mathfrak{Y}$ is $\mathrm{QR}_0\,\mathfrak{Y}$, composed of all rings that can be expressed as quotients of subdirect products of a finite number of rings in $\mathfrak{Y}$. When $\mathfrak{Y} = (R)$ consists only of the rings isomorphic to $R$, we obtain that the smallest formation containing $R$ is $\mathrm{QR}_0(R)$;

such a formation is called *one-generated*. Let $\mathfrak{F}$ be a class of finite rings, then $\mathfrak{F}$ is a formation iff $\mathfrak{F} = \mathrm{QR}_0\,\mathfrak{F}$.

In the scope of groups, formations generalize some notions as solvability, supersolvability and nilpotency of groups. Let us consider an example for formations of rings.

**Example 2.52** (Locally defined formations of finite rings [32]). For any ring $R$, the intersection $\Phi(R)$ of its maximal ideals, when such exist, is called the *Frattini subring* of $R$. For finite rings $\Phi(R)$ is contained in the Jacobson radical $J(R)$ of $R$. We are concerned with classes of rings that contain a ring $R$ whenever they contain it Frattini factor ring $R/\Phi(R)$. Such classes are said to be *Frattini closed*.

A nontrivial examples of a Frattini closed formation of finite rings is the class $\mathfrak{N}$ of all finite nilpotent rings. This class can be described *locally* in the sense that $R \in \mathfrak{N}$ iff the minimal ideals of its factor rings $R/K$ are trivial left $R$-modules.

Following [32] we refer to the minimal ideals of the factor rings of a finite ring $R$ as *chief factors* of $R$. Since each chief factor has prime characteristic it can be classified, according to which prime $p$ is involved, as a *p-chief factor*. Denote for each chief factor $H/K$ of $R$ its left annihilator $\{r \mid r \in R \text{ and } rH \subseteq K\}$ in $R$ by $A_R(H/K)$. Given a set of primes $\pi$ and a function $f$ with domain $\pi$ whose images are formations of finite rings. The class $\mathfrak{F}$ of $\pi$-rings whose $p$-chief factors $H/K$ have the property

$$R/A_R(H/K) \in f(p) \text{ for each } p \in \pi$$

is a formation. Such a formation is called the *local formation* defined by the *formation function* $f$ with *support* $\pi$.

In view of the primary decomposition of finite rings, we see that for any $p \in \pi$, the class $\mathfrak{F}_p$ of $p$-rings in $\mathfrak{F}$ is a formation and is defined locally by the formation function $f_p$ with support $\{p\}$ and image $\{f(p)\}$. The most elementary nontrivial local formations are the formations $\mathfrak{N}_p$ of finite nilpotent $p$-rings in the sense that they contain no proper local formations.

### 2.4.2 Lattices of formations of finite rings

All rings considered are finite. Complete lattices of formations of finite rings are defined analogously to the case of formations of finite groups. We note that $\emptyset$ and $(0)$ are formations and the set of all formations of finite rings is the complete lattice of formations. Let $\Theta$ be a complete lattice of formations of finite rings, and let $\mathfrak{M}$ and $\mathfrak{H}$ belong to $\Theta$. Then $\mathrm{QR}_0(\mathfrak{M} \cup \mathfrak{H})$ is the least upper bound for $\{\mathfrak{M}, \mathfrak{H}\}$ in $\Theta$, and $\mathfrak{M} \cap \mathfrak{H}$ is the greatest lower bound for $\{\mathfrak{M}, \mathfrak{H}\}$ in $\Theta$.

**Definition 2.53** ([21]). An element $a$ of a lattice $\Theta$ is *compact* if $a \leq \vee(x_j \mid j \in S)$ holds for $a \leq \vee(x_j \mid j \in J)$ and some finite subset $S \subset J$. Compact elements are important in domain theory [2] which has major applications for functional programming languages. A complete lattice is called *algebraic* if each element of it is the union (i.e. the least upper bound) of some set of compact elements.

The notation $J \lhd R$ means that $J$ is an ideal of a ring $R$, and we use the notation $R/I$ for a quotient ring of $R$ modulo $I$ if $I \lhd R$.

*Remark* 2.54. We observe that a class of rings $\mathfrak{F}$ is a formation iff it satisfies the following two conditions:

1. if $R \in \mathfrak{F}$ and $J \lhd R$, then $R/J \in \mathfrak{F}$; and

2. if $R/I_1$, $R/I_2 \in \mathfrak{F}$, then $R/I_1 \cap I_2 \in \mathfrak{F}$ for any $I_1$, $I_2 \lhd R$.

**Theorem 2.55** ([125]). *The lattice of all formations of finite rings is algebraic and modular.*

*Proof.* Step 1 (algebraicity). We show first that each one-generated formation $\mathfrak{F} = \mathrm{QR}_0(R)$ is a compact element in the lattice of all formations of rings.

Let $\mathfrak{F} \subseteq \mathrm{QR}_0(\bigcup_{i\in I} \mathfrak{F}_i)$, where $\{\mathfrak{F}_i \mid i \in I\}$ is a set of formations. Then $R \in \mathrm{QR}_0(\bigcup_{i\in I} \mathfrak{F}_i)$. Hence $R \simeq T/J$, where $J \lhd T \in \mathrm{R}_0(\cup_{i\in I}\mathfrak{F}_i)$. Then there are some $J_k \lhd T$ $(k = 1, \dots, r)$ such that $T/J_k \in \cup_{i\in I}\mathfrak{F}_i$ and $J_1 \cap \cdots \cap J_r = \{0\}$. Consequently, $T/J_1 \in \mathfrak{F}_{i_1}$, $\dots$, $T/J_r \in \mathfrak{F}_{i_r}$ for some $i_1$, $\dots$, $i_r \in I$.

Thus for any $k \in \{1, \dots, r\}$, we have $T/J_k \in \mathfrak{F}_{i_1} \cup \cdots \cup \mathfrak{F}_{i_r}$. Therefore $T \in \mathrm{R}_0(\mathfrak{F}_{i_1} \cup \cdots \cup \mathfrak{F}_{i_r})$. From $R \simeq T/J$ and $J \lhd T$, we have $R \in \mathrm{QR}_0(\mathfrak{F}_{i_1} \cup \cdots \cup \mathfrak{F}_{i_r})$, then

$$\mathfrak{F} = \mathrm{QR}_0(R) \subseteq \mathrm{QR}_0(\mathfrak{F}_{i_1} \cup \cdots \cup \mathfrak{F}_{i_r}).$$

We show next that any nonempty formation of rings $\mathfrak{M}$ is the union (in the lattice of all formations of rings) of its one-generated subformations $\mathfrak{M}_l = \mathrm{QR}_0(R_l)$, where $l \in L$. Let $\mathfrak{Y} = \mathrm{QR}_0(\cup_{l \in L}\mathfrak{M}_l)$. We show now that $\mathfrak{M} = \mathfrak{Y}$. Let $R \in \mathfrak{M}$. Then

$$R \in \mathrm{QR}_0(R) \subseteq \bigcup_{i \in L} \mathfrak{M}_i \subseteq \mathrm{QR}_0(\bigcup_{i \in L} \mathfrak{M}_i) = \mathfrak{Y}.$$

Consequently, $\mathfrak{M} \subseteq \mathfrak{Y}$. The inverse inclusion is obvious; $\mathfrak{M}_i \subseteq \mathfrak{M}$ implies $\cup_{i \in L}\mathfrak{M}_i \subseteq \mathfrak{M}$, and, consequently, $\mathfrak{Y} \subseteq \mathfrak{M}$.

Step 2 (modularity). We wish to show that the following equality holds, for any formations of rings $\mathfrak{X} \subseteq \mathfrak{Y}$ and $\mathfrak{F}$:

$$\mathfrak{Y} \cap \mathrm{QR}_0(\mathfrak{X} \cup \mathfrak{F}) = \mathrm{QR}_0(\mathfrak{X} \cup (\mathfrak{Y} \cap \mathfrak{F})).$$

The inclusion "$\supseteq$" is trivial. Let $A \in \mathfrak{Y} \cap \mathrm{QR}_0(\mathfrak{X} \cup \mathfrak{F})$. Then $A$ is a homomorphic image of some ring $R \in \mathrm{R}_0(\mathfrak{X} \cup \mathfrak{F})$, and we can find some ideals $J_1$ and $J_2$ of the ring $R$ such that $R/J_1 \in \mathfrak{X}$ and $R/J_2 \in \mathfrak{F}$ with $J_1 \cap J_2 = \{0\}$.

Let $A \cong R/I$, where $I \lhd R$. It is well known that the set of all ideals of a ring forms a complete modular lattice with respect to set inclusion. Thus, by modular law, we have $J_1 \cap ((J_1 \cap I) + J_2) = (J_1 \cap I) + (J_1 \cap J_2) = J_1 \cap I$. We note that

$$(R/(J_1 \cap I))/(J_1/(J_1 \cap I)) \cong R/J_1 \in \mathfrak{X}, \text{ and}$$

$$(R/(J_1 \cap I))/((J_1 \cap I) + J_2/(J_1 \cap I)) \cong R/(J_1 \cap I) + J_2 \in \mathfrak{F}.$$

Hence, $R/(J_1 \cap I) \in \mathrm{R}_0(\mathfrak{X} \cup \mathfrak{F})$. From $R/I \in \mathfrak{Y}$ and $\mathfrak{X} \subseteq \mathfrak{Y}$, we conclude that $R/(J_1 \cap I) \in \mathfrak{Y}$.

Consequently, $R/(J_1 \cap I) \in \mathrm{R}_0(\mathfrak{X} \cup (\mathfrak{Y} \cap \mathfrak{F}))$ implies $A \in \mathrm{QR}_0(\mathfrak{X} \cup (\mathfrak{Y} \cap \mathfrak{F}))$. This proves the theorem. □

Let $\mathfrak{F}$ and $\mathfrak{H}$ be formations such that $\mathfrak{H} \subseteq \mathfrak{F}$. We denote by $\mathfrak{F}/\mathfrak{H}$ the lattice of all formations $\mathfrak{M}$ such that $\mathfrak{H} \subseteq \mathfrak{M} \subseteq \mathfrak{F}$. As an immediate corollary from the modularity of the lattice of all formations of rings, we obtain the following result.

**Corollary 2.56.** *For any two formations $\mathfrak{M}$ and $\mathfrak{F}$ the lattices $\mathit{QR}_0(\mathfrak{M} \cup \mathfrak{F})/\mathfrak{M}$ and $\mathfrak{F}/(\mathfrak{F} \cap \mathfrak{M})$ are isomorphic.*

Fuzzy sets, which have been introduced by Zadeh [141] and Klaua [65], have found many applications in fields such as data mining, machine learning, and pattern recognition; see [60, 22]. Focusing on the structure of ring, Liu [70] introduced and studied the notions of fuzzy subrings and fuzzy ideals, and showed that the images and preimages under onto homomorphisms of fuzzy ideals are fuzzy ideals. Many authors have developed the fuzzy ring theory. However, as we may see that not each result on rings can be fuzzified. For instance, Dixit, Kumar, and Ajmal [35] discussed the conditions under which a given fuzzy ideal can or cannot be expressed as a union of two proper fuzzy ideals.

**Definition 2.57** ([141])**.** A *fuzzy subset* of a set $X$ is a function from $X$ into the closed interval $[0, 1]$. Let $X$ and $X'$ be any two sets, and $f : X \to X'$ be any function. A fuzzy subset $\mu$ of $X$ is called *$f$-invariant* if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$, where $x, y \in X$.

**Definition 2.58** ([35])**.** Let '$\cdot$' be a binary composition in a set $X$, and $\mu$ and $\mu'$

be any two fuzzy subsets of $X$. The *product* $\mu\mu'$ is defined by

$$\mu\mu'(z) = \begin{cases} \sup(\min\{\mu(x), \mu'(y)\}), & \text{for } x, y \in X, \text{ and } z = x \cdot y; \\ 0, & \text{if } z \text{ is not expressible as } z = x \cdot y \text{ for all } x, y \in X. \end{cases}$$

Clearly, $\mu\mu'$ is a fuzzy subset of $X$.

**Definition 2.59** ([35]). A fuzzy subset $\mu$ of a ring $R$ is called a *fuzzy ideal* of $R$ if it has the following two properties:

1. $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ for any $x, y \in R$; and
2. $\mu(xy) \geq \max\{\mu(x), \mu(y)\}$ for any $x, y \in R$.

In the sequel, by a ring we shall always mean a finite commutative ring with identity. A fuzzy ideal $\mu$ of a ring $R$ is called *fuzzy prime* if for any fuzzy ideals $\mu$ and $\mu'$ of $R$, the condition $\mu\mu' \subseteq \mu$ implies that either $\mu \subseteq \mu'$ or $\mu' \subseteq \mu$.

We shall write $\operatorname{form} R$ instead of $\mathrm{QR}_0(R)$ for the formation generated by $R$.

**Lemma 2.60** ([125]). *Let $R$ be a ring, and $\mathfrak{F} = \operatorname{form} R$. Then the following two conditions hold.*

1. *Any invariant fuzzy prime ideal of $R$ corresponds in a natural way to a fuzzy prime ideal of each member of $\mathfrak{F}$.*
2. *Any fuzzy prime ideal of each member of $\mathfrak{F}$ corresponds in a natural way to a fuzzy prime ideal of $R$.*

*Proof.* We note that the formation $\mathfrak{F}$ consists of all quotients of subdirect products of copies of $R$. Let $f$ be any homomorphism from the ring $R$ onto a ring $A \in \operatorname{form} R$. Then $f(R) = A$.

(1) Let $\mu$ be an $f$-invariant fuzzy prime ideal of $R$. Then by [35, Theorem 4.4], $f(\mu)$ (see [35, Lemma 4.1]) is a fuzzy prime ideal of $A$.

(2) Let $\nu$ be a fuzzy prime ideal of $A$. Then $f^{-1}(\nu)$ (see [35, Lemma 4.1]) is a fuzzy prime ideal of $R$ by [35, Theorem 4.5]. □

The proved lemma implies the following result.

**Proposition 2.61** ([125]). *Let $R$ be a ring, and $\mathfrak{F} = \operatorname{form} R$. Then there is a one-to-one correspondence between the set of all invariant fuzzy prime ideals of $R$ and the set of all fuzzy prime ideals of each ring of $\mathfrak{F}$.*

We note that by Theorem 2.55 *every formation of finite rings is the join of some one-generated formations*.

### 2.4.3 Classes of $T$-groups

Methods and results of formation theory have found fruitful applications in studies of finite and infinite groups, rings, and modules. There is the kind of duality in the research of formations of finite groups and formations of other algebraic systems. That rises up the motivation to study formations of different algebraic structures from a unified viewpoint, and to introduce formations of multioperator $T$-groups.

**Definition 2.62** ([34]). Let $G$ be an additive (not necessarily commutative) group with zero $0$. We say that $G$ is a *multioperator $T$-group* whenever we have a system $T$ of $k$-ary algebraic operations on $G$ for $k > 0$, while $t(0, \ldots, 0) = 0$ for all $t \in T$, where $0$ appears on the left $k$ times if $t$ is an $k$-ary operation; see Chapter III of the book [69], [58] and Chapter VI of the book [116].

*Remark* 2.63. We note that groups, modules, rings and multirings are particular cases of multioperator $T$-groups. *Multiring* is a multioperator $T$-group with the condition every $t \in T$ on $G$ is distributive with respect to addition; see [99]. Special types of multioperator $T$-group are discussed in Chapter 4 in [116] and Section 4 in [58].

**Definition 2.64** ([116]). Let $G$ be a multioperator $T$-group. A normal subgroup $N$ of $G$ is called an *ideal* of $G$ if for any positive integer $n$, any $t \in T_n$, any $i \in \{1, \ldots, n\}$, for arbitrary elements $g_1, \ldots, g_n \in G$ and $x \in N$, we have

$$t(g_1, \ldots, g_n) - t(g_1, \ldots, g_{i-1}, g_i + x, g_{i+1}, \ldots, g_n) \in N.$$

We write $K \lhd G$ to denote that $K$ is an ideal of a multioperator $T$-group $G$.

**Definition 2.65** ([34]). A *formation of multioperator $T$-groups* is a class $\mathfrak{F}$ of multioperator $T$-groups satisfying the following two conditions:

1. if $G \in \mathfrak{F}$ and $N \lhd G$, then $G/N \in \mathfrak{F}$; and

2. if $N_1, N_2 \lhd G$ and $G/N_1$, $G/N_2 \in \mathfrak{F}$, then $G/N_1 \cap N_2 \in \mathfrak{F}$.

By $\mathfrak{C}$, we denote the class of all multioperator $T$-groups with finite composition series.

*Remark* 2.66 ([34]). A variety of multioperator $T$-groups is a class of multioperator $T$-groups closed under taking multioperator $T$-subgroups, quotients and finite direct products. Thus, formations of multioperator $T$-groups of extend the notion of a variety of multioperator $T$-groups. The class $\mathfrak{C}$ is a formation but it is not a variety, but any variety of multioperator $T$-groups is a formation of multioperator $T$-groups.

Let $\mathfrak{M}$ be the class of all multioperator $T$-groups satisfying the minimality and maximality conditions for multioperator $T$-subgroups. Following [34], we write $\mathfrak{I}_1$ to denote the class of all simple $\mathfrak{M}$-groups, i.e., those nonzero multioperator $T$-groups $P$ whose only ideals are $\{0\}$ and $P$ itself. In the sequel, all considered multioperator $T$-groups are in the class $\mathfrak{C}$.

**Definition 2.67** ([34])**.** A class $\mathfrak{F}$ of multioperator $T$-groups is called a *Fitting class* whenever

1. if $G \in \mathfrak{F}$ and $N \lhd G$, then $N \in \mathfrak{F}$; and
2. if $N_1, N_2 \lhd G$ and $N_1, N_2 \in \mathfrak{F}$, then $N_1 + N_2 \in \mathfrak{F}$.

Let $\mathfrak{F}$ be a Fitting class of multioperator $T$-groups. The symbol $G_{\mathfrak{F}}$ denotes the $\mathfrak{F}$-*radical* of $G$, i.e., the sum of all $\mathfrak{F}$-ideals of a multioperator $T$-group $G$; see [34].

A *Fitting formation* is a Fitting class of multioperator $T$-groups which is at the same time a formation of multioperator $T$-groups.

### 2.4.4 $\Omega_1$-Foliated formations

Vedernikov [131] introduced an elegant concept of partially foliated formation. The idea have led to the necessity of considering the $\Omega$-satellites of various directions to construct new types of formations. The direction of an $\Omega$-satellites $f$ has been defined as a mapping of the class $\mathfrak{I}$ of all finite simple groups into the set of all nonempty Fitting formations. Obviously, we may find infinitely many such directions. For a fixed nonempty class $\Omega$, we can form infinitely many new types of classes, which are called $\Omega$-foliated formations of finite groups.

We use notations and terminologies from [34, 131]. Let $\Omega_1$ be a nonempty subclass of $\mathfrak{I}_1$, $\Omega_1' = \mathfrak{I}_1 \setminus \Omega_1$. The symbol $\mathcal{K}(G)$ denotes the class of all simple $\mathfrak{M}$-groups isomorphic to the composition factors of an $\mathfrak{M}$-group $G$. The group $G$ is called an $\Omega_1$*-group* if $\mathcal{K}(G) \subseteq \Omega_1$. The symbol $\mathfrak{M}_{\Omega_1}$ means the class of all $\Omega_1$-groups belonging to $\mathfrak{M}$; $\{0\} \in \mathfrak{M}_{\Omega_1}$. We set

$$O_{\Omega_1}(G) = G_{\mathfrak{M}_{\Omega_1}} \text{ and } O_{\Omega_1',\Omega_1}(G) = G_{\mathfrak{M}_{\Omega_1'}\mathfrak{M}_{\Omega_1}}.$$

**Definition 2.68** ([131]). A function $f$ of the form

$$f : \Omega_1 \cup \{\Omega_1'\} \to \{\text{formations of } T\text{-groups}\}$$

is called an $\Omega_1 F$-*function*, and a function $\varphi$ of the form

$$\varphi : \mathfrak{I}_1 \to \{\text{nonepty Fitting formations of } T\text{-groups}\}$$

is called an $FR$-*function*. We introduce a partial order $\leqslant$ on the set of all $\Omega_1 F$-functions and all $FR$-functions. For every two functions $\mu_1$ and $\mu_2$, put $\mu_1 \leqslant \mu_2$ if $\mu_1(A) \subseteq \mu_2(A)$ for all $A \in \Omega_1 \cup \{\Omega_1'\}$ $(A \in \mathfrak{I}_1)$. The set of all functions $\varphi$ with $\mu_1 \leqslant \varphi \leqslant \mu_2$ is called a segment and is denoted by $[\mu_1, \mu_2]$. If $\mu_1 \leqslant \mu_2$ and $\mu_1 \neq \mu_2$ then write $\mu_1 < \mu_2$. The class $\Omega_1 F(f, \varphi) =$

$$(G \in \mathfrak{M} \mid G/O_{\Omega_1}(G) \in f(\Omega'),\, G/G_{\varphi(A)} \in f(A)\,\forall\, A \in \Omega_1 \cap \mathcal{K}(G))$$

is called an $\Omega_1$-*foliated formation* of multioperator $T$-groups with $\Omega_1$-satellite $f$ and a direction $\varphi$ (or an $\Omega_1 F$-formation). A formation $\Omega_1 F(f, \varphi)$ is called $\Omega_1$-*free* if $\varphi(A) = \mathfrak{M}_{A'}$ for any $A \in \mathfrak{I}_1$ (we denote the direction of this formation by $\varphi_0$).

**Definition 2.69** ([34]). For any multioperator $T$-group $G$, we select a system of multioperator $T$-subgroups $\tau(G)$, and say that $\tau$ is a $T$-*subgroup functor* if the following two conditions hold:

1. $G \in \tau(G)$ for each multioperator $T$-group $G$; and
2. for every epimorphism $\varrho : A \to B$ and any $H \in \tau(A)$ and $K \in \tau(B)$, the following holds: $H^{\varrho} \in \tau(B)$ and $K^{\varrho^{-1}} \in \tau(A)$.

We say that $\tau$ is *trivial* if $\tau(G) = \{G\}$.

Let $\mathfrak{Y}$ be a set of multioperator $T$-groups. Then the symbol $\mathrm{s}_\tau(\mathfrak{Y})$ denotes the set of multioperator $T$-groups $H$ such that $H \in \tau(G)$ for some multioperator $T$-group $G$ in $\mathfrak{Y}$.

A class of multioperator $T$-groups $\mathfrak{F}$ is called *$\tau$-closed* if $\mathrm{s}_\tau(\mathfrak{F}) = \mathfrak{F}$. In particular, a formation $\mathfrak{F}$ is said to be *$\tau$-closed* if $\tau(G) \subseteq \mathfrak{F}$ for every multioperator $T$-group $G$ of $\mathfrak{F}$.

*Remark* 2.70 ([34]). It is easy to see that the class $\mathfrak{C}$ of all multioperator $T$-groups with finite composition series is not a $\tau$-closed formation, in the general case. For instance, consider a $\mathfrak{C}$-group $G$ such that $\tau(G) \nsubseteq \mathfrak{C}$ for every subgroup $\mathfrak{C}$-functor with $\tau(G) \nsubseteq \{G, \{0\}\}$. By Theorem 28.3 in [76], $G$ is a simple group of Olshanskii with infinite cyclic proper subgroups.

In the forthcoming chapters, we assume that $\mathfrak{M}$ is a subclass of the class $\mathfrak{C}$, and $\mathfrak{M}$ is a class of multioperator $T$-groups of one of the following two types:

1. the class of all finite multioperator $T$-groups, or

2. the class of all multioperator $T$-groups satisfying the conditions of minimality and maximality for $T$-subgroups.

We note that for any subgroup $\mathfrak{M}$-functor $\tau$, the class $\mathfrak{M}$ is a $\tau$-closed $\mathfrak{C}$-formation of multioperator $T$-groups. The $\Omega_1 F$-function is said to be $\tau$-closed (or $\tau\Omega_1 F$-function), if all its values are $\tau$-closed $\mathfrak{M}$-formations of multioperator $T$-groups. We suppose later on that any $\Omega_1$-foliated $\tau$-closed $\mathfrak{M}$-formation of multioperator $T$-groups is $\tau\Omega_1$-foliated $\mathfrak{M}$-formation of multioperator $T$-groups. We study these formations in Chapter 6.

## Conclusion

The basic concepts of formation theory are introduced. It is shown that the lattice of all formations of finite rings is algebraic and modular. Let $R$ be a finite commutative ring with an identity element. It is established that there is a one-to-one correspondence between the set of all invariant fuzzy prime ideals of $R$ and the set of all fuzzy prime ideals of each ring of the formation generated by $R$.

# Chapter 3

# Lattices of Saturated Formations and Group Languages

## 3.1 Languages associated with multiply local formations of finite groups

We borrow notations and terminology from the papers of Ballester-Bolinches, Pin, and Soler-Escrivà [13, 14]. All considered monoids are finite in this section. The Formation Theorem, and Lemmas 2.28 and 2.30 imply the following two results.

**Proposition 3.1.** *Let $\mathfrak{F}$ be an $n$-multiply local formation of groups with canonical local satellite $F$, and let $\mathcal{F}$ be a formation of languages associated with $\mathfrak{F}$. Then $\mathcal{F}$ is the join of the formations of languages $\mathcal{F}_p$ associated with $(n-1)$-multiply local* *formations $F(p)$ for all primes $p$.*

**Proposition 3.2.** *Let $\mathfrak{F}$ be a totally local formation of groups with canonical local satellite $F$, and let $\mathcal{F}$ be a formation of languages associated with $\mathfrak{F}$. Then $\mathcal{F}$ is the join of the formations of languages $\mathcal{F}_p$ associated with totally local formations $F(p)$ for all primes $p$.*

These results show that computing the formation of languages $\mathcal{F}$ reduces to computing $\mathcal{F}_p$ for all primes $p$. Value of the canonical definition $F$ of $n$-multiply (totally) local formation $\mathfrak{F} = LF(f)$ can be compute by the following formula

$$F(p) = \mathfrak{N}_p(f(p) \cap \mathfrak{F})$$

for all primes $p$.

**Definition 3.3** (Section 5 [14])**.** Given a prime $p$. Let $L_0, \ldots, L_k$ be languages of $A^*$; $a_1, \ldots, a_k$ be letters of $A$ and let $r < p$ be a nonnegative integer. Define $(L_0a_1L_1 \ldots a_kL_k)_{r,p}$, — the *modular product* [14] of the languages $L_0, \ldots, L_k$ *with respect to $r$ and $p$*, as the set of all words $u$ in $A^*$ such that the number of factorizations of $u$ in the form $u = u_0a_1u_1 \ldots a_ku_k$, with $u_i \in L_i$ for $0 \le i \le k$, is congruent to $r$ modulo $p$. A language is a *$p$-modular product* of the languages $L_0, \ldots, L_k$ if it is of the form $(L_0a_1L_1 \ldots a_kL_k)_{r,p}$ for some $r$.

Following [14], we use next the modular product to describe the formation of languages corresponding to $\mathbb{L}\mathfrak{N}_p \bullet \mathfrak{M}$.

*Remark* 3.4 (Proposition 3.16 [14] ). For any formation $\mathfrak{M}$ of finite groups, the formation product $\mathfrak{N}_p\mathfrak{M}$ coincides with the Mal'cev product $\mathbb{L}\mathfrak{N}_p \bullet \mathfrak{M}$, where $\mathbb{L}\mathfrak{N}_p$ is the class of semigroups which are locally a $p$-group.

Let $\mathcal{C}_p$ be the formation of languages associated with $(n-1)$-multiply (totally) local formation of groups $\mathfrak{M} = f(p) \cap \mathfrak{F}$. Then by Theorem 6.2 in [14] and the propositions above we obtain the following proposition.

**Proposition 3.5.** *Let $\mathfrak{F}$ be an $n$-multiply (totally) local formation, and $\mathcal{F}$ be the formation of languages associated with $\mathfrak{F}$. Then for each alphabet $A$, $\mathcal{F}(A^*)$ is the Boolean algebra generated by the languages of the form*

$$(L_0 a_1 L_1 \ldots a_k L_k)_{r,p},$$

*where $L_i \in \mathcal{C}_p(A^*)$, $0 \leq i \leq k$, $0 \leq r < p$, and $p$ runs over all primes.*

Analogously we obtain a dual result for $\tau$-closed local formations.

**Proposition 3.6** ([118])**.** *Given $\mathfrak{F} = l^{\tau}\,\mathrm{form}\,\mathfrak{Y}$. Let $\mathcal{F}$ be the formation of languages associated with $\mathfrak{F}$, and let $\mathcal{C}_p$ be the formation of languages associated with $\mathfrak{Y}(p)$. Then for each alphabet $A$, $\mathcal{F}(A^*)$ is the Boolean algebra generated by the languages of the form*

$$(L_0 a_1 L_1 \ldots a_k L_k)_{r,p},$$

*where $L_i \in \mathcal{C}_p(A^*)$, $0 \leq i \leq k$, $0 \leq r < p$, and $p$ runs over all primes.*

**Corollary 3.7.** *Given* $\mathfrak{F} = l\text{form}\,\mathfrak{Y}$ *and* $\pi(\mathfrak{Y}) = \cup_{G \in \mathfrak{Y}} \pi(|G|)$. *Let* $\mathcal{F}$ *be the formation of languages associated with* $\mathfrak{F}$, *and let* $\mathcal{C}_p$ *be the formation of languages associated with*

$$\mathfrak{Y}(p) = \begin{cases} \text{form}(G/O_{p',p}(G) \mid G \in \mathfrak{Y}) & \text{if } p \in \pi(\mathfrak{Y}); \\ \emptyset & \text{if } p \notin \pi(\mathfrak{Y}). \end{cases}$$

*Then for each alphabet* $A$, $\mathcal{F}(A^*)$ *is the Boolean algebra generated by the languages of the form* $(L_0 a_1 L_1 \ldots a_k L_k)_{r,p}$, *where* $L_i \in \mathcal{C}_p(A^*)$, $0 \leq i \leq k$, $0 \leq r < p$, *and* $p$ *runs over all primes.*

**Corollary 3.8.** *Given* $\mathfrak{F} = l_{n+1}\text{form}\,\mathfrak{Y}$. *Let* $\mathcal{F}$ *be the formation of languages associated with* $\mathfrak{F}$, *and let* $\mathcal{C}_p$ *be the formation of languages associated with*

$$\mathfrak{Y}_n(p) = \begin{cases} l_n\text{form}\,\mathfrak{Y}(p) & \text{if } p \in \pi(\mathfrak{Y}); \\ \emptyset & \text{if } p \notin \pi(\mathfrak{Y}). \end{cases}$$

*Then for each alphabet* $A$, $\mathcal{F}(A^*)$ *is the Boolean algebra generated by the languages of the form* $(L_0 a_1 L_1 \ldots a_k L_k)_{r,p}$, *where* $L_i \in \mathcal{C}_p(A^*)$, $0 \leq i \leq k$, $0 \leq r < p$, *and* $p$ *runs over all primes.*

**Corollary 3.9.** *Given* $\mathfrak{F} = l_\infty\text{form}\,\mathfrak{Y}$. *Let* $\mathcal{F}$ *be the formation of languages associated with* $\mathfrak{F}$, *and let* $\mathcal{C}_p$ *be the formation of languages associated with*

$$\mathfrak{Y}_\infty(p) = \begin{cases} l_\infty\text{form}\,\mathfrak{Y}(p) & \text{if } p \in \pi(\mathfrak{Y}); \\ \emptyset & \text{if } p \notin \pi(\mathfrak{Y}). \end{cases}$$

*Then for each alphabet $A$, $\mathcal{F}(A^*)$ is the Boolean algebra generated by the languages of the form $(L_0a_1L_1\dots a_kL_k)_{r,p}$, where $L_i \in \mathcal{C}_p(A^*)$, $0 \le i \le k$, $0 \le r < p$, and $p$ runs over all primes.*

## 3.2 Lattices of $\sigma$-local formations

Notations and terminology are borrowed from [29], where $\sigma$-local formations have been introduced. We refer to the mentioned paper for more details and definitions on the scope of the topic. The symbol $l_n^\sigma\mathrm{form}\,\mathfrak{Y}$ denotes the intersection of all $n$-multiply $\sigma$-local formations containing the set of groups $\mathfrak{Y}$.

*Remark* 3.10 (Theorem 1.15 [29]). The set $l_n^\sigma$ of all $n$-multiply $\sigma$-local formations is partially ordered by the set inclusion and forms a complete lattice in which $\bigcap_{j\in J}\mathfrak{F}_j$ is the greates lower bound and $\bigvee_n^\sigma(\mathfrak{F}_j \mid j \in J) = l_n^\sigma\mathrm{form}\,(\bigcup_{j\in J}\mathfrak{F}_j)$ is the smallest upper bound.

**Definition 3.11** ([29]). A formation $\sigma$-function $f$ is said to be $l_{n-1}^\sigma$*-valued* if $f(\sigma_i)$ is an $(n-1)$-multiply $\sigma$-local formation for every $\sigma_i \in Supp(f)$.

The *smallest* $l_{n-1}^\sigma$*-valued definition* of a formation $\mathfrak{F}$ have been introduced in Lemma 2.6 in [29].

### 3.2.1 Laws of the lattices of multiply $\sigma$-local formations

Let $\xi$ be a term of the signature $\{\bigcap, \bigvee_n^\sigma\}$. By $\bar{\xi}$, it is denoted the term of the signature $\{\bigcap, \bigvee_{n-1}^\sigma\}$, which we obtain from $\xi$ by replacing of each symbol $\bigvee_n^\sigma$ by the symbol $\bigvee_{n-1}^\sigma$.

**Lemma 3.12** **([124]).** *Let $\xi(x_{i_1}, \dots, x_{i_m})$ be a term of signature $\{\bigcap, \bigvee_n^\sigma\}$, and let $f_i$ be an integrated $l_{n-1}^\sigma$-valued definition of an $n$-multiply $\sigma$-local formation $\mathfrak{F}_i$, where $i = 1, \dots, m$ and $n \geqslant 1$. Then*

$$\xi(\mathfrak{F}_1, \dots, \mathfrak{F}_m) = LF_\sigma(\bar{\xi}(f_1, \dots, f_m)).$$

*Proof.* Following [120], we proceed by induction on the number $r$ of occurrences of the symbols of $\{\bigcap, \bigvee_n^\sigma\}$ into $\xi$. The case $r = 1$ follows by Lemmas 2.2 and 3.1 in [29].

Let $\xi$ have $r > 1$ occurrences of the symbols $\{\bigcap, \bigvee_n^\sigma\}$. We set

$$\xi(x_1, \dots, x_m) = \xi_1(x_{i_1}, \dots, x_{i_a}) \triangle \xi_2(x_{j_1}, \dots, x_{j_b}),$$

where $\triangle \in \{\bigcap, \bigvee_n^\sigma\}$ and $\{x_{i_1}, \dots, x_{i_a}\} \cup \{x_{j_1}, \dots, x_{j_b}\} = \{x_1, \dots, x_m\}$. We suppose that the assertion is true for the terms $\xi_1$ and $\xi_2$. Then

$$\xi_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a}) = LF_\sigma(\bar{\xi}(f_{i_1}, \dots, f_{i_a}))$$

and

$$\xi_1(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b}) = LF_\sigma(\bar{\xi}(f_{j_1}, \dots, f_{j_b})).$$

For every $i$ we have $\overline{\xi}(f_{i_1}, \ldots, f_{i_a})(\sigma_i) \subseteq \xi_1(\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a})$ and

$$\overline{\xi}(f_{j_1}, \ldots, f_{j_b})(\sigma_i) \subseteq \xi_1(\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b}).$$

Hence

$$\xi(\mathfrak{F}_1, \ldots, \mathfrak{F}_m) =$$

$$\xi_1(\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a}) \triangle \xi_1(\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b}) = LF_\sigma(\overline{\xi}(f_{i_1}, \ldots, f_{i_a}) \bar{\triangle} \overline{\xi}(f_{j_1}, \ldots, f_{j_b}))$$

$$= LF_\sigma(\overline{\xi}(f_1, \ldots, f_m)),$$

where $\bar{\triangle} = \bigcap$ if $\triangle = \bigcap$, and $\bar{\triangle} = \bigvee_{n-1}^\sigma$ if $\triangle = \bigvee_n^\sigma$. The result is now immediate.

□

**Theorem 3.13** ([124]). *Let $n > 0$. Then every law of the lattice of all formations $l_0^\sigma$ is fulfilled in the lattice of all $n$-multiply $\sigma$-local formations $l_n^\sigma$.*

*Proof.* Fix a law

$$\xi_1(x_{i_1}, \ldots, x_{i_a}) = \xi_2(x_{j_1}, \ldots, x_{j_b}) \tag{3.1}$$

of signature $\{\bigcap, \bigvee_n^\sigma\}$. Let

$$\overline{\xi}_1(x_{i_1}, \ldots, x_{i_a}) = \overline{\xi}_2(x_{j_1}, \ldots, x_{j_b}) \tag{3.2}$$

be the same law of signature $\{\bigcap, \bigvee_{n-1}^\sigma\}$.

Suppose that law (3.2) fulfilled in the lattice $l_{n-1}^\sigma$. Given arbitrary $n$-multiply $\sigma$-local formations $\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a}; \mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b}$, we show that

$$\xi_1(\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a}) = \xi_2(\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b}).$$

Let $f_{i_c}$ be the smallest $l^{\sigma}_{n-1}$-valued definition of $\mathfrak{F}_{i_c}$, where $c = 1, \dots, a$, and let $f_{j_d}$ be the smallest $l^{\sigma}_{n-1}$-valued definition of $\mathfrak{F}_{j_d}$, where $d = 1, \dots, b$. Then applying Lemma 6.4, we obtain

$$\xi_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a}) = LF_{\sigma}(\overline{\xi}_1(f_{i_1}, \dots, f_{i_a})),$$

$$\xi_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b}) = LF_{\sigma}(\overline{\xi}_2(f_{j_1}, \dots, f_{j_b})).$$

By Lemma 2.6 in [29], $f_{i_1}, \dots, f_{i_a}; f_{j_1}, \dots, f_{j_b}$ are $l^{\sigma}_{n-1}$-valued. Then by induction for any index $i$ we obtain the following equality:

$$\overline{\xi}_1(f_{i_1}, \dots, f_{i_a})(\sigma_i) =$$

$$\overline{\xi}_1(f_{i_1}(\sigma_i), \dots, f_{i_a}(\sigma_i)) = \overline{\xi}_2(f_{j_1}(\sigma_i), \dots, f_{j_b}(\sigma_i)) =$$

$$\overline{\xi}_2(f_{j_1}, \dots, f_{j_b})(\sigma_i).$$

Hence $\xi_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a}) = \xi_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b})$. So, law (3.1) is fulfilled in the lattice $l^{\sigma}_n$, as required. □

By Theorem 1.15 in [29] the lattice $l^{\sigma}_n$ is modular. We obtain the same result as an immediate corollary of Theorem 3.13.

**Corollary 3.14** ([124]). *The lattice of all $n$-multiply $\sigma$-local formations $l^{\sigma}_n$ is modular but not distributive for any nonnegative integer $n$.*

*Proof.* Corollary 4.2.8 in [107] states that the lattice of all formations $l^{\sigma}_0$ is modular, and then applying Theorem 3.13, we conclude that the lattice $l^{\sigma}_n$ is modular for any nonnegative integer $n$.

Let us show now that the lattice $l_n^\sigma$ is not distributive. Given the class $\mathfrak{M}$ of locally finite groups whose exponents divide a given prime $p \neq 2$, which is a variety by [66]. Let $L(\mathfrak{M})$ be the lattice of the subvarieties of $\mathfrak{M}$. By Higman [59], $L(\mathfrak{M})$ is not distributive, and Lemma 2.17 implies that the map $\mathrm{fin} : \mathfrak{M} \to \mathrm{fin}\,\mathfrak{M}$ is an embedding of the lattice and the semigroup of locally finite varieties into the algebra of all formations of finite groups $l_0^\sigma$. Then again applying Theorem 3.13, we see that the lattice $l_n^\sigma$ is not distributive for any nonnegative integer $n$. □

When $\sigma = \{\{2\}, \{3\}, \ldots\}$, a formation $\sigma$-function and a $\sigma$-local formation are a formation function and a local formation, respectively; see Example 1.2 in [29]. Then Theorem 3.13 implies the following two results.

**Corollary 3.15** (Chapter 4 [107]). *Let $n$ be a positive integer. Then every law of the lattice of all formations of finite groups is fulfilled in the lattice of all $n$-multiply local formations of finite groups.*

**Corollary 3.16** (Corollary 4.2.8 [107]). *The lattice of all $n$-multiply local formations of finite groups is modular but not distributive for any nonnegative integer $n$.*

A formation of groups is said to be *totally $\sigma$-local* if it is $n$-multiply $\sigma$-local for every positive integer $n$ [29]. Theorem 3.13 rises up the motivation for the following question.

**Question 3.1** ([124]). *Is it true that every law of the lattice of all formations is fulfilled in the lattice of all totally $\sigma$-local formations?*

Safonov [89] showed that the lattice of all totally saturated formations of finite groups is distributive.

**Question 3.2** ([124]). *Is it true that the lattice of all totally $\sigma$-local formations is distributive (or modular at least)?*

*Remark* 3.17. We note that Guo, Zhang and N.T. Vorob'ev [53] have described the properties of $\sigma$-local Fitting classes of finite grops.

### 3.2.2 Frattini subformations

Given $n$-multiply $\sigma$-local formations $\mathfrak{F}$ and $\mathfrak{H}$ such that $\mathfrak{H} \subseteq \mathfrak{F}$. We write $\mathfrak{F}/_n^\sigma\mathfrak{H}$ to denote the lattice of all $n$-multiply $\sigma$-local formations of finite groups $\mathfrak{M}$ such that $\mathfrak{H} \subseteq \mathfrak{M} \subseteq \mathfrak{F}$. Corollary 3.14 implies the following result.

**Corollary 3.18** ([124]). *The lattices $(\mathfrak{M} \bigvee_n^\sigma \mathfrak{F})/_n^\sigma\mathfrak{M}$ and $\mathfrak{F}/_n^\sigma(\mathfrak{F} \bigcap \mathfrak{M})$ are isomorphic for any $n$-multiply $\sigma$-local formations $\mathfrak{M}$ and $\mathfrak{F}$*

**Definition 3.19.** If $\mathfrak{M} \subset \mathfrak{F}$ and the lattice $\mathfrak{F}/_n^\sigma\mathfrak{M}$ consists of only two elements then $\mathfrak{M}$ is called a *maximal* $n$-multiply $\sigma$-local *subformation* of $\mathfrak{F}$.

**Lemma 3.20.** *Let $G$ be a group and $\mathfrak{Y}$ be a nonempty set of groups. Then the formation $\mathfrak{F} = l_n^\sigma\mathrm{form}(\mathfrak{Y} \cup \{G\})$ contains a maximal $l_n^\sigma$-subformation containing $l_n^\sigma\mathrm{form}\,\mathfrak{Y} \neq \mathfrak{F}$ for every $n \geq 0$.*

*Proof.* The assertion follows by the Kuratowski–Zorn lemma (for instance see the proof of [122, Lemma 3.3] or [119, Lemma 3.1]). □

The study of maximal subformations of saturated formations and their intersections was originated by Skiba [104, 105, 107], Förster [40], and Herzfeld [55, 56, 57].

**Definition 3.21.** We denote the intersection of all maximal $n$-multiply $\sigma$-local subformations of $\mathfrak{F}$ by the symbol $\Phi_n^\sigma(\mathfrak{F})$, and call it the $n$-multiply $\sigma$-local *Frattini subformation* of $\mathfrak{F}$ (we set $\Phi_n^\sigma(\mathfrak{F}) = \mathfrak{F}$ if there are no such subformations).

Let $\mathfrak{Y}$ be a nonempty set of groups. If $\mathfrak{F} = l_n^\sigma\mathrm{form}\,(\mathfrak{Y} \cup \{G\})$ always implies that $\mathfrak{F} = l_n^\sigma\mathrm{form}\,\mathfrak{Y}$ then we say that $G$ is an $l_n^\sigma$*-nongenerator* of $\mathfrak{F}$.

**Proposition 3.22** ([124]). *Given $n$-multiply $\sigma$-local formations $\emptyset \neq \mathfrak{F}_0 \subseteq \mathfrak{F} \neq (1)$. Then*

1. $\Phi_n^\sigma(\mathfrak{F}_0) \subseteq \Phi_n^\sigma(\mathfrak{F})$, *and*
2. $\Phi_n^\sigma(\mathfrak{F})$ *consists of all $l_n^\sigma$-nongenerators of $\mathfrak{F}$.*

*Proof.* (1) Suppose that $\Phi_n^\sigma(\mathfrak{F}_0) \nsubseteq \Phi_n^\sigma(\mathfrak{F})$. Let $\mathfrak{M}$ be a maximal $l_n^\sigma$-subformation of $\mathfrak{F}$ with $\Phi_n^\sigma(\mathfrak{F}_0) \nsubseteq \mathfrak{M}$. Thus $\mathfrak{F}_0 \nsubseteq \mathfrak{M}$. By Corollary 6.12 we have $\mathfrak{F}/_n^\sigma\mathfrak{M} = (\mathfrak{M} \bigvee_n^\sigma \mathfrak{F}_0)/_n^\sigma\mathfrak{M} \simeq \mathfrak{F}_0/_n^\sigma(\mathfrak{F}_0 \bigcap \mathfrak{M})$. We see that the lattice from the left side consists of only two elements. Thus, $\mathfrak{F}_0 \bigcap \mathfrak{M}$ is the maximal $l_n^\sigma$-subformation of $\mathfrak{F}_0$. Hence $\Phi_n^\sigma(\mathfrak{F}_0) \subseteq \mathfrak{M}$. A contradiction is obtained. Thus, $\Phi_n^\sigma(\mathfrak{F}_0) \subseteq \Phi_n^\sigma(\mathfrak{F})$, as asserted.

(2) Let $G$ be a $l_n^\sigma$-nongenerator, and $\mathfrak{L}$ be a maximal $l_n^\sigma$-subformation of $\mathfrak{F}$. Suppose $G \notin \mathfrak{L}$. Consequently, $l_n^\sigma\mathrm{form}(\mathfrak{L} \cup \{G\}) = \mathfrak{F} = l_n^\sigma\mathrm{form}\,\mathfrak{L} = \mathfrak{L}$. We have a contradiction. Then $G \in \mathfrak{L}$. Denote by $\mathfrak{Y}$ be a nonempty set of groups contained in

$\mathfrak{F}$ and $G \in \Phi_n^\sigma(\mathfrak{F})$. Assume that $l_n^\sigma \mathrm{form}(\mathfrak{Y} \cup \{G\}) = \mathfrak{F} \neq l_n^\sigma \mathrm{form}\,\mathfrak{Y}$. Lemma 6.17 implies that $\mathfrak{F}$ has a maximal $l_n^\sigma$-subformation $\mathfrak{M}$ with the property $l_n^\sigma \mathrm{form}\,\mathfrak{Y} \subseteq \mathfrak{M}$. Because $G \in \Phi_n^\sigma(\mathfrak{F})$, we obtain $\mathfrak{M} = \mathfrak{F}$. Again we have a contradiction. Finally, $\mathfrak{F} = l_n^\sigma \mathrm{form}\,\mathfrak{Y}$. □

### 3.2.3 Languages associated with $\sigma$-local formations

We write $\mathcal{L}_m^\sigma$ to denote the lattice of the formations of languages corresponding to the lattice $l_m^\sigma$, where $m \geq 0$. Then by Theorem 3.13, we have the following corollary.

**Corollary 3.23.** *Let $n$ be a positive integer. Then every law of the lattice $\mathcal{L}_0^\sigma$ is fulfilled in the lattice $\mathcal{L}_n^\sigma$.*

**Question 3.3.** *How to describe the languages corresponding to $\sigma$-local ($n$-multiply $\sigma$-local, totally $\sigma$-local) formations?*

By Corollary 2.6(1) in [111], for every formation $\sigma$-function $f$ the class $LF_\sigma(f)$ is a nonempty saturated formation. Thus using Theorem 6.2 in [14] and Proposition 5.1 in [118], the problem above can be solved in some fashion for $\sigma$-local formations as follows.

**Proposition 3.24** ([124]). *Given $\mathfrak{F} = l^\sigma \mathrm{form}\,\mathfrak{Y}$ where $\mathfrak{Y} \neq \emptyset$ and $\pi(\mathfrak{Y}) = \cup_{G\in\mathfrak{Y}}\pi(G)$. Let $\mathcal{F}$ be the formation of languages associated with $\mathfrak{F}$, and let $\mathcal{C}_p$ be the formation* *of languages associated with*

$$\mathfrak{Y}(p) = \begin{cases} \mathrm{form}(G/O_{p',p}(G) \mid G \in \mathfrak{Y}) & \text{if } p \in \pi(\mathfrak{Y}); \\ \emptyset & \text{if } p \notin \pi(\mathfrak{Y}). \end{cases}$$

*Then for each alphabet $A$, $\mathcal{F}(A^*)$ is the Boolean algebra generated by the languages of the form $(L_0a_1L_1 \dots a_kL_k)_{r,p}$, where $L_i \in \mathcal{C}_p(A^*)$, $0 \leq i \leq k$, $0 \leq r < p$, and $p$ runs over all primes.*

However, it will be interesting to find a description of the group languages corresponding to multiply $\sigma$-local formation $\mathfrak{F} = LF_\sigma(F)$ using properties of the canonical $\sigma$-local definition $F$ of $\mathfrak{F}$ which is unique by Corollary 2.6(2) in [111].

# Conclusion

The languages corresponding to to $\tau$-closed local, multiply local and totally local formations are described. Let $\sigma$ be a partition of the set of all primes. It is shown that every law of the lattice of all formations is fulfilled in the lattice of all multiply $\sigma$-local formations of finite groups. Some properties of Frattini subformations of multiply $\sigma$-local formations are discussed. The main contributions have been published in the papers [118, 124].

# Chapter 4

# Lattices of Partially Composition Formations of Finite Groups

In the present chapter, we use only subgroup functors $\tau$ such that for every finite group $G$ all subgroups of $\tau(G)$ are subnormal in $G$.

## 4.1 Inductive lattices

Inductive lattices of formations of finite groups have been introduced in the book [107]. The property of the lattice to being inductive is a very important one in the study of formation lattices. In 2001 at the Gomel Algebraic Seminar (Belarus), Skiba has proposed the following problem related to the lattices of Baer-local formations of finite groups.

**Question 4.1** (Skiba). *Is it true that the lattice of all $\tau$-closed $n$-multiply $\omega$-composition formations inductive?*

Assuming that $\Theta$ is a complete lattice of formations, denote by the symbol $\Theta^{\omega_c}$ the set of all formations having an $\omega$-composition $\Theta$-valued satellite; see [114, 113]. In [114, p. 901] it is shown that $\Theta^{\omega_c}$ is a complete lattice of formations.

**Definition 4.1** ([107]). A lattice $\Theta^{\omega_c}$ is called *inductive* if for any set of partially composition formations of finite groups $\{\mathfrak{F}_i = CF_\omega(f_i) \mid i \in I\}$, where $f_i$ is an integrated satellite of $\mathfrak{F}_i \in \Theta^{\omega_c}$, the following equality holds: $\bigvee_{\Theta^{\omega_c}}(\mathfrak{F}_i \mid i \in I) = CF_\omega(\bigvee_\Theta(f_i \mid i \in I))$.

*Remark* 4.2. The inductance of a lattice $\Theta^{\omega_c}$ implies that instead of the study of the operation $\bigvee_{\Theta^{\omega_c}}$ on the set $\Theta^{\omega_c}$, we can reduce it to a study of the operation $\bigvee_\Theta$ on the set $\Theta$. So, the inductance is a very important property of $\Theta^{\omega_c}$.

In the present section, we establish the inductance of the lattices $c^\tau_{\omega_n}$ and $c^\tau_\infty$.

### 4.1.1 Minimal satellites

**Definition 4.3** ([114]). Let $\Theta$ be a complete lattice of formations. A satellite $f$ is called $\Theta$-*valued* if all its values belong to $\Theta$. If $\mathfrak{F} = CF_\omega(f)$ and $f(a) \subseteq \mathfrak{F}$ for all $a \in \omega \cup \{\omega'\}$, then $f$ is called an *integrated* satellite of $\mathfrak{F}$.

The set of all $\tau$-closed $n$-multiply $\omega$-composition formations $c^\tau_{\omega_n}$ is a complete lattice by an inclusion $\subseteq$; see [138]. By $c^\tau_{\omega_0}$ and $c^\omega_n$ we denote the lattice of all $\tau$-closed formations and the lattice of all $n$-multiply $\omega$-composition formations, respectively.

Let $\{f_i \mid i \in I\}$ be a set of $\omega$-composition satellites. Following [114]), we write $\bigcap_{i\in I} f_i$ to denote the $\omega$-composition satellite $f$ such that $f(a) = \bigcap_{i\in I} f_i(a)$ for every $a \in \omega \cup \{\omega'\}$.

**Lemma 4.4** (Lemma 2[114]). *Let $\mathfrak{F} = \bigcap_{i\in I} \mathfrak{F}_i$ where $\mathfrak{F}_i = CF_\omega(f_i)$. Then $\mathfrak{F} = CF_\omega(f)$ where $f = \bigcap_{i\in I} f_i$.*

Let $\{f_i \mid i \in I\}$ be the set of all $\omega$-composition $c^\tau_{\omega_{n-1}}$-valued satellites of a formation $\mathfrak{F}$. Because $c^\tau_{\omega_n}$ is a complete complete lattice of formation of finite groups, applying Lemma 4.7, we see that $f = \bigcap_{i\in I} f_i$ is an $\omega$-composition $c^\tau_{\omega_{n-1}}$-valued satellite of $\mathfrak{F}$. The satellite $f$ is called *minimal*.

**Lemma 4.5** (Corollary 4.2.8[107]). *The lattice of all $\tau$-closed $n$-multiply saturated formations is modular but not distributive for all $n \geq 0$.*

Let $\Theta$ be a complete lattice of formations, and let $\{\mathfrak{F}_i \mid i \in I\}$ be a set of $\Theta$-formations. Then we use the notion $\bigvee_\Theta(\mathfrak{F}_i \mid i \in I) = \Theta\mathrm{form}\left(\bigcup_{i\in I} \mathfrak{F}_i\right)$. In particular, if $\Theta = c^\tau_{\omega_n}$ we write $\bigvee^\tau_{\omega_n}(\mathfrak{F}_i \mid i \in I) = c^\tau_{\omega_n}\mathrm{form}\left(\bigcup_{i\in I} \mathfrak{F}_i\right)$. Let $\{f_i \mid i \in I\}$ be a set of $\Theta$-valued functions of the form (2.3). Then we write $\bigvee_\Theta(f_i \mid i \in I)$ to denote a function $f$ such that $f(a) = \Theta\mathrm{form}\left(\bigcup_{i\in I} f_i(a)\right)$ for all $a \in \omega \cup \{\omega'\}$. Applying Lemmas 2.1 and 3.1 in [138], we obtain the following lemma.

**Lemma 4.6.** *Let $n$ be a natural number. Then* $(c^{\tau}_{\omega_{n-1}})^{\omega_c} = c^{\tau}_{\omega_n}$.

**Lemma 4.7** (Lemma 2 [114]). *Let* $\mathfrak{F} = \bigcap_{i\in I} \mathfrak{F}_i$, *where* $\mathfrak{F}_i = CF_\omega(f_i)$. *Then* $\mathfrak{F} = CF_\omega(f)$, *where* $f = \bigcap_{i\in I} f_i$.

Let $G$ be a finite group. Then following [37, p. 66], we write $Z_p \wr G$ to denote the regular wreath product of groups $Z_p$ and $G$. Here $p$ be a prime.

**Lemma 4.8** (Lemma 2 [112]). *Let $Z_p$ be a group of a prime order $p$, $G$ be a group with $O_p(G) = 1$, and let $T = Z_p \wr G = [K]G$ is the regular wreath product, where $K$ is the base group of $T$. Then $K = C^p(T) = O_p(T)$.*

**Lemma 4.9** (Lemma 4 [114]). *Let $\mathfrak{F} = CF_\omega(f)$. If $G/O_p(G) \in f(p) \cap \mathfrak{F}$ for some prime $p \in \omega$, then $G \in \mathfrak{F}$.*

The lemma below describes the minimal $c^{\tau}_{\omega_{n-1}}$-valued satellite of a formation.

**Lemma 4.10** (Lemma 8 [128]). *Let $\mathfrak{Y}$ be a nonempty set of groups, $\mathfrak{F} = c^{\tau}_{\omega_n}\mathrm{form}\,\mathfrak{Y}$, where $n \geqslant 1$, let $\pi = \omega \cap \pi(\mathrm{Com}(\mathfrak{X}))$, and let $f$ the minimal $c^{\tau}_{\omega_{n-1}}$-valued $\omega$-composition satellite of $\mathfrak{F}$. Then:*

1. $f(\omega') = c^{\tau}_{\omega_{n-1}}\mathrm{form}(G/R_\omega(G) \mid G \in \mathfrak{Y})$;
2. $f(p) = c^{\tau}_{\omega_{n-1}}\mathrm{form}(G/C^p(G) \mid G \in \mathfrak{Y})$ *for all* $p \in \pi$;
3. $f(p) = \varnothing$ *for all* $p \in \omega \setminus \pi$;

4. *if* $\mathfrak{F} = CF_\omega(h)$, *where* $h$ *is a* $c^\tau_{\omega_{n-1}}$*-valued* $\omega$*-composition satellite, then*

$$f(p) = c^\tau_{\omega_{n-1}}\mathrm{form}(G \mid G \in h(p) \cap \mathfrak{F} \textit{ and } O_p(G) = 1)$$

*for all* $p \in \pi$ *and*

$$f(\omega') = c^\tau_{\omega_{n-1}}\mathrm{form}(G \mid G \in h(\omega') \cap \mathfrak{F} \textit{ and } R_\omega(G) = 1).$$

The lemma below is obtained by direct calculation.

**Lemma 4.11** ([128]). *Let* $n \geqslant 1$, $f_i$ *be the minimal* $c^\tau_{\omega_{n-1}}$*-valued* $\omega$*-composition satellite of a formation* $\mathfrak{F}_i$, $i \in I$. *Then* $\bigvee^\tau_{\omega_{n-1}}(f_i \mid i \in I)$ *is the minimal* $c^\tau_{\omega_{n-1}}$*-valued* $\omega$*-composition satellite of* $\mathfrak{F} = \bigvee^\tau_{\omega_n}(\mathfrak{F}_i \mid i \in I)$.

**Definition 4.12.** Let $\mathfrak{F} = CF_\omega(f)$, and $f(a) \subseteq \mathfrak{F}$ for all $a \in \omega \cup \{\omega'\}$. Then the composition satellite $f$ is called an *integrated* satellite of formation $\mathfrak{F}$.

Following the paper [114], we put for any set of groups $\mathfrak{Y}$:

$$\mathfrak{Y}(C^p) = \begin{cases} \mathrm{form}\ \ (G/C^p(G) \mid G \in \mathfrak{Y}) & \text{if } p \in \pi(\mathrm{Com}(\mathfrak{Y})); \\ \varnothing & \text{if } p \in \mathbb{P} \setminus \pi(\mathrm{Com}(\mathfrak{Y})). \end{cases}$$

**Definition 4.13** ([114]). Let $\mathfrak{F} = CLF(F)$, where $F(0) = \mathfrak{F}$ and $F(p) = \mathfrak{N}_p\mathfrak{F}(C^p)$ for all $p \in \mathbb{P}$. Then the satellite $F$ is called a *canonical composition satellite* of the formation $\mathfrak{F}$. By [114, Remark 1], every composition formation possesses a canonical composition satellite.

**Lemma 4.14** (Lemma 8 [114]). *Let $\Theta$ be a complete lattice of formations such that $\Theta^c \subseteq \Theta$; and let the formation $\mathfrak{N}_p\mathfrak{H}$ belongs to $\Theta$ for each formation $\mathfrak{H} \in \Theta$, and every $p \in \omega$. If $\mathfrak{F} = CLF(F) \in \Theta^c$, then the satellite $F$ is $\Theta$-valued.*

**Corollary 4.15.** *The following equality holds $(c_\infty^\tau)^c = c_\infty^\tau$.*

*Proof.* The inclusion $(c_\infty^\tau)^c \subseteq c_\infty^\tau$ is obvious. Let $\mathfrak{F} \in c_\infty^\tau$ and $F$ be a canonical composition satellite of $\mathfrak{F}$. Then by Lemmas 4.14 and 6.19 for all $a \in \mathbb{P} \cup \{0\}$ and each positive integer $n$, we see that $F(a)$ is $\tau$-closed $n$-multiply composition formation of finite groups. So, the composition satellite $F$ is $c_\infty^\tau$-valued. Thus, we have $\mathfrak{F} \in (c_\infty^\tau)^c$, and, finally, $c_\infty^\tau \subseteq (c_\infty^\tau)^c$. □

**Lemma 4.16** (Lemma 2.1 [138]). *Let $\mathfrak{F} = CLF(F)$ be a $\tau$-closed $n$-multiply composition formation, where $n$ is a positive integer. Then the satellite $F$ is $c_n^\tau$-valued.*

Lemma 4.16 implies the following corollary.

**Corollary 4.17.** *Let $\mathfrak{F} = CLF(F)$ be a $\tau$-closed totally composition formation. Then the satellite $F$ is $c_\infty^\tau$-valued.*

**Definition 4.18.** Let $\{f_i \mid i \in I\}$ be the set of all composition $c_\infty^\tau$-valued satellites of a formation $\mathfrak{F}$. Because $c_\infty^\tau$ is a complete lattice of formations of finite groups, then Lemma 4.7 implies that $f = \bigcap_{i\in I} f_i$ is a composition $c_\infty^\tau$-valued satellite of formation $\mathfrak{F}$. This composition satellite $f$ is called *minimal*.

For a complete lattice of formations $\Theta$, we write $\Theta\mathrm{form}\,\mathfrak{Y}$ to denote the intersection of all $\Theta$-formations containing a set of groups $\mathfrak{Y}$. So, $c_\infty^\tau\mathrm{form}\,\mathfrak{Y}$ is the intersection of all $\tau$-closed totally composition formations containing a set of groups $\mathfrak{Y}$. The lemma below, which immediately follows by Lemma 5 in [114] and Corollary 4.15, describes the minimal $c_\infty^\tau$-valued satellite of a formation $c_\infty^\tau\mathrm{form}\,\mathfrak{Y}$.

**Lemma 4.19** (Lemma 2.2 [121]). *Let $\mathfrak{Y}$ be a nonempty set of finite groups, and let $\mathfrak{F} = c_\infty^\tau\mathrm{form}\,\mathfrak{Y}$. Denote $\pi(\mathrm{Com}(\mathfrak{Y}))$ by $\pi$, and let $f$ be the minimal $c_\infty^\tau$-valued composition satellite of $\mathfrak{F}$. Then the following statements hold:*

1. $f(0) = c_\infty^\tau\mathrm{form}\,(G/R(G) \mid G \in \mathfrak{Y})$;
2. $f(p) = c_\infty^\tau\mathrm{form}\,(G/C^p(G) \mid G \in \mathfrak{Y})$ *for all* $p \in \pi$;
3. $f(p) = \varnothing$ *for all* $p \in \mathbb{P} \setminus \pi$;
4. *if* $\mathfrak{F} = CLF(h)$ *and the satellite* $h$ *is* $c_\infty^\tau$*-valued, then for all* $p \in \pi$ *we have*

$$f(p) = c_\infty^\tau\mathrm{form}\,(G \mid G \in h(p) \cap \mathfrak{F} \text{ and } O_p(G) = 1); \text{ and}$$

$$f(0) = c_\infty^\tau\mathrm{form}\,(G \mid G \in h(0) \cap \mathfrak{F} \text{ and } R(G) = 1).$$

Lemma 4.19 implies the following assertion.

**Corollary 4.20.** *Let $f_1$ and $f_2$ be the minimal composition $c_\infty^\tau$-valued satellites of formations $\mathfrak{F}_1$ and $\mathfrak{F}_2$ respectively. Then $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ if and only if $f_1 \leq f_2$.*

### 4.1.2 Inductance of the lattice $c^{\tau}_{\omega_n}$

We say that a group $G$ is *monolithic* if it possesses the unique minimal normal subgroup (*monolith*), which is contained in each nontrivial normal subgroup of $G$.

**Lemma 4.21** (Corollary 1.2.26 [107]). *Let $\mathfrak{Y}$ be a $\tau$-closed semiformation, a fininite group $A$ belongs to $\mathfrak{F} = \tau\mathrm{form}\,\mathfrak{Y}$. Let $A$ be a monolithic group, and $A \notin \mathfrak{Y}$. Then there exists a group $H$ in $\mathfrak{F}$ and normal subgroups $N, N_1, \ldots, N_t$; $M, M_1, \ldots, M_t$ $(t \geqslant 2)$ of $H$ such that the following statements hold:*

1. *$H/N \cong A$, $M/N = \mathrm{Soc}(H/N)$;*
2. *$N_1 \cap \ldots \cap N_t = 1$;*
3. *$H/N_i$ is a monolithic $\mathfrak{Y}$-group and $M_i/N_i$ is the socle of $H/N_i$ which is $H$-isomorhpic to $M/N$;*
4. *$M_1 \cap \ldots \cap M_t \subseteq M$.*

**Lemma 4.22** (Lemma 4.1.3 [107]). *Let $N_1 \times \ldots \times N_t = \mathrm{Soc}(G)$, where $N_i$ is a minimal normal subgroup of $G$ $(i = 1, \ldots, t)$, $t > 1$, and $O_p(G) = 1$. Let $M_i$ be the largest normal subgroup in $G$ containing $N_1 \times \ldots \times N_{i-1} \times N_{i+1} \times \ldots \times N_t$, but not containing $N_i$, $i = 1, \ldots, t$. Then*

1. *for every $i \in \{1, \ldots, t\}$, $O_p(G/M_i) = 1$, $G/M_i$ is monolithic and its socle $N_iM_i/M_i$ is $G$-isomorphic to $N_i$;*
2. *$M_1 \cap \ldots \cap M_t = 1$.*

**Lemma 4.23** (Lemma 9 [128]). *Let $A$ be a monolithic group with a nonabelian socle $R$, and let $\mathfrak{M}$ be a $\tau$-closed semiformation and $A \in c^{\tau}_{\omega_n}\mathrm{form}\mathfrak{M}$, $n \geqslant 0$. Then $A \in \mathfrak{M}$.*

**Lemma 4.24** (Lemma 10 [128]). *Let $\mathfrak{M}$ be a semiformation. Suppose than a finite group $A$ belongs to the formation $c^{\tau}_{\omega_n}\mathrm{form}\mathfrak{M}$, $n \geqslant 0$. Then:*

1. *if $O_p(A) = 1$ and $p \in \omega$, then $A \in c^{\tau}_{\omega_n}\mathrm{form}\mathfrak{M}_1$, where*

$$\mathfrak{M}_1 = (G/O_p(G) \mid G \in \mathfrak{M});$$

2. *if $R_\omega(A) = 1$, then $A \in c^{\tau}_{\omega_n}\mathrm{form}\mathfrak{M}_2$, where $\mathfrak{M}_2 = (G/R_\omega(G) \mid G \in \mathfrak{M})$.*

**Theorem 4.25** (Theorem [128]). *Let $n$ be a positive integer, and let $\omega \neq \varnothing$ be a set of primes. Then the lattice of all $\tau$-closed $n$-multiply $\omega$-composition formations $c^{\tau}_{\omega_n}$ is inductive.*

*Proof.* Let $\{\mathfrak{F}_i \mid i \in I\}$ be a set of $\tau$-closed $n$-multiply $\omega$-composition formations of finite groups, and let $f_i$ be an integrated $c^{\tau}_{\omega_{n-1}}$-valued $\omega$-composition satellite of formation $\mathfrak{F}_i$.

Denote $\mathfrak{M} = CF_\omega(\bigvee^{\tau}_{\omega_{n-1}}(f_i \mid i \in I))$, and $\mathfrak{F} = \bigvee^{\tau}_{\omega_n}(\mathfrak{F}_i \mid i \in I)$. Let $h_i$ be the minimal $c^{\tau}_{\omega_{n-1}}$-valued $\omega$-composition satellite of formation $\mathfrak{F}_i$. Applying Lemma 4.11, we see that $h = \bigvee^{\tau}_{\omega_{n-1}}(h_i \mid i \in I)$ is the minimal $c^{\tau}_{\omega_{n-1}}$-valued $\omega$-composition satellite of formation $\mathfrak{F}$. Because $h_i \leq f_i$ for all $i \in I$, it holds $h \leq f = \bigvee^{\tau}_{\omega_{n-1}}(f_i \mid i \in I)$. Hence, $\mathfrak{F} \subseteq \mathfrak{M}$.

Assume $\mathfrak{M} \not\subseteq \mathfrak{F}$. Consider a group of minimal order in $\mathfrak{M} \backslash \mathfrak{F}$, say $G$. Thus, $G$ is a monolithic group, and $R = G^{\mathfrak{F}}$ is the socle of this group. If $\omega \cap \pi(\mathrm{Com}(R)) = \varnothing$, then $R_\omega(G) = 1$.

Thus, $G \cong G/1 = G/R_\omega(G) \in f(\omega') = (\bigvee^{\tau}_{\omega_{n-1}}(f_i \mid i \in I))(\omega') = c^{\tau}_{\omega_{n-1}}\mathrm{form}(\bigcup_{i\in I} f_i(\omega')) = c^{\tau}_{\omega_{n-1}}\mathrm{form}(\bigcup_{i\in I} \mathfrak{F}_i)$. But Lemma 4.23 implies that

$$G \in \bigcup_{i\in I} \mathfrak{F}_i \subseteq c^{\tau}_{\omega_{n-1}}\mathrm{form}(\bigcup_{i\in I} \mathfrak{F}_i) = \mathfrak{F},$$

i.e., we have a contradiction. So, $\omega \cap \pi(\mathrm{Com}(R)) \neq \varnothing$.

Let $R$ be a nonabelian group. Then $\pi(\mathrm{Com}(R)) = \varnothing$. But $\omega \cap \pi(\mathrm{Com}(R)) = \varnothing$, and again we have a contradiction. Thus, $R$ is a $p$-group, where prime $p$ belongs to $\omega \cap \pi(\mathrm{Com}(R))$. Because $G \in \mathfrak{M} = CF_\omega(f)$, we have $G/R \in \mathfrak{M}$. Then by induction because $|G/R| < |G|$, we see that $G/R \in \mathfrak{F} = CF_\omega(h)$. So,

$$(G/R)/R_\omega(G/R) = (G/R)/(R_\omega(G)/R) \cong G/R_\omega(G) \in h(\omega'),$$

$$(G/R)/C^q(G/R) = (G/R)/(C^q(G)/R) \cong G/C^q(G) = h(q)$$

for any $q \in \omega \cap \pi(\mathrm{Com}(G/R)) \setminus \{p\}$. However, $G \in \mathfrak{M} = CF_\omega(f)$. Thus,

$$G/C^p(G) \in f(p) = c^{\tau}_{\omega_{n-1}}\mathrm{form}(\bigcup_{i\in I} f_i(p)).$$

Because $O_p(G/C^p(G)) = 1$, then applying Lemmas 4.10 and 4.24, we obtain

$$G/C^p(G) \in c^{\tau}_{\omega_{n-1}}\mathrm{form}(A/O_p(A) \mid A \in \bigcup_{i\in I} f_i(p)) =$$

$$c^{\tau}_{\omega_{n-1}}\mathrm{form}(\bigcup_{i\in I}(A/O_p(A) \mid A \in f_i(p))) =$$

$$c^{\tau}_{\omega_{n-1}}\mathrm{form}(\bigcup_{i\in I} c^{\tau}_{\omega_{n-1}}\mathrm{form}(A/O_p(A) \mid A \in f_i(p))) = c^{\tau}_{\omega_{n-1}}\mathrm{form}(\bigcup_{i\in I} h_i(p)).$$

however, $c^{\tau}_{\omega_{n-1}}\mathrm{form}(\bigcup_{i\in I} h_i(p)) = (\bigvee^{\tau}_{\omega_{n-1}}(h_i \mid i \in I))(p) = h(p)$. So,

$$G/R_{\omega}(G) \in h(\omega') \qquad \text{and} \qquad G/C^{r}(G) \in h(r)$$

for any $r \in \omega \cap \pi(\mathrm{Com}(G))$. Thus, $G \in \mathfrak{F}$, and we otbtain a contradiction. Finally, $\mathfrak{F} = \mathfrak{M}$. The theorem is proved. □

For the trivial subgroup functor, we obtain the following corollary.

**Corollary 4.26.** *Let $n > 0$ and $\omega$ be nonempty set of primes. Then the lattice of all $n$-multiply $\omega$-composition formations is inductive.*

Let $\omega = \mathbb{P}$, then we obtain the following result.

**Corollary 4.27.** *Let $n > 0$. Then the lattice of all $n$-multiply composition formations is inductive.*

### 4.1.3 Inductance of the lattice $c^{\tau}_{\infty}$

We obtain the following lemma by direct calculation.

**Lemma 4.28.** *Let $f_i$ be the minimal $c^{\tau}_{\infty}$-valued composition satellite of a formation $\mathfrak{F}_i$, where $i \in I$. Then $f = \bigvee^{\tau}_{\infty}(f_i \mid i \in I)$ is the minimal $c_{\infty}$-valued composition satellite of formation $\mathfrak{F} = \bigvee^{\tau}_{\infty}(\mathfrak{F}_i \mid i \in I)$.*

**Theorem 4.29** (Theorem 1.1. [117]). *The lattice of all $\tau$-closed totally composition formations $c^{\tau}_{\infty}$ is inductive.*

*Proof.* Let $\{\mathfrak{F}_i \mid i \in I\}$ be a set of $\tau$-closed totally composition formations, and $f_i$ be an integrated $c_\infty^\tau$-valued composition satellite of $\mathfrak{F}_i$. Let $\mathfrak{F} = CLF(f) = \bigvee_\infty^\tau(\mathfrak{F}_i \mid i \in I)$, and $\mathfrak{M} = CLF(\bigvee_\infty^\tau(f_i \mid i \in I))$. We shall show that $\mathfrak{F} = \mathfrak{M}$ proceeding by induction on $i$.

**Step 1.** Let $i = 2$, $p \in \mathbb{P}$, and $h_j$ be the minimal $c_\infty^\tau$-valued composition satellite of the formation $\mathfrak{F}_j = CLF(f_j)$, where $j = 1, 2$. Then by Corollary 4.17, we have

$$h_j(p) \subseteq f_j(p) \subseteq \mathfrak{N}_p h_j(p) = F_j(p) \in c_\infty^\tau,$$

where $F_j$ is the canonical $c_\infty^\tau$-valued composition satellite of the formation $\mathfrak{F}_j$. Let $\mathfrak{F} = CLF(F)$, where $F$ is the canonical $c_\infty^\tau$-valued composition satellite of the formation $\mathfrak{F}$. Then by Lemma 4.19, we have

$$h(p) = c_\infty^\tau\text{form}\,((\mathfrak{F}_1 \cup \mathfrak{F}_2)(C^p)) = c_\infty^\tau\text{form}\,(\mathfrak{F}_1(C^p) \cup \mathfrak{F}_2(C^p)) =$$

$$c_\infty^\tau\text{form}\,(h_1(p) \cup h_2(p)) \subseteq f(p) \subseteq$$

$$\mathfrak{N}_p c_\infty^\tau\text{form}\,(h_1(p) \cup h_2(p)) = \mathfrak{N}_p h(p) = F(p).$$

Thus, we have $h(p) \subseteq f(p) \subseteq F(p)$ for all $p \in \mathbb{P}$; moreover, it holds $h(0) \subseteq f(0) \subseteq F(0)$. Hence, $h(a) \subseteq f(a) \subseteq F(a)$ for all $a \in \mathbb{P} \cup \{0\}$ implies $h \leq f \leq F$. Consequently, we have $\mathfrak{F}_1 \bigvee_\infty^\tau \mathfrak{F}_2 = CLF(f_1 \bigvee_\infty^\tau f_2)$.

**Step 2.** Let $i > 2$, and the assertion is true for $i = r - 1$ by induction. Then $\mathfrak{F}_1 \bigvee_\infty^\tau \cdots \bigvee_\infty^\tau \mathfrak{F}_{r-1} = CLF(f_1 \bigvee_\infty^\tau \cdots \bigvee_\infty^\tau f_{r-1})$.

By Step 1, we have $\mathfrak{F} = c_\infty^\tau \mathrm{form}\,((\mathfrak{F}_1 \bigvee_\infty^\tau \dots \bigvee_\infty^\tau \mathfrak{F}_{r-1}) \cup \mathfrak{F}_r) = CLF(f)$, and $f(a) = c_\infty^\tau \mathrm{form}\,((f_1(a) \bigvee_\infty^\tau \dots \bigvee_\infty^\tau f_{r-1}(a)) \cup f_r(a)) = f_1(a) \bigvee_\infty^\tau \dots \bigvee_\infty^\tau f_r(a) = (f_1 \bigvee_\infty^\tau \dots \bigvee_\infty^\tau f_r)(a)$ for each $a \in \mathbb{P} \cup \{0\}$. Therefore, $f = f_1 \bigvee_\infty^\tau \dots \bigvee_\infty^\tau f_r$. This proves the theorem. □

Each complete sublattice of the inductive lattice is an inductive lattice. Thus, we obtain the following result.

**Corollary 4.30.** *Let $\theta$ be a complete sublattice of the lattice $c_\infty^\tau$. Then $\theta$ is inductive.*

If $\tau$ is trivial, we have the corollary.

**Corollary 4.31.** *The lattice of all totally composition formations is inductive.*

## 4.2 Algebraic lattices of formations

Skiba posed the following question:

**Question 4.2** (Question 4.4.6 [107]). *Let $\tau$ be a subgroup functor. Is it true that the lattice of all $\tau$-closed totally local formations algebraic?*

In [87], it is given the solution of the mentioned problem. We known that the following lattices of formations of finite groups are alebraic:

- the lattice of all $\tau$-closed $n$-multiply $\omega$-composition formations [138];
- the lattice of all solvable totally local formations [136];

- the lattice of all $\tau$-closed totally $\omega$-saturated formations [93];
- the lattice of all $n$-multiply $\sigma$-local formations [29];
- the lattice of all $n$-multiply $\mathfrak{L}$-composition formations [114].

Recently, the author [121] solved the following related problem:

**Question 4.3** (Problem 1 [114]). *Is the lattice of all totally composition formations of finite groups algebraic?*

The main goal of this section is to generalize this solution for $\tau$-closed formations.

**Lemma 4.32** (Lemma 7 [127]). *Let $\mathfrak{F}$ be a nonempty $\tau$-closed formation of finite groups. Then $\mathfrak{S}_\pi\mathfrak{F}$ is a $\tau$-closed totally composition formation of finite groups, where $\pi(\mathfrak{F}) \subseteq \pi \subseteq \mathbb{P}$.*

**Lemma 4.33** (Lemma 8 [127]). *Let $\mathfrak{F} = \bigvee_\infty^\tau(\mathfrak{F}_i \mid i \in I)$, $\mathfrak{F}_i \in c_\infty^\tau$ for any $i \in I$. Suppose that $A$ is a monolithic $\mathfrak{F}$-group with a nonabelian socle $R$. Then we have $A \in \bigcup_{i\in I}\mathfrak{F}_i$.*

*Proof.* Denote $\pi = \pi(\mathfrak{F})$. By Lemma 4.32 the following holds:

$$\mathfrak{F} \subseteq \mathfrak{M} = \mathfrak{S}_\pi c_0^\tau \mathrm{form}(\bigcup_{i\in I}\mathfrak{F}_i).$$

So, we have $A \in \mathfrak{M}$. We note that $A \in c_0^\tau\mathrm{form}(\bigcup_{i\in I}\mathfrak{F}_i)$, because the socle $R = \mathrm{Soc}(A)$ is nonabelian, and $A$ belongs to $\bigcup_{i\in I}\mathfrak{F}_i$ by Lemma 4.23. $\square$

**Proposition 4.34** (Proposition 1 [127]). *Let $G$ be a finite group. Then one-generated totally composition formation $\mathfrak{F} = c_{\infty}^{\tau}\mathrm{form}\, G$ is a compact element of the lattice $c_{\infty}^{\tau}$.*

*Proof.* We shall proof the proposition using induction on $|G|$. Let $A$ be a counterexample of the minimal order, and let $\mathfrak{F} = c_{\infty}^{\tau}\mathrm{form}\, A \subseteq \mathfrak{M} = c_{\infty}^{\tau}\mathrm{form}(\bigcup_{i\in I}\mathfrak{F}_i) = \bigvee_{\infty}^{\tau}(\mathfrak{F}_i \mid i \in I)$, where $\mathfrak{F}_i \in c_{\infty}^{\tau}$ for any $i \in I$. We shall show that the group $A$ is monolithic. Let us consider the cases below:

$(i)$ Let $M_1$ and $M_2$ be two distinct minimal normal subgroups of the group $A$. Assume first that $\mathfrak{M}_j = c_{\infty}^{\tau}\mathrm{form}(A/M_j)$ for $j = 1, 2$. So, $|A/M_j| < |A|$. Then, using induction, we have $\mathfrak{M}_j \subseteq \mathfrak{M}$. But then

$$\mathfrak{M}_1 \subseteq c_{\infty}^{\tau}\mathrm{form}(\mathfrak{F}_{i_1}\bigcup\ldots\bigcup\mathfrak{F}_{i_t}), \mathfrak{M}_2 \subseteq c_{\infty}^{\tau}\mathrm{form}(\mathfrak{F}_{i_{t+1}}\bigcup\ldots\bigcup\mathfrak{F}_{i_s})$$

for some $i_1, \ldots, i_s$. Consequently, $\mathfrak{F} = \mathfrak{M}_1 \bigvee_{\infty}^{\tau} \mathfrak{M}_2$ is a subformation of formation

$$c_{\infty}^{\tau}\mathrm{form}(\mathfrak{F}_{i_1}\bigcup\ldots\bigcup\mathfrak{F}_{i_t}\bigcup\mathfrak{F}_{i_{t+1}}\bigcup\ldots\bigcup\mathfrak{F}_{i_s}).$$

The contradiction obtained.

$(ii)$ Assume that $R = \mathrm{Soc}(A)$. If $R$ is nonabelian, then by Lemma 4.33 we see that $A \in \bigcup_{i\in I}\mathfrak{F}_i$. So, $A \in \mathfrak{M}$, and we obtain a contradiction again.

$(iii)$ Let $R$ be an abelian $p$-group for some prime $p \in \pi(\mathrm{Com}(A))$. In this case, we have $A/\Phi(A) \in \mathrm{form}\, A$. So, $c_{\infty}^{\tau}\mathrm{form}(A/\Phi(A)) = c_{\infty}^{\tau}\mathrm{form}\, A$. But $|A| > |A/\Phi(A)|$; using induction, we see that $R \not\subseteq \Phi(A)$.

Suppose $B$ is a subgroup of $A$, such that $R \cap B = 1$, and $O_p(B) = 1$. Then, $A = Z_p \wr B = [R]B$, and applying Lemma 4.8 we obtain $R = C^p(A) = O_p(A)$.

Let $m$, $f_i$, and $f$ be the minimal $c_\infty^\tau$-valued composition satellites of formations $\mathfrak{M}$, $\mathfrak{F}$, and $\mathfrak{F}_i$, respectively.

Then Lemma 4.28 ensures that $m = \bigvee_\infty^\tau(f_i \mid i \in I)$. So, applying the properties of regular wreath products, we see that $B \cong A/O_p(A) = A/R = A/C^p(A)$ is in $m(p)$. Because of $|B| < |A|$, for some $j_1, j_2, \dots, j_k \in J \subseteq I$, it follows that $B \cong A/C^p(A) \in f_{j_1}(p) \bigvee_\infty^\tau \dots \bigvee_\infty^\tau f_{j_k}(p)$.

Now Lemma 4.28 ensures that $m_3 = \bigvee_\infty^\tau(f_j \mid j \in J)$ is the minimal $c_\infty^\tau$-valued composition satellite of formation $\mathfrak{M}_3 = \bigvee_\infty^\tau(\mathfrak{F}_j \mid j \in J)$. Therefore, $A/O_p(A) \cong B \in m_3(p)$. Applying Lemma 4.9 we see that $A$ belongs to formation $\mathfrak{M}_3$. So, $\mathfrak{F} = c_\infty^\tau \mathrm{form}\, A \subseteq \mathfrak{M}_3$, and it is a contradiction. □

**Theorem 4.35** (Theorem [127]). *The lattice $c_\infty^\tau$ of all $\tau$-closed totally composition formations of finite groups is algebraic.*

*Proof.* Let $\mathfrak{F}$ be a $\tau$-closed totally composition formation of finite groups. It is easy to see that $\mathfrak{F} = c_\infty^\tau \mathrm{form}(\bigcup_{i \in I} \mathfrak{F}_i) = \bigvee_\infty^\tau(\mathfrak{F}_i \mid i \in I)$, where $\mathfrak{F}_i = c_\infty^\tau \mathrm{form}\, G_i$ for some group $G_i$ $(i \in I)$. We shall show that every one-generated formation of finite groups $\mathfrak{F}_i$ is a compact element of the lattice of all $\tau$-closed totally composition formations of finite groups. However, it immediately follows by Proposition 4.34. □

For trivial subgroup functor $\tau$ we have

**Corollary 4.36** ([121]). *The lattice of all totally composition formations is algebraic.*

## 4.3 Separated lattices of formations

**Definition 4.37** ([107]). Fix a nonempty class $\mathfrak{Y}$ of finite groups. Let $\Theta$ be a lattice of formations of finite groups. Then $\Theta$ is called $\mathfrak{Y}$*-separated* if for any term $\xi(x_1, \dots, x_m)$ of signature $\{\cap, \bigvee_\Theta\}$, any formations of finite groups $\mathfrak{F}_1, \dots, \mathfrak{F}_m$ of $\Theta$, and any finite group $A \in \mathfrak{Y} \cap \xi(\mathfrak{F}_1, \dots, \mathfrak{F}_m)$, there exist $\mathfrak{Y}$-groups $A_1 \in \mathfrak{F}_1, \dots, A_m \in \mathfrak{F}_m$ such that $A \in \xi(\Theta\text{form}\, A_1, \dots, \Theta\text{form}\, A_m)$.

**Lemma 4.38** (Lemma 17 [137]). *Let $\Theta$ be an $\mathfrak{Y}$-separated lattice of formations of finite groups and let $\eta$ be a sublattice of $\Theta$ such that $\eta$ contains all one-generated $\Theta$-subformations of the form $\Theta\text{form}\, A$, where $A \in \mathfrak{Y}$, of every formation of finite groups $\mathfrak{F} \in \eta$. Let a law $\xi_1 = \xi_2$ of signature $\{\cap, \bigvee_\Theta\}$ is true for all one-generated $\Theta$-formations belonging to $\eta$. Then the law $\xi_1 = \xi_2$ is true for all $\Theta$-subformations belonging to $\eta$.*

Lemma 2.14 imply the following result.

**Lemma 4.39.** *Let $\mathfrak{R}_i$ be a $\tau$-closed semiformation generated by some finite groups $G_1$ and $G_2$. Then $\mathfrak{R}_1 \cup \mathfrak{R}_2$ is a $\tau$-closed semiformation, and $\mathfrak{R}_1 = (B_1, \dots, B_t)$, $\mathfrak{R}_2 = (C_1, \dots, C_s)$ for some finite groups $B_1, \dots, B_t \in QS_{\overline{\tau}}(G_1)$ and $C_1, \dots, C_s \in QS_{\overline{\tau}}(G_2)$.*

**Lemma 4.40** (Lemma 3.4. [129]). *Let $n$ be a nonnegative integer, and let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be $\tau$-closed $n$-multiply $\omega$-composition formations of finite groups, and $A$ be a finite* *group such that* $A \in c^{\tau}_{\omega_n}\text{form}\,(\mathfrak{F}_1 \cup \mathfrak{F}_2)$, *Then there exist finite groups* $A_1 \in \mathfrak{F}_1$ *and* $A_2 \in \mathfrak{F}_2$ *such that* $A \in (c^{\tau}_{\omega_n}\text{form}\,A_1) \bigvee^{\tau}_{\omega_n} (c^{\tau}_{\omega_n}\text{form}\,A_2)$.

*Proof.* We shall use the induction on $n$. Let $n = 0$. We note that the formations $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are $\tau$-closed. Then by Lemmas 2.16 and 2.15

$$A \in c^{\tau}_{\omega_0}\text{form}\,(\mathfrak{F}_1 \cup \mathfrak{F}_2) = \text{form}\,(\mathfrak{F}_1 \cup \mathfrak{F}_2) = \mathrm{QR}_0(\mathfrak{F}_1 \cup \mathfrak{F}_2).$$

Consequently $A \cong H/N$ where $H \in \mathrm{R}_0(\mathfrak{F}_1 \cup \mathfrak{F}_2)$. Using Lemma 15 in [137] we see that $A \in \text{form}\,(H/H^{\mathfrak{F}_1}) \bigvee \text{form}\,(H/H^{\mathfrak{F}_2}) \subseteq \mathfrak{F}_1 \bigvee^{\tau}_{\omega_0} \mathfrak{F}_2$.

Let $n > 0$, $\{p_1, \dots, p_t\} = \omega \cap \pi(\text{Com}(A))$ and $A \in \mathfrak{F}_1 \bigvee^{\tau}_{\omega_n} \mathfrak{F}_2$. Then by [128, Lemma 8] and Lemma 4.11, $A/C^{p_i}(A) \in f_1(p_i) \bigvee^{\tau}_{\omega_{n-1}} f_2(p_i)$ and $A/R_{\omega}(A) \in f_1(\omega') \bigvee^{\tau}_{\omega_{n-1}} f_2(\omega')$ where $f_j$ is the minimal $c^{\tau}_{\omega_{n-1}}$-valued $\omega$-composition satellite of $\mathfrak{F}_j$ $(j = 1, 2)$. By induction, we may find finite groups

$$A_{i_1} \in f_1(p_i),\ A_{i_2} \in f_2(p_i),\ T_1 \in f_1(\omega'),\ T_2 \in f_2(\omega')$$

such that $A/C^{p_i}(A) \in (c^{\tau}_{\omega_{n-1}}\text{form}\,A_{i_1}) \bigvee^{\tau}_{\omega_{n-1}} (c^{\tau}_{\omega_{n-1}}\text{form}\,A_{i_2})$ and $A/R_{\omega}(A) \in (c^{\tau}_{\omega_{n-1}}\text{form}\,T_1) \bigvee^{\tau}_{\omega_{n-1}} (c^{\tau}_{\omega_{n-1}}\text{form}\,T_2)$.

We claim that $c^{\tau}_{\omega_{n-1}}\text{form}\,(A_{i_1}, A_{i_2}) = (c^{\tau}_{\omega_{n-1}}\text{form}\,A_{i_1}) \bigvee^{\tau}_{\omega_{n-1}} (c^{\tau}_{\omega_{n-1}}\text{form}\,A_{i_2})$, and $c^{\tau}_{\omega_{n-1}}\text{form}\,(T_1, T_2) = (c^{\tau}_{\omega_{n-1}}\text{form}\,T_1) \bigvee^{\tau}_{\omega_{n-1}} (c^{\tau}_{\omega_{n-1}}\text{form}\,T_2)$. Thus $A/C^{p_i}(A) \in c^{\tau}_{\omega_{n-1}}\text{form}\,(A_{i_1}, A_{i_2})$ and $A/R_{\omega}(A) \in c^{\tau}_{\omega_{n-1}}\text{form}\,(T_1, T_2)$.

Let $\mathfrak{R}_k$ be a $\tau$-closed semiformation generated by the group $A_{i_k}$ and $\mathfrak{Y}_k$ be a $\tau$-closed semiformation generated by the group $T_k$ where $k = 1, 2$. By Lemma 4.39 the semiformations $\mathfrak{R}_1 \cup \mathfrak{R}_2$ and $\mathfrak{Y}_1 \cup \mathfrak{Y}_2$ are $\tau$-closed, and $\mathfrak{R}_1 = (B_1, \dots, B_t)$ and

$\mathfrak{R}_2 = (C_1, \dots, C_s)$ for some $B_1, \dots, B_t \in \mathrm{QS}_{\overline{\tau}}(A_{i_1})$ and $C_1, \dots, C_s \in \mathrm{QS}_{\overline{\tau}}(A_{i_2})$; $\mathfrak{Y}_1 = (U_1, \dots, U_m)$ and $\mathfrak{Y}_2 = (V_1, \dots, V_q)$ for some $U_1, \dots, U_m \in \mathrm{QS}_{\overline{\tau}}(T_1)$ and $V_1, \dots, V_q \in \mathrm{QS}_{\overline{\tau}}(T_2)$. Since $A_{i_k} \in \mathfrak{R}_k$ $(k = 1, 2)$, then $c^{\tau}_{\omega_{n-1}}\mathrm{form}\,(A_{i_1}, A_{i_2}) \subseteq c^{\tau}_{\omega_{n-1}}\mathrm{form}\,(\mathfrak{R}_1 \cup \mathfrak{R}_2)$.

We prove the inverse inclusion. Since $\mathrm{S}_{\overline{\tau}}(A_{i_k}) \subseteq \mathrm{R}_0\mathrm{S}_{\overline{\tau}}(A_{i_k})$ $(k = 1, 2)$, then by Lemma 2.15, $\mathfrak{R}_k = \mathrm{QS}_{\overline{\tau}}(A_{i_k}) \subseteq \mathrm{QR}_0\mathrm{S}_{\overline{\tau}}(A_{i_k}) = \tau\mathrm{form}\,(A_{i_k})$ where $k = 1,\, 2$. Thus, we have the inclusion $\tau\mathrm{form}\,(\mathfrak{R}_1 \cup \mathfrak{R}_2) \subseteq \tau\mathrm{form}\,(A_{i_1}, A_{i_2})$.

Hence $c^{\tau}_{\omega_{n-1}}\mathrm{form}\,(\mathfrak{R}_1 \cup \mathfrak{R}_2) \subseteq c^{\tau}_{\omega_{n-1}}\mathrm{form}\,(A_{i_1}, A_{i_2})$. Thus,

$$c^{\tau}_{\omega_{n-1}}\mathrm{form}\,(A_{i_1}, A_{i_2}) = c^{\tau}_{\omega_{n-1}}\mathrm{form}\,(\mathfrak{R}_1 \cup \mathfrak{R}_2).$$

Analogously $c^{\tau}_{\omega_{n-1}}\mathrm{form}\,(T_1,\, T_2) = c^{\tau}_{\omega_{n-1}}\mathrm{form}\,(\mathfrak{Y}_1 \cup \mathfrak{Y}_2)$. Thus

$$A/C^{p_i}(A) \in c^{\tau}_{\omega_{n-1}}\mathrm{form}\,(A_{i_1}, A_{i_2}) = c^{\tau}_{\omega_{n-1}}\mathrm{form}\,(B_1, \dots, B_t; C_1, \dots, C_s),$$

$$A/R_{\omega}(A) \in c^{\tau}_{\omega_{n-1}}\mathrm{form}\,(T_1, T_2) = c^{\tau}_{\omega_{n-1}}\mathrm{form}\,(U_1, \dots, U_m; V_1, \dots, V_q).$$

Since $O_{p_i}(A/C^{p_i}(A)) = 1$ and $R_{\omega}(A/R_{\omega}(A)) = 1$, then by Lemma 4.24

$$A/C^{p_i}(A) \in c^{\tau}_{\omega_{n-1}}\mathrm{form}\,(G/O_{p_i}(G) \mid G \in \mathfrak{R}_1 \cup \mathfrak{R}_2) =$$

$$c^{\tau}_{\omega_{n-1}}\mathrm{form}\,(B_1/O_{p_i}(B_1), \dots, B_t/O_{p_i}(B_t); C_1/O_{p_i}(C_1), \dots, C_s/O_{p_i}(C_s)),$$

$$A/R_{\omega}(A) \in c^{\tau}_{\omega_{n-1}}\mathrm{form}\,(G/R_{\omega}(G) \mid G \in \mathfrak{Y}_1 \cup \mathfrak{Y}_2) =$$

$$c^{\tau}_{\omega_{n-1}}\mathrm{form}\,(U_1/R_{\omega}(U_1), \dots, U_m/R_{\omega}(U_m); V_1/R_{\omega}(V_1), \dots, V_q/R_{\omega}(V_q)).$$

Thus we have the inclusions $c^{\tau}_{\omega_{n-1}}\mathrm{form}\,(B_1, \dots, B_t; C_1, \dots, C_s) \subseteq$

$$c^{\tau}_{\omega_{n-1}}\mathrm{form}\,(B_1/O_{p_i}(B_1), \dots, B_t/O_{p_i}(B_t); C_1/O_{p_i}(C_1), \dots, C_s/O_{p_i}(C_s)),$$

$$c^{\tau}_{\omega_{n-1}}\text{form}\,(U_1,\ldots,U_m;V_1,\ldots,V_q)\subseteq$$

$$c^{\tau}_{\omega_{n-1}}\text{form}\,(U_1/R_\omega(U_1),\ldots,U_m/R_\omega(U_m);V_1/R_\omega(V_1),\ldots,V_q/R_\omega(V_q)).$$

On the other hand, since $\mathfrak{R}_1\cup\mathfrak{R}_2$ and $\mathfrak{Y}_1\cup\mathfrak{Y}_2$ are semiformations, then for any finite group $G$ it holds:

if $G\in\mathfrak{R}_1\cup\mathfrak{R}_2$, then $G/O_{p_i}(G)\in\mathfrak{R}_1\cup\mathfrak{R}_2$;

if $G\in\mathfrak{Y}_1\cup\mathfrak{Y}_2$, then $G/R_\omega(G)\in\mathfrak{Y}_1\cup\mathfrak{Y}_2$.

Consequently

$$c^{\tau}_{\omega_{n-1}}\text{form}\,(B_1/O_{p_i}(B_1),\ldots,B_t/O_{p_i}(B_t);C_1/O_{p_i}(C_1),\ldots,C_s/O_{p_i}(C_s))=$$

$$c^{\tau}_{\omega_{n-1}}\text{form}\,(G/O_{p_i}(G)\mid G\in\mathfrak{R}_1\cup\mathfrak{R}_2)\subseteq$$

$$c^{\tau}_{\omega_{n-1}}\text{form}\,(\mathfrak{R}_1\cup\mathfrak{R}_2)=c^{\tau}_{\omega_{n-1}}\text{form}\,(B_1,\ldots,B_t;C_1,\ldots,C_s),$$

$$c^{\tau}_{\omega_{n-1}}\text{form}\,(U_1/R_\omega(U_1),\ldots,U_m/R_\omega(U_m);V_1/R_\omega(V_1),\ldots,V_q/R_\omega(V_q))=$$

$$c^{\tau}_{\omega_{n-1}}\text{form}\,(G/R_\omega(G)\mid G\in\mathfrak{Y}_1\cup\mathfrak{Y}_2)\subseteq$$

$$c^{\tau}_{\omega_{n-1}}\text{form}\,(\mathfrak{Y}_1\cup\mathfrak{Y}_2)=c^{\tau}_{\omega_{n-1}}\text{form}\,(U_1,\ldots,U_m;V_1,\ldots,V_q).$$

Thus

$$A/C^{p_i}(A)\in c^{\tau}_{\omega_{n-1}}\text{form}\,(B_1,\ldots,B_t;C_1,\ldots,C_s)=$$

$$c^{\tau}_{\omega_{n-1}}\text{form}\,(B_1/O_{p_i}(B_1),\ldots,B_t/O_{p_i}(B_t);C_1/O_{p_i}(C_1),\ldots,C_s/O_{p_i}(C_s)),$$

$$A/R_\omega(A)\in c^{\tau}_{\omega_{n-1}}\text{form}\,(U_1,\ldots,U_m;V_1,\ldots,V_q)=$$

$$c^{\tau}_{\omega_{n-1}}\text{form}\,(U_1/R_\omega(U_1),\ldots,U_m/R_\omega(U_m);V_1/R_\omega(V_1),\ldots,V_q/R_\omega(V_q)).$$

Hence we may suppose that $O_{p_i}(B_k) = 1 = O_{p_i}(C_l)$ and $R_\omega(U_x) = 1 = R_\omega(V_z)$ for all $k = 1, \dots, t$ and $l = 1, \dots, s$; $x = 1, \dots, m$ and $z = 1, \dots, w$.

Let $D_{i_1} = B_1 \times \dots \times B_t$ and $D_{i_2} = C_1 \times \dots \times C_s$; $U = U_1 \times \dots \times U_m$ and $V = V_1 \times \dots \times V_q$. Then $O_{p_i}(D_{i_1}) = 1 = O_{p_i}(D_{i_2})$ and $R_\omega(U) = 1 = R_\omega(V)$.

It is clear that $D_{i_1} \in \mathrm{D}_0\mathfrak{R}_1 \subseteq \mathrm{D}_0(c^\tau_{\omega_{n-1}}\mathrm{form}\, A_{i_1})$. Since $\mathrm{D}_0 \leqslant \mathrm{R}_0$ (see II, p. 267 in [37]), then $D_{i_1} \in \mathrm{R}_0(c^\tau_{\omega_{n-1}}\mathrm{form}\, A_{i_1}) = c^\tau_{\omega_{n-1}}\mathrm{form}\, A_{i_1}$. Analogously $D_{i_2} \in c^\tau_{\omega_{n-1}}\mathrm{form}\, A_{i_2}$. Consequently $c^\tau_{\omega_{n-1}}\mathrm{form}\,(D_{i_1}, D_{i_2}) \subseteq c^\tau_{\omega_{n-1}}\mathrm{form}\,(A_{i_1}, A_{i_2})$. Since $B_k \leqslant D_{i_1}$ for all $k = 1, \dots, t$, then $\mathfrak{R}_1 \subseteq c^\tau_{\omega_{n-1}}\mathrm{form}\,(D_{i_1}, D_{i_2})$. Analogously $\mathfrak{R}_2 \subseteq c^\tau_{\omega_{n-1}}\mathrm{form}\,(D_{i_1}, D_{i_2})$. Consequently $\mathfrak{R}_1 \cup \mathfrak{R}_2 \subseteq c^\tau_{\omega_{n-1}}\mathrm{form}\,(D_{i_1}, D_{i_2})$. Thus $A/C^{p_i}(A) \in c^\tau_{\omega_{n-1}}\mathrm{form}\,(\mathfrak{R}_1 \cup \mathfrak{R}_2) \subseteq c^\tau_{\omega_{n-1}}\mathrm{form}\,(A_{i_1}, A_{i_2})$.

Let $Z_{p_i}$ be a group of order $p_i$, $W_{i_1} = Z_{p_i} \wr D_{i_1}$ and $W_{i_2} = Z_{p_i} \wr D_{i_2}$. We show that $W_{i_1} \in \mathfrak{F}_1$. Let $B = Z_{p_i}^\natural$ be the base group of the wreath product $W_{i_1}$. Applying the properties of wreath product, we see

$$W_{i_1}/O_{p_i}(W_{i_1}) = W_{i_1}/B = (Z_{p_i} \wr D_{i_1})/B \cong D_{i_1}.$$

Since $A_{i_1} \in f_1(p_i)$ and $\mathrm{QS}_{\overline{\tau}}(A_{i_1}) = (B_1, \dots, B_t)$ where $B_1, \dots, B_t \in \mathrm{QS}_{\overline{\tau}}(A_{i_1})$, then $\mathrm{QS}_{\overline{\tau}}(A_{i_1}) \subseteq \mathrm{QS}_{\overline{\tau}}(f_1(p_i)) = f_1(p_i)$ and $B_1, \dots, B_t \in f_1(p_i)$. Consequently

$$D_{i_1} = B_1 \times \dots \times B_t \in \mathrm{D}_0 f_1(p_i) = f_1(p_i).$$

Since $f_1$ is the minimal $c^\tau_{\omega_{n-1}}$-valued $\omega$-composition satellite of $\mathfrak{F}_1$, then $D_{i_1} \in f_1(p_i) \cap \mathfrak{F}_1$. Thus $W_{i_1}/O_{p_i}(W_{i_1}) \cong D_{i_1} \in f_1(p_i) \cap \mathfrak{F}_1$ for all $p_i \in \omega \cap \pi(\mathrm{Com}(A))$. By Lemma 4 in [114] $W_{i_1} \in \mathfrak{F}_1$. Analogously $W_{i_2} \in \mathfrak{F}_2$. Since $T_1 \in f_1(\omega')$ and $f_1$

is an integrated $\omega$-composition satellite of $\mathfrak{F}_1$, then $T_1 \in \mathfrak{F}_1$. Hence $U_x \in \mathfrak{F}_1$ for all $x = 1, \ldots, m$. Analogously $V_z \in \mathfrak{F}_2$ for all $z = 1, \ldots, q$.

Let $A_1 = W_{1_1} \times W_{2_1} \times \ldots \times W_{t_1} \times U$ and $A_2 = W_{1_2} \times W_{2_2} \times \ldots \times W_{t_2} \times V$. Then $A_1 \in \mathfrak{F}_1$ and $A_2 \in \mathfrak{F}_2$. Using Lemma 15 in [137], we see that $A \in \mathfrak{F} = (c^{\tau}_{\omega_n}\mathrm{form}\, A_1) \bigvee^{\tau}_{\omega_n} (c^{\tau}_{\omega_n}\mathrm{form}\, A_2)$. The proof completed. □

**Theorem 4.41** (Theorem 3.2. [129]). *Let $n$ be a nonnegative integer. Then the lattice of all $\tau$-closed $n$-multiply $\omega$-composition formations of finite groups is $\mathfrak{G}$-separated.*

*Proof.* Let $\xi(x_1, \ldots, x_m)$ be a term of signature $\{\cap, \bigvee^{\tau}_{\omega_n}\}$, $\mathfrak{F}_1, \ldots, \mathfrak{F}_m$ be $\tau$-closed $n$-multiply $\omega$-composition formations and $A \in \xi(\mathfrak{F}_1, \ldots, \mathfrak{F}_m)$. We proceed by induction on the number $r$ of occurences of the symbols in $\{\cap, \bigvee^{\tau}_{\omega_n}\}$ into the term $\xi$. We shall show that there exist groups $A_i \in \mathfrak{F}_i$ $(i = 1, \ldots, m)$ such that $A \in \xi(c^{\tau}_{\omega_n}\mathrm{form}\, A_1, \ldots, c^{\tau}_{\omega_n}\mathrm{form}\, A_m)$. Let $r = 0$. It is clear that $A \in c^{\tau}_{\omega_n}\mathrm{form}\, A$.

We shall establish the assertion for $r = 1$. There are only two cases: either $A \in \mathfrak{F}_1 \cap \mathfrak{F}_2$ or $A \in \mathfrak{F}_1 \bigvee^{\tau}_{\omega_0} \mathfrak{F}_2 = c^{\tau}_{\omega_0}\mathrm{form}\,(\mathfrak{F}_1 \cup \mathfrak{F}_2) = \tau\mathrm{form}\,(\mathfrak{F}_1 \cup \mathfrak{F}_2)$. In the first case we have $A \in \tau\mathrm{form}\, A \cap \tau\mathrm{form}\, A$. In the second case by Lemma 4.40 there are finite groups $A_i \in \mathfrak{F}_i$ $(i = 1, 2)$ such that $A \in (c^{\tau}_{\omega_0}\mathrm{form}\, A_1) \bigvee^{\tau}_{\omega_0} (c^{\tau}_{\omega_0}\mathrm{form}\, A_2) = (\tau\mathrm{form}\, A_1) \bigvee^{\tau}_{\omega_0} (\tau\mathrm{form}\, A_2)$. The assertion of the theorem for $r = 1$ is true.

Let a term $\xi$ have $r > 1$ occurences of the symbols in $\{\cap, \bigvee^{\tau}_{\omega_n}\}$. We suppose proving by induction that the theorem holds for term with less number of occurences. Let $\xi$ be of the form $\xi_1(x_{i_1}, \ldots, x_{i_a}) \triangle \xi_2(x_{j_1}, \ldots, x_{j_b})$ where $\triangle \in \{\cap, \bigvee^{\tau}_{\omega_n}\}$ and $\{x_{i_1}, \ldots, x_{i_a}\} \cup \{x_{j_1}, \ldots, x_{j_b}\} = \{x_1, \ldots, x_m\}$. By $\mathfrak{H}_1$ we denote the formation $\xi_1(\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a})$, and by $\mathfrak{H}_2$ the formation $\xi_2(\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b})$. There exist groups $A_1 \in \mathfrak{H}_1$, $A_2 \in \mathfrak{H}_2$ such that $A \in c^{\tau}_{\omega_n}\operatorname{form} A_1 \triangle c^{\tau}_{\omega_n}\operatorname{form} A_2$. On the other hand, by induction, there exist groups $B_1, \ldots$, $B_a$; $C_1, \ldots, C_b$ such that $B_k \in \mathfrak{F}_{i_k}$, $C_k \in \mathfrak{F}_{j_k}$,

$$A_1 \in \xi_1(c^{\tau}_{\omega_n}\operatorname{form} B_1, \ldots, c^{\tau}_{\omega_n}\operatorname{form} B_a),$$

$$A_2 \in \xi_2(c^{\tau}_{\omega_n}\operatorname{form} C_1, \ldots, c^{\tau}_{\omega_n}\operatorname{form} C_b).$$

Suppose that $x_{i_1}, \ldots, x_{i_t}$ are not contained in $\xi_2$, but $x_{i_{t+1}}, \ldots, x_{i_a}$ are contained in $\xi_2$. Let $D_{i_k} = B_k$ if $k < t+1$, $D_{i_k} = B_k \times C_q$ where $q$ satisfies $x_{i_k} = x_{j_q}$ for all $k \geqslant t+1$. Let $D_{j_k} = C_k$ if $x_{j_k} \notin \{x_{i_{t+1}}, \ldots, x_{i_a}\}$. We denote by $\mathfrak{R}_p$ the formation $c^{\tau}_{\omega_n}\operatorname{form} D_{i_p}$ and by $\mathfrak{Y}_c$ we denote the formation $c^{\tau}_{\omega_n}\operatorname{form} D_{j_c}$, $p = 1, \ldots, a$; $c = 1, \ldots, b$. It follows that $A_1 \in \xi_1(\mathfrak{R}_1, \ldots, \mathfrak{R}_a)$, $A_2 \in \xi_2(\mathfrak{Y}_1, \ldots, \mathfrak{Y}_b)$. There exist the formations $\mathfrak{H}_1, \ldots, \mathfrak{H}_m$ such that $A \in \xi_1(\mathfrak{H}_{i_1}, \ldots, \mathfrak{H}_{i_a}) \triangle \xi_2(\mathfrak{H}_{j_1}, \ldots, \mathfrak{H}_{j_b}) = \xi(\mathfrak{H}_1, \ldots, \mathfrak{H}_m)$ where $\mathfrak{H}_i = c^{\tau}_{\omega_n}\operatorname{form} K_i$, $K_i \in \mathfrak{F}_i$. Thus the lattice $c^{\tau}_{\omega_n}$ is $\mathfrak{G}$-separated. The theorem is proved. $\square$

**Corollary 4.42.** *Let $n$ be a nonnegative integer. Then the lattice of all $n$-multiply $\mathfrak{L}$-composition formations of finite groups is $\mathfrak{G}$-separated.*

**Corollary 4.43** (Proposition [137]). *Let $n$ be a nonnegative integer. Then the lattice of all $n$-multiply $\omega$-composition formations of finite groups is $\mathfrak{G}$-separated.*

## 4.4 Laws of the lattices of partially composition formations

In [114], Skiba and Shemetkov proposed the question on the laws of the lattices of multiply $\mathfrak{L}$-composition formations. We study in this section the following more general question:

**Question 4.4.** *Let $m$ and $n$ be nonnegative integers. Does it true that for any sugroup functor $\tau$ and any nonempty set of primes $\omega$ the lattices $c^{\tau}_{\omega_m}$ and $c^{\tau}_{\omega_n}$ have the same system of laws?*

We shall solve the mentioned problem for an infinite set of primes $\omega$. An important step towards this task is the theorem on the $\mathfrak{G}$-separability of the lattice of all $\tau$-closed $n$-multiply $\omega$-composition formations of finite groups established in the previous section; see Theorem 4.41.

For every term $\xi$ of signature $\{\cap, \bigvee^{\tau}_{\omega_n}\}$ we write $\overline{\xi}$ to denote the term of signature $\{\cap, \bigvee^{\tau}_{\omega_{n-1}}\}$ obtained from $\xi$ by replacing of every symbol $\bigvee^{\tau}_{\omega_n}$ by the symbol $\bigvee^{\tau}_{\omega_{n-1}}$.

**Lemma 4.44** (Lemma 16 [137]). *Let $\xi(x_{i_1}, \ldots, x_{i_m})$ be a term of signature $\{\cap, \bigvee^{\tau}_{\omega_n}\}$ and let $f_i$ be an inner $c^{\tau}_{\omega_{n-1}}$-valued $\omega$-composition satellite of a formation $\mathfrak{F}_i$ where $i = 1, \ldots, m$ and $n \geqslant 1$. Then $\xi(\mathfrak{F}_1, \ldots, \mathfrak{F}_m) = CF_{\omega}(\overline{\xi}(f_1, \ldots, f_m))$.*

The following result is a special case of Theorem 6.5.

**Theorem 4.45** (Theorem 3.1. [129]). *Let $n$ be a positive integer. Then every law of the lattice of all $\tau$-closed formations $c_0^\tau$ is fulfilled in the lattice of all $\tau$-closed $n$-multiply $\omega$-composition formations $c_{\omega_n}^\tau$.*

**Corollary 4.46** (Corollary 2.5 [138]). *The lattice of all $\tau$-closed $n$-multiply $\omega$-composition formations of finite groups $c_{\omega_n}^\tau$ is modular, but not distributive for any nonnegative integer $n$.*

*Proof.* We apply Theorem 4.45 and proceed as in the proof of Corollary 3.14. □

**Theorem 4.47** (Theorem 3.3. [129]). *Let $n$ be a positive integer. If $\omega$ is an infinite set, then the law system of the lattice of all $\tau$-closed formations of finite groups $c_0^\tau$ coinsides with the law system of the lattice of all $\tau$-closed $n$-multiply $\omega$-composition formations of finite groups $c_{\omega_n}^\tau$.*

*Proof.* Fix a law

$$\xi_1(x_{i_1}, \ldots, x_{i_a}) = \xi_2(x_{j_1}, \ldots, x_{j_b}) \tag{4.1}$$

of signature $\{\cap, \bigvee_{\omega_n}^\tau\}$. Let

$$\overline{\xi}_1(x_{i_1}, \ldots, x_{i_a}) = \overline{\xi}_2(x_{j_1}, \ldots, x_{j_b}) \tag{4.2}$$

be the same law of signature $\{\cap, \bigvee_{\omega_{n-1}}^\tau\}$.

Assume that law (4.1) is true in the lattice $c_{\omega_n}^\tau$. We shall show that law (4.2) is true in the lattice $c_{\omega_{n-1}}^\tau$. By Lemma 4.38 and Theorem 4.41, it suffices to prove that if $\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a}; \mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b}$ are arbitrary one-generated $\tau$-closed $(n-1)$-multiply $\omega$-composition formations, then $\overline{\xi}_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a}) = \overline{\xi}_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b})$. Let

$$\mathfrak{F}_{i_1} = c^{\tau}_{\omega_{n-1}}\text{form } A_{i_1}, \dots, \mathfrak{F}_{i_a} = c^{\tau}_{\omega_{n-1}}\text{form } A_{i_a},$$

$$\mathfrak{F}_{j_1} = c^{\tau}_{\omega_{n-1}}\text{form } A_{j_1}, \dots, \mathfrak{F}_{j_b} = c^{\tau}_{\omega_{n-1}}\text{form } A_{j_b}.$$

We choose prime $p \in \omega$ such that $p \notin \pi(A_{i_1}, \dots, A_{i_a}; A_{j_1}, \dots, A_{j_b})$.

Let $B_{i_1} = Z_p \wr A_{i_1}, \dots, B_{i_a} = Z_p \wr A_{i_a}$; $B_{j_1} = Z_p \wr A_{j_1}, \dots, B_{j_b} = Z_p \wr A_{j_b}$ where $Z_p$ is a group of order $p$. Since formations $\mathfrak{M}_{i_1} = c^{\tau}_{\omega_n}\text{form } B_{i_1}, \dots, \mathfrak{M}_{i_a} = c^{\tau}_{\omega_n}\text{form } B_{i_a}$; $\mathfrak{M}_{j_1} = c^{\tau}_{\omega_n}\text{form } B_{j_1}, \dots, \mathfrak{M}_{j_b} = c^{\tau}_{\omega_n}\text{form } B_{j_b}$ belong to the lattice $c^{\tau}_{\omega_n}$, then $\mathfrak{F} = \mathfrak{H}$ where $\mathfrak{F} = \xi_1(\mathfrak{M}_{i_1}, \dots, \mathfrak{M}_{i_a})$ and $\mathfrak{H} = \xi_2(\mathfrak{M}_{j_1}, \dots, \mathfrak{M}_{j_b})$. Let $f_{i_c}$ be the minimal $c^{\tau}_{\omega_{n-1}}$-valued $\omega$-composition satellite of $\mathfrak{M}_{i_c}$ (where $c = 1, \dots, a$); $f_{j_d}$ be the minimal $c^{\tau}_{\omega_{n-1}}$-valued $\omega$-composition satellite of $\mathfrak{M}_{j_d}$ (where $d = 1, \dots, b$). By Lemma 6.4,

$$\xi_1(\mathfrak{M}_{i_1}, \dots, \mathfrak{M}_{i_a}) = CF_{\omega}(\overline{\xi}_1(f_{i_1}, \dots, f_{i_a}));$$

$$\xi_2(\mathfrak{M}_{j_1}, \dots, \mathfrak{M}_{j_b}) = CF_{\omega}(\overline{\xi}_2(f_{j_1}, \dots, f_{j_b})).$$

Let $f$ and $h$ be the minimal $c^{\tau}_{\omega_{n-1}}$-valued $\omega$-composition satellites of $\mathfrak{F}$ and $\mathfrak{H}$, respectively. Then by Lemmas 4.10 and 4.11,

$$f(p) = \overline{\xi}_1(f_{i_1}, \dots, f_{i_a})(p) = \overline{\xi}_1(f_{i_1}(p), \dots, f_{i_a}(p)),$$

$$h(p) = \overline{\xi}_2(f_{j_1}, \dots, f_{j_b})(p) = \overline{\xi}_2(f_{j_1}(p), \dots, f_{j_b}(p)).$$

Hence $\overline{\xi}_1(f_{i_1}(p), \dots, f_{i_a}(p)) = \overline{\xi}_2(f_{j_1}(p), \dots, f_{j_b}(p))$. Since $O_p(A_{i_c}) = 1$, then by Lemma 4.10 $f_{i_c}(p) = \mathfrak{F}_{i_c}$ where $c = 1, \dots, a$. Analogously $f_{j_d}(p) = \mathfrak{F}_{j_d}$ where $d = 1, \dots, b$. It follows that $\overline{\xi}_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a}) = \overline{\xi}_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b})$, i.e., the law (4.2) is true in the lattice $c^{\tau}_{\omega_{n-1}}$. Thus every law of the lattice $c^{\tau}_{\omega_n}$ is true in the lattice of all $\tau$-closed formations $c^{\tau}_{0}$. Applying Theorem 4.45, we complete the proof. □

**Corollary 4.48.** *Let $\omega$ be an infinite set. Let $m$ and $n$ be nonnegative integers. Then the law systems of lattices $c^{\tau}_{\omega_m}$ and $c^{\tau}_{\omega_n}$ coincide.*

*Proof.* Assume that a law is true in the lattice $c^{\tau}_{\omega_n}$. Applying Theorem 4.47, we see that the law is true in $c^{\tau}_{0}$. Thus, using Theorem 4.45, we conclude that the law is true in $c^{\tau}_{\omega_m}$. □

*Remark* 4.49. Let $m$ and $n$ be nonnegative integers. We note that Vedernikov [103] showed that the law system of the lattice of all $m$-multiply canonical formations coinsides with the law system of the lattice of the lattice of all $n$-multiply canonical formations.

## 4.5 Comments & further research

Let us take attention to some open questions related to lattices of composition formations. Note that some of them are analogues of the corresponding problems in [107, 113, 114, 101, 100].

**Question 4.5.** *Does it true that for all nonnegative integers $n$, the lattice of all $n$-multiply composition formations and the lattice of all $\tau$-closed $n$-multiply composition formations have the same system of laws?*

**Question 4.6.** *Does it true that the lattices $c_\infty$ and $c_\infty^\tau$ have the same system of laws?*

Theorem 4.45 gives a motivation to the following two questions.

**Question 4.7.** *Does it true that every law of the lattice of all $\tau$-closed formations $c_0^\tau$ is fulfilled in the lattice of all $\tau$-closed totally $\omega$-composition formations $c_{\omega_\infty}^\tau$?*

**Question 4.8.** *Let $m$ and $n$ be nonnegative integers with $m > n$. Does it true that the lattice of all $\tau$-closed $m$-multiply $\omega$-composition formations is not a sublattice of the lattice of all $\tau$-closed $n$-multiply $\omega$-composition formations?*

Note that the answer to an analogue of the question above is positive for the lattices of all $\tau$-closed multiply $\omega$-saturated formations (see [101]). However in [115] it was shown that the lattice of all saturated formations is a complete sublattice of the lattice of all composition formations. Safonov proved that the lattice of all $\tau$-closed totally saturated formations is a complete sublattice of the lattice of totally saturated formations (see [92]).

In [136] it was established that the lattice of all soluble totally saturated formations is algebraic and distributive. Independently Reifferscheid solved the problem of distributivity of the lattice of all soluble totally saturated formations (see

[82]). Safonov proved that the lattice of all $\tau$-closed totally saturated formations is $\mathfrak{G}$-separated [90], algebraic [87] and modular [84, 85] (moreover this lattice is distributive [86]). However the following questions are still open now.

**Question 4.9.** *Is it true that the lattice of all $\tau$-closed totally $\omega$-composition formations is $\mathfrak{G}$-separated?*

**Question 4.10.** *Is it true that the lattice of all $\tau$-closed totally $\omega$-composition formations is distributive (or modular at least)?*

## Conclusion

New series of inductive, separated, algebraic and modular lattices of multiply composition formations are described. Laws of lattices of functor-closed multiply $\omega$-composition formations of finite groups are studied. The contributions have been published in the papers [117, 121, 127, 128, 129].

# Chapter 5

# Lattices of $\mathfrak{X}$-Local Formations of Finite Groups

By an inclusion $\subseteq$, the set of all $\mathfrak{X}$-local formations of finite groups forms a complete lattice. Let $\mathfrak{X}' \neq \emptyset$. Then we write $f(\mathfrak{X}')$ to denote the common value of $f$ at the $\mathfrak{X}'$-groups. In the present chapter, the notation $\mathfrak{F} = LF_{\mathfrak{X}}(f)$ always means that $f$ is an $\mathfrak{X}$-local satellite of the formation of finnite groups $\mathfrak{F}$.

## 5.1 Algebraic lattices of $\mathfrak{X}$-local formations

**Theorem 5.1** (Theorem [123]). *The lattice of all $\mathfrak{X}$-local formations is algebraic.*

**Lemma 5.2** (Lemma 4.11 [9], Remark 3.1.7 [12]). *Let $f$ and $f_i$ be $\mathfrak{X}$-local satellites for all $i \in I$. Then $\bigcap_{i\in I} LF_{\mathfrak{X}}(f_i) = LF_{\mathfrak{X}}(g)$, where $g(x) = \cap_{i\in I} f_i(x)$ for all*

$x \in (\mathrm{char}\,\mathfrak{X}) \cup \mathfrak{X}'$.

We write $\mathrm{form}_{\mathfrak{X}}\,\mathfrak{Y}$ to denote the intersection of all $\mathfrak{X}$-local formations of finite groups containing a set of finite groups $\mathfrak{Y}$. When $\mathfrak{Y} = \{G\}$, we have a *one-generated* $\mathfrak{X}$-local formation: $\mathrm{form}_{\mathfrak{X}}\,G$. Following [12], we write $\mathcal{K}(G)$ to denote the class all of simple groups isomorphic to composition factors of a group $G$; and by $\mathrm{E}\mathfrak{X}$ it is denoted the class of all finite groups $G$ such that $\mathcal{K}(G) \subseteq \mathfrak{X}$. Let $\mathfrak{Y}$ be a class of finite groups. Then we put:

1. $\mathfrak{Y}(p) = \mathrm{form}(G/G_{cp} \mid G \in \mathfrak{Y}$ and $Z_p \in \mathcal{K}(G))$ if $p \in \pi(\mathfrak{X})$;
2. $\mathfrak{Y}(\mathfrak{X}') = \mathrm{form}(G/G_{\mathrm{E}\mathfrak{X}} \mid G \in \mathfrak{Y}$ and $G \neq G_{\mathrm{E}\mathfrak{X}})$ if $S \in \mathfrak{X} \neq \emptyset$.

*Remark* 5.3 (p. 128 [12]). By Lemma 5.2 every $\mathfrak{X}$-local formation $\mathfrak{F}$ possesses the unique $\mathfrak{X}$-local satellite $\underline{f}$, called *minimal $\mathfrak{X}$-local satellite* of formation $\mathfrak{F}$, such that $\underline{f} \leq f$ for every $\mathfrak{X}$-formation satellite $f$ with $\mathfrak{F} = LF_{\mathfrak{X}}(f)$.

**Lemma 5.4** (Lemma 4.14 [9], Theorem 3.1.11 [12]). *Let $\mathfrak{Y}$ be a class of finite groups. Then we have $\mathfrak{F} = \mathrm{form}_{\mathfrak{X}}(\mathfrak{Y}) = LF_{\mathfrak{X}}(\underline{f})$, where*

$\underline{f}(p) = \mathfrak{Y}(p)$, *if $p \in \pi(\mathfrak{X})$; and*

$\underline{f}(\mathfrak{X}') = \mathfrak{Y}(\mathfrak{X}')$, *if $\mathfrak{X}' \neq \emptyset$.*

**Definition 5.5** ([12]). Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation of finite groups with an $\mathfrak{X}$-local satellite $f$. Then $f$ is said to be *integrated* if $f(x) \subseteq \mathfrak{F}$ for any $x \in \pi(\mathfrak{X}) \cup \mathfrak{X}'$, and *full* if $\mathfrak{N}_p f(p) = f(p)$ for any $p \in \pi(\mathfrak{X})$.

**Lemma 5.6** (Lemma 2.2 [9]). *Let $\mathfrak{X}$ be a nonempty class of simple groups. If a formation $\mathfrak{F}$ has an $\mathfrak{X}$-local satellite, then it has an integrated $\mathfrak{X}$-local satellite.*

For a nonempty class of simple groups $\mathfrak{L}$, we write $\mathfrak{E}_{c\mathfrak{L}}$ to denote the class of finite groups whose chief E$\mathfrak{L}$-factors are central. Note that the class $\mathfrak{E}_{c\mathfrak{L}}$ is a Fitting formation. Let $G$ be a finite group. By $G_{c\mathfrak{L}}$ is denoted the$\mathfrak{E}_{c\mathfrak{L}}$-radical of $G$, and the symbol $G_{c\mathfrak{L}}$ denotes the $\mathfrak{E}_{c\mathfrak{L}}$-radical of $G$. Let $\mathfrak{L} = (Z_p)$. In this case we have $\mathfrak{E}_{c\mathfrak{L}} = \mathfrak{E}_{cp}$, $G_{c\mathfrak{L}} = G_{cp} = C^p(G)$; see p. 371 in [37] for more details.

Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation of finite groups. The following lemma describes a particular full and integrated $\mathfrak{X}$-local satellite of $\mathfrak{F}$. It is called *canonical $\mathfrak{X}$-local satellite* of formation $\mathfrak{F}$.

**Lemma 5.7** (Lemmas 4.7, 4.10 and 4.13 [9]). *Let $\mathfrak{X} \neq \emptyset$ be class of simple groups. Let $\mathfrak{F}$ be a formation with an $\mathfrak{X}$-local satellite. Then $\mathfrak{F}$ also possesses the unique integrated $\mathfrak{X}$-local satellite $F$ such that $F(\mathfrak{X}') = \mathfrak{F}$ if $\mathfrak{X}' \neq \emptyset$ and $F(p) = \mathfrak{N}_p F(p)$ for any $p \in \pi(\mathfrak{X})$. In addition, for each integrated $\mathfrak{X}$-local satellite $f$ of $\mathfrak{F}$ we have $f \leq F$, and*

$$\mathfrak{N}_p f(p) = F(p) = \mathfrak{N}_p \mathrm{form}(G/C^p(G) \mid G \in \mathfrak{F},\, Z_p \in \mathcal{K}(G))$$

*for any $p \in \pi(\mathfrak{X})$.*

Let $f_1$ and $f_2$ be some $\mathfrak{X}$-local satellites. We write $f_1 \leq f_2$ if $f_1(x) \subseteq f_2(x)$ for all $x \in (\mathrm{char}\,\mathfrak{X}) \cup \mathfrak{X}'$. In this case, we have $LF_{\mathfrak{X}}(f_1) \subseteq LF_{\mathfrak{X}}(f_2)$.

**Lemma 5.8** (Corollary 3.1.20 [12]). *Let $\mathfrak{F} = LF_{\mathfrak{X}}(\underline{f}) = LF_{\mathfrak{X}}(F)$ and $\mathfrak{H} = LF_{\mathfrak{X}}(\underline{h}) = LF_{\mathfrak{X}}(H)$ be $\mathfrak{X}$-local formations of finite groups. Then every two of the following statements are equivalent:*

1. $\mathfrak{F} \subseteq \mathfrak{H}$;
2. $F \leq H$;
3. $\underline{f} \leq \underline{h}$.

**Lemma 5.9** (Lemma 2.1 [123]). *Let $\mathfrak{F}_j = LF_{\mathfrak{X}}(f_j)$, where $f_j$ is an integrated $\mathfrak{X}$-local satellite of formation $\mathfrak{F}_j$, such that $f_j(\mathfrak{X}') = \mathfrak{F}_j$ if $\mathfrak{X}' \neq \emptyset$; $j = 1, 2$. If*

$$\mathfrak{F} = \mathrm{form}_{\mathfrak{X}}(\mathfrak{F}_1 \cup \mathfrak{F}_2),$$

*then $\mathfrak{F} = LF_{\mathfrak{X}}(g)$, where $g(p) = \mathrm{form}(f_1(p) \cup f_2(p))$ for each $p \in \pi(\mathfrak{X})$, and $g(\mathfrak{X}') = \mathrm{form}(f_1(\mathfrak{X}') \cup f_2(\mathfrak{X}'))$ if $\mathfrak{X}' \neq \emptyset$.*

*Proof.* Let $\mathfrak{M} = LF_{\mathfrak{X}}(g)$ and $\mathfrak{F} = LF_{\mathfrak{X}}(\underline{f})$. We shall show that the following equality holds: $\mathfrak{F} = \mathfrak{M}$.

Let $\underline{f}_j$ be the minimal $\mathfrak{X}$-local satellite of the formation $\mathfrak{F}_j$ for $j = 1, 2$. Applying Lemmas 5.8 and 5.7 for every $p \in \pi(\mathfrak{X})$, we obtain

$$\underline{f}_j(p) \subseteq f_j(p) \subseteq \mathfrak{N}_p \underline{f}_j(p) = F_j(p),$$

$$\underline{f}_j(\mathfrak{X}') \subseteq f_j(\mathfrak{X}') \subseteq F_j(\mathfrak{X}') = \mathfrak{F}_j,$$

where $F_j$ is the canonical $\mathfrak{X}$-local satellite of $\mathfrak{F}_j$.

Let $\mathfrak{F} = LF_{\mathfrak{X}}(F)$, where $F$ is the canonical $\mathfrak{X}$-local satellite of the formation $\mathfrak{F}$. Applying Lemma 5.4, we have for $p \in \pi(\mathfrak{X})$ the following inclusion:

$$f(p) = \mathrm{form}((\mathfrak{F}_1 \cup \mathfrak{F}_2)(p)) =$$

$$\mathrm{form}(\mathfrak{F}_1(p) \cup \mathfrak{F}_2(p)) = \mathrm{form}(f_1(p) \cup f_2(p)) \subseteq$$

$$g(p) \subseteq \mathfrak{N}_p\mathrm{form}(f_1(p) \cup f_2(p)) = \mathfrak{N}_p f(p) = F(p).$$

Consequently, $f(p) \subseteq g(p) \subseteq F(p)$ if $p \in \pi(\mathfrak{X})$, and $f_j(\mathfrak{X}') = \mathfrak{F}_j$ if $\mathfrak{X}' \neq \emptyset$, $j = 1, 2$, implies the following inclusion: $f(\mathfrak{X}') \subseteq g(\mathfrak{X}') \subseteq F(\mathfrak{X}')$. Thus, $f \leq g \leq F$. Finally, $\mathfrak{F} = \mathfrak{M}$.

□

**Lemma 5.10** (Lemma 2.2 [123]). *Let $\{\mathfrak{F}_i = LF_{\mathfrak{X}}(f_i) \mid i \in I\}$ be a set of $\mathfrak{X}$-local formations of finite groups, where $f_i$ is an integrated $\mathfrak{X}$-local satellite of $\mathfrak{F}_i$. Let $\mathfrak{F} = \bigvee_{\mathfrak{X}}(\mathfrak{F}_i \mid i \in I)$. Then $\mathfrak{F} = LF_{\mathfrak{X}}(g)$, where $g(x) = \vee(f_i(x) \mid i \in I)$ for all $x \in (\mathrm{char}\,\mathfrak{X}) \cup \mathfrak{X}'$.*

*Proof.* Let $\mathfrak{M} = LF_{\mathfrak{X}}(\vee(f_i \mid i \in I))$ and $\mathfrak{F} = LF_{\mathfrak{X}}(f)$. We shall proceed by induction on $i$ to show that the following equality holds: $\mathfrak{F} = \mathfrak{M}$

**Step 1.** Let $i = 2$. Then applying Lemma 5.9, we immediately obtain the required equality: $\mathfrak{F}_1 \bigvee_{\mathfrak{X}} \mathfrak{F}_2 = LF_{\mathfrak{X}}(f_1 \vee f_2)$.

**Step 2.** Let $i > 2$, and the assertion is true for $i = r - 1$ by induction, i.e., it holds $\mathfrak{F}_1 \bigvee_{\mathfrak{X}} ... \bigvee_{\mathfrak{X}} \mathfrak{F}_{r-1} = LF_{\mathfrak{X}}(f_1 \vee ... \vee f_{r-1})$. Using Step 1, we see that

$\mathfrak{F} = \mathrm{form}_{\mathfrak{X}}((\mathfrak{F}_1 \bigvee_{\mathfrak{X}} ... \bigvee_{\mathfrak{X}} \mathfrak{F}_{r-1}) \cup \mathfrak{F}_r) = LF_{\mathfrak{X}}(g)$, where

$$g(x) = \mathrm{form}((f_1(x) \vee ... \vee f_{r-1}(x)) \cup f_r(x)) =$$

$$f_1(x) \vee ... \vee f_r(x) = (f_1 \vee ... \vee f_r)(x)$$

for all $x \in (\mathrm{char}\,\mathfrak{X}) \cup \mathfrak{X}'$. Thus, $g = f_1 \vee ... \vee f_r$. This proves the lemma. □

**Lemma 5.11** (Lemma 3.1 [123]). *Each one-generated formation $\mathfrak{F} = \mathrm{form}\, G$ is a compact element in the lattice of all formations of finite groups.*

*Proof of Theorem 5.1.* **Step 1.** It will be shown first that in the lattice of all $\mathfrak{X}$-local formations of finite groups any nonempty $\mathfrak{X}$-local formation $\mathfrak{F}$ is the join of its one-generated $\mathfrak{X}$-local subformations $\mathfrak{F}_i = \mathrm{form}_{\mathfrak{X}}\, G_i$, where $i \in I$. Let $\mathfrak{Y} = \mathrm{form}_{\mathfrak{X}}(\cup_{i\in I}\mathfrak{F}_i)$. Let us show that the following equality holds: $\mathfrak{F} = \mathfrak{Y}$. Assume $G \in \mathfrak{F}$. Then we see that

$$G \in \mathrm{form}_{\mathfrak{X}}\, G \subseteq \bigcup_{i\in I} \mathfrak{F}_i \subseteq \mathrm{form}_{\mathfrak{X}}(\bigcup_{i\in I} \mathfrak{F}_i) = \mathfrak{Y}.$$

So, $\mathfrak{F} \subseteq \mathfrak{Y}$. It is easy to see that the inverse inclusion is trivial: $\mathfrak{F}_i \subseteq \mathfrak{F}$ implies $\cup_{i\in I}\mathfrak{F}_i \subseteq \mathfrak{F}$,. Thus, $\mathfrak{Y} \subseteq \mathfrak{F}$, and we obtain the required equality $\mathfrak{F} = \mathfrak{Y}$.

**Step 2.** We are showing now that every one-generated $\mathfrak{X}$-local formation $\mathfrak{H}$ is a compact element in the lattice of all $\mathfrak{X}$-local formations of finite groups.

Let

$$\mathfrak{H} = \mathrm{form}_{\mathfrak{X}}\, G \subseteq \mathfrak{M} = \mathrm{form}_{\mathfrak{X}}(\bigcup_{i\in I} \mathfrak{H}_i),$$

where $\mathfrak{H}_i$ is an $\mathfrak{X}$-local formation, $i \in I$. Let $\underline{h}_i$ be the minimal $\mathfrak{X}$-local satellite of $\mathfrak{H}_i$, $\underline{h}$ be the minimal $\mathfrak{X}$-local satellite of $\mathfrak{H}$ and let $\underline{m}$ be the minimal $\mathfrak{X}$-local satellite of formation $\mathfrak{M}$. Applying Lemma 5.4, we see that

$$\underline{h}(p) = \mathrm{form}(G/C^p(G) \mid Z_p \in \mathcal{K}(G)), \text{ if } p \in \pi(\mathfrak{X});$$

$$\underline{h}(\mathfrak{X}') = \mathrm{form}(G/G_{\mathrm{E}\mathfrak{X}} \mid G \neq G_{\mathrm{E}\mathfrak{X}}), \quad \text{if } \mathfrak{X}' \neq \emptyset.$$

Lemmas 5.8 and 5.10 imply $\underline{h} \leq \underline{m} \leq \vee(\underline{h}_i \mid i \in I)$.

By Remark 4.4.4 in [107] the lattice of all formations of finite groups is algebraic. Then using Lemma 5.11 we have $i_1, \ldots, i_k, j_1, \ldots, j_l \in I$ such that

$$G/C^p(G) \in \mathrm{form}(\underline{h}_{i_1}(p) \cup \cdots \cup \underline{h}_{i_k}(p)), \text{ if } p \in \pi(\mathfrak{X});$$

$$G/G_{\mathrm{E}\mathfrak{X}} \in \mathrm{form}(\underline{h}_{j_1}(\mathfrak{X}') \cup \cdots \cup \underline{h}_{j_l}(\mathfrak{X}')), \text{ if } \mathfrak{X}' \neq \emptyset.$$

Put $\{r_1, \ldots, r_t\} = \{i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_l\}$.

Thus, $\mathfrak{H}_{r_1} \bigvee_{\mathfrak{X}} \cdots \bigvee_{\mathfrak{X}} \mathfrak{H}_{r_t} = LF_{\mathfrak{X}}(\underline{h}_{r_1} \vee \cdots \vee \underline{h}_{r_t})$. So, we obtain

$$G/C^p(G) \in \underline{h}(p) \subseteq \mathrm{form}(\underline{h}_{r_1}(p) \cup \cdots \cup \underline{h}_{r_t}(p)),$$

$$G/G_{\mathrm{E}\mathfrak{X}} \in \underline{h}(S) \subseteq \mathrm{form}(\underline{h}_{r_1}(\mathfrak{X}') \cup \cdots \cup \underline{h}_{r_t}(\mathfrak{X}')).$$

Consequently, $G \in \mathrm{form}_{\mathfrak{X}}(\mathfrak{H}_{r_1} \cup \cdots \cup \mathfrak{H}_{r_t})$. Thus, $\mathfrak{H} = \mathrm{form}_{\mathfrak{X}}\, G \subseteq \mathfrak{H}_{r_1} \bigvee_{\mathfrak{X}} \cdots \bigvee_{\mathfrak{X}} \mathfrak{H}_{r_t}$.

□

**Corollary 5.12** (Theorem 4 [114]). *The lattice of all composition formations of finite groups is algebraic.*

## 5.2 Modular lattices of $\mathfrak{X}$-local formations

**Lemma 5.13** ([107]). *The lattice of all formations of finite groups is modular but not distributive.*

**Theorem 5.14** (Theorem 3.1. [126]). *The lattice of all $\mathfrak{X}$-local formations of finite groups is modular.*

*Proof.* Let $\mathfrak{F}_i = LF_{\mathfrak{X}}(f_i)$, where $i = 1, 2, 3$ and $\mathfrak{F}_2 \subseteq \mathfrak{F}_1$. We shall show that $\mathfrak{F} = \mathfrak{F}_1 \cap (\mathfrak{F}_2 \bigvee_{\mathfrak{X}} \mathfrak{F}_3) = \mathfrak{F}_2 \bigvee_{\mathfrak{X}} (\mathfrak{F}_1 \cap \mathfrak{F}_3) = \mathfrak{M}$.

Lemmas 5.6 and 5.7 imply $\mathfrak{F}_i = LF_{\mathfrak{X}}(F_i)$ for $i = 1, 2, 3$, where $F_i(\mathfrak{X}') = \mathfrak{F}_i$ and

$$F_i(p) = \mathrm{form}(G/G_{cp} \mid G \in \mathfrak{F}_i,\, Z_p \in \mathcal{K}(G))$$

for all $p \in \pi(\mathfrak{X})$.

Denote $g = F_2 \vee F_3$. Then by Lemma 5.10 we have $\mathfrak{F}_2 \bigvee_{\mathfrak{X}} \mathfrak{F}_3 = LF_{\mathfrak{X}}(g)$.

Denote $h = F_1 \cap g$. Applying Lemma 5.2 we obtain $\mathfrak{F} = LF_{\mathfrak{X}}(h)$.

Note that $F_2 \leq F_1$ by Lemma 5.8. Then Lemma 5.13 implies for all $p \in \pi(\mathfrak{X})$

$$F_2(p) \vee (F_1(p) \cap F_3(p)) = F_1(p) \cap (F_2(p) \vee F_3(p)),$$

$$F_2(\mathfrak{X}') \vee (F_1(\mathfrak{X}') \cap F_3(\mathfrak{X}')) = F_1(\mathfrak{X}') \cap (F_2(\mathfrak{X}') \vee F_3(\mathfrak{X}')).$$

Consequently for all $x \in \pi(\mathfrak{X}) \cup \mathfrak{X}'$ we have the equality

$$h(x) = F_2(x) \vee (F_1(x) \cap F_3(x)).$$

Note that $F_1 \cap F_3$ is integrated. Thus Lemma 5.10 implies $\mathfrak{M} = LF_{\mathfrak{X}}(h)$. $\square$

Let $\mathfrak{F}$ and $\mathfrak{H}$ be $\mathfrak{X}$-local formations of finite groups such that the inclusion $\mathfrak{H} \subseteq \mathfrak{F}$ holds. We write $\mathfrak{F}/_{\mathfrak{X}}\mathfrak{H}$ to denote the lattice of all $\mathfrak{X}$-local formations of finite groups $\mathfrak{M}$ such that $\mathfrak{H} \subseteq \mathfrak{M} \subseteq \mathfrak{F}$.

**Corollary 5.15.** *For any $\mathfrak{X}$-local formations $\mathfrak{M}$ and $\mathfrak{F}$ of finite groups, the following lattice isomorphism holds:* $(\mathfrak{M} \bigvee_{\mathfrak{X}} \mathfrak{F})/_{\mathfrak{X}}\mathfrak{M} \simeq \mathfrak{F}/_{\mathfrak{X}}(\mathfrak{F} \cap \mathfrak{M})$.

**Corollary 5.16** (Corollary 4.2.8 [107]). *The lattice of all local formations of finite groups is modular.*

**Corollary 5.17** (Theorem 4 [114]). *The lattice of all composition formations of finite groups is modular.*

**Lemma 5.18** (Proposition 3.1.39 [12]). *Let $\mathfrak{F} = LF_{\mathfrak{X}}(f)$ be an $\mathfrak{X}$-local formation and let $c$ be one of the closure operations $s_n$ or $N_0$. If $f(x) = cf(x)$ for all $x \in$* $(\mathrm{char}\,\mathfrak{X}) \cup \mathfrak{X}'$, *then $\mathfrak{F} = c\mathfrak{F}$.*

Lemma 5.18 immediately implies the following corollary.

**Corollary 5.19.** *Let $c$ be one of the closure operations $s_n$ or $N_0$. Then the lattice of all $c$-closed $\mathfrak{X}$-local formations of finite groups is modular.*

**Corollary 5.20.** *The lattice of all $\mathfrak{X}$-local Fitting formations of finite groups is modular.*

**Lemma 5.21** (Corollary 5 [137]). *The lattice of all $\omega$-composition formations of finite groups is not distributive.*

**Lemma 5.22** ([97]). *Let $\mathfrak{X} \neq \emptyset$ be a class of simple groups such that $\omega = \pi(\mathfrak{X}) = \operatorname{char} \mathfrak{X}$. Then any $\mathfrak{X}$-local formation of finite groups is an $\omega$-composition formation of finite groups.*

Applying Lemmas 5.21 and 5.22 we obtain the following theorem.

**Theorem 5.23** (Theorem 4.1. [126]). *The lattice of all $\mathfrak{X}$-local formations of finite groups is not distributive.*

## Conclusion

Let $\mathfrak{X}$ be a class of simple groups with a completeness property $\pi(\mathfrak{X}) = \operatorname{char} \mathfrak{X}$. Förster introduced the concept of $\mathfrak{X}$-local formation in order to obtain a common extension of well-known theorems of Baer and Gaschütz–Lubeseder–Schmid. In the present chapter, it is proved that the lattice of all $\mathfrak{X}$-local formations of finite groups is algebraic and modular. The results have been published in the papers [123, 126].

# Chapter 6

# Lattices of Formations of Multioperator $T$-groups

## 6.1 Laws of the lattices of foliated formations of $T$-groups

**Definition 6.1** ([34]). Any $\tau$-closed $\mathfrak{M}$-formation is 0-multiply $\tau\Omega_1$-foliated with an arbitrary direction $\varphi$, by definition. Let $\tau\Omega_1 F_0^{\varphi}$ be the lattice of all $\tau\Omega_1$-formations. Given a sequence of the lattices of $\tau$-closed $\Omega_1$-foliated $\mathfrak{M}$-formations

$$\tau\Omega_1 F_1^{\varphi}, \dots, \tau\Omega_1 F_n^{\varphi}, \dots,$$

where $\tau\Omega_1 F_1^{\varphi}$ is the set of all $\tau$-closed $\Omega_1$-foliated $\mathfrak{M}$-formations possessing a direction $\varphi$ and a $\tau\Omega_1$-satellite. Let $n > 0$. Then $\tau\Omega_1 F_n^{\varphi}$ is the set of all *$\tau$-closed $n$-multiply $\Omega_1$-foliated $\mathfrak{M}$-formations*, i.e., for each $\mathfrak{F}$ in $\tau\Omega_1 F_n^{\varphi}$ there exists a $\tau\Omega_1 F_{n-1}^{\varphi}$-satellite of $\mathfrak{F}$.

*Remark* 6.2 (Theorem 1 [34]). The set $\tau\Omega_1 F_n^{\varphi}$ is a complete lattice in $\mathfrak{M}$ for any nonnegative integer $n$ and any direction $\varphi$.

Let $\mathfrak{Y} \neq \emptyset$ be a set of $\mathfrak{M}$-groups. We write $\tau\mathrm{form}\,\mathfrak{Y}$ to denote the intersection of all $\tau$-closed $\mathfrak{M}$-formations containing $\mathfrak{Y}$. The notion $\tau\Omega_1 F_n(\mathfrak{Y}, \varphi)$ means the intersection of all $\tau\Omega_1 F_n^{\varphi}$-formations containing $\mathfrak{Y}$. If $\mathfrak{Y} = \{G\}$, then we write $\tau\Omega_1 F_n(G, \varphi)$ to denote a *one-generated* $\tau\Omega_1 F_n(\mathfrak{Y}, \varphi)$*-formation*.

*Remark* 6.3 ([34]). Let $\mathfrak{L}, \mathfrak{H} \in \tau\Omega_1 F_n^{\varphi}$. Then $\mathfrak{L} \bigvee^{\tau}_{\Omega_1 F_n^{\varphi}} \mathfrak{H} = \tau\Omega_1 F_n^{\varphi}(\mathfrak{L} \cup \mathfrak{H})$ is the least upper bound for $\{\mathfrak{L}, \mathfrak{H}\}$ in $\tau\Omega_1 F_n^{\varphi}$, and $\mathfrak{L} \cap \mathfrak{H}$ is the greatest lower bound for $\{\mathfrak{L}, \mathfrak{H}\}$ in $\tau\Omega_1 F_n^{\varphi}$.

For every term $\xi$ of the signature $\{\cap, \vee^{\tau}_{\Omega_1 F_n^{\varphi}}\}$ we write $\overline{\xi}$ to denote the term of signature $\{\cap, \vee^{\tau}_{\Omega_1 F_{n-1}^{\varphi}}\}$ obtained from $\xi$ by replacing of each symbol $\vee^{\tau}_{\Omega_1 F_n^{\varphi}}$ by the symbol $\vee^{\tau}_{\Omega_1 F_{n-1}^{\varphi}}$.

**Lemma 6.4** (Lemma 3.1 [120]). *Let $\xi(x_1, \ldots, x_m)$ be a term of signature $\{\cap, \vee^{\tau}_{\Omega_1 F_n^{\varphi}}\}$ and let $f_i$ be the minimal $\tau\Omega_1 F_{n-1}^{\varphi}$-satellite of a $\mathfrak{M}$-formation $\mathfrak{F}_i$ $(i = 1, \ldots, m)$. Then for any positive integer $n$ it holds*

$$\xi(\mathfrak{F}_1, \ldots, \mathfrak{F}_m) = \Omega_1 F(\overline{\xi}(f_1, \ldots, f_m), \varphi).$$

*Proof.* Let $r$ be a number of occurrences of the symbols of $\{\cap, \vee^{\tau}_{\Omega_1 F_n^{\varphi}}\}$ in $\xi$. We proceed the proof by induction on $r$.

If $r = 1$, then the assertion follows by Lemma 5 in [34], and Lemma 8 in

[131]. Let $\xi$ has $r > 1$ occurrences of the symbols $\{\cap, \vee^{\tau}_{\Omega_1 F_n^{\varphi}}\}$. We put

$$\xi(x_1, \ldots, x_m) = \xi_1(x_{i_1}, \ldots, x_{i_a}) \triangle \xi_2(x_{j_1}, \ldots, x_{j_b}),$$

where $\triangle \in \{\cap, \vee^{\tau}_{\Omega_1 F_n^{\varphi}}\}$ and $\{x_{i_1}, \ldots, x_{i_a}\} \cup \{x_{j_1}, \ldots, x_{j_b}\} = \{x_1, \ldots, x_m\}$. Suppose that the assertion holds for $\xi_1$ and $\xi_2$. Thus,

$$\xi_1(\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a}) = \Omega_1 F(\bar{\xi}(f_{i_1}, \ldots, f_{i_a}), \varphi)$$

and

$$\xi_1(\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b}) = \Omega_1 F(\bar{\xi}(f_{j_1}, \ldots, f_{j_b}), \varphi).$$

For every $A \in \Omega_1 \cup \{\Omega_1'\}$, we have

$$\bar{\xi}(f_{i_1}, \ldots, f_{i_a})(A) \subseteq \xi_1(\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a})$$

and

$$\bar{\xi}(f_{j_1}, \ldots, f_{j_b})(A) \subseteq \xi_1(\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b}).$$

Hence,

$$\xi(\mathfrak{F}_1, \ldots, \mathfrak{F}_m) = \xi_1(\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a}) \triangle \xi_1(\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b}) =$$

$$\Omega_1 F(\bar{\xi}(f_{i_1}, \ldots, f_{i_a}) \bar{\triangle} \bar{\xi}(f_{j_1}, \ldots, f_{j_b}), \varphi) = \Omega_1 F(\bar{\xi}(f_1, \ldots, f_m), \varphi),$$

where $\bar{\triangle} = \cap$ if $\triangle = \cap$, and $\bar{\triangle} = \vee^{\tau}_{\Omega_1 F_{n-1}^{\varphi}}$ if $\triangle = \vee^{\tau}_{\Omega_1 F_n^{\varphi}}$. $\square$

**Theorem 6.5** (Theorem 1.1 [120]). *Let $\mathfrak{M}$ be the class of all $T$-groups satisfying the minimality and maximality conditions for multioperator $T$-subgroups, and let $n > 0$. Then every law of the lattice of all $\tau$-closed $\mathfrak{M}$-formations (denoted by $\tau\Omega_1 F_0^{\varphi}$) is* *fulfilled in the lattice* $\tau\Omega_1 F_n^{\varphi}$ *(of all* $\tau$*-closed* $n$*-multiply* $\Omega_1$*-foliated* $\mathfrak{M}$*-formations with direction* $\varphi$*, such that* $\varphi_0 \leqslant \varphi$*).*

*Proof.* We fix a law

$$\xi_1(x_{i_1}, \dots, x_{i_a}) = \xi_2(x_{j_1}, \dots, x_{j_b}) \tag{6.1}$$

of signature $\{\cap, \vee^{\tau}_{\Omega_1 F_n^{\varphi}}\}$. Let

$$\overline{\xi}_1(x_{i_1}, \dots, x_{i_a}) = \overline{\xi}_2(x_{j_1}, \dots, x_{j_b}) \tag{6.2}$$

be the same law of signature $\{\cap, \vee^{\tau}_{\Omega_1 F_{n-1}^{\varphi}}\}$. Suppose that law (6.2) is true in the lattice $\tau\Omega_1 F_{n-1}^{\varphi}$. Let $\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a}$ and $\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b}$ be arbitrary $n$-multiply $\tau\Omega_1$-foliated $\mathfrak{M}$-formations with direction $\varphi$, such that $\varphi_0 \leqslant \varphi$. We shall show that

$$\xi_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a}) = \xi_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b}).$$

Let $f_{i_c}$ be the minimal $\tau\Omega_1 F_{n-1}^{\varphi}$-satellite of the formation $\mathfrak{F}_{i_c}$ (where $c = 1, \dots, a$) and let $f_{j_d}$ be the minimal $\tau\Omega_1 F_{n-1}^{\varphi}$-satellite of $\mathfrak{F}_{j_d}$ (where $d = 1, \dots, b$). By Lemma 6.4, $\xi_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a}) = \Omega_1 F(h_1, \varphi)$ and $\xi_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b}) = \Omega_1 F(h_2, \varphi)$, where $h_1 = \overline{\xi}_1(f_{i_1}, \dots, f_{i_a})$ and $h_2 = \overline{\xi}_2(f_{j_1}, \dots, f_{j_b})$. By Lemma 4 from [34] for every $A \in \Omega_1 \cup \{\Omega_1'\}$, the formations $f_{i_1}(A), \dots, f_{i_a}(A)$ and $f_{j_1}(A), \dots, f_{j_b}(A)$ belong to the lattice $\tau\Omega_1 F_{n-1}^{\varphi}$. Then by induction, we conclude that

$$h_1(A) = \overline{\xi}_1(f_{i_1}(A), \dots, f_{i_a}(A)) = \overline{\xi}_2(f_{j_1}(A), \dots, f_{j_b}(A)) = h_2(A).$$

Hence, $\xi_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a}) = \xi_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b})$. Thus, law (6.1) is true in the lattice $\tau\Omega_1 F_n^{\varphi}$. □

**Corollary 6.6.** *Every law of the lattice of all $\tau$-closed $\mathfrak{M}$-formations is fulfilled in the lattice of all $\tau$-closed $n$-multiply $\Omega_1$-free $\mathfrak{M}$-formations $\tau\Omega_1 F_n^{\varphi_0}$.*

**Definition 6.7** ([34])**.** A formation $\mathfrak{F} = \Omega_1 F(f, \varphi)$ is called $\Omega_1$-*bicanonical* (and $\Omega_1$-*composition*, respectively) if $\varphi(A) = \mathfrak{C}_{A'}$ for any nonabelian $T$-group $A \in \mathfrak{I}_1$ and $\varphi(A) = \mathfrak{C}_{A'}\mathfrak{C}_A$ ($\varphi(A) = \mathfrak{S}_{cA}$) for any abelian $A \in \mathfrak{I}_1$. We write $\varphi_2$ and $\varphi_3$, respectively, to denote the directions of the mentioned formations.

**Corollary 6.8.** *Every law of the lattice of all $\tau$-closed $\mathfrak{M}$-formations is fulfilled in the lattice of all $\tau$-closed $n$-multiply $\Omega_1$-bicanonical $\mathfrak{M}$-formations $\tau\Omega_1 F_n^{\varphi_2}$.*

For the trivial subgroup $\mathfrak{C}$-functor $\tau$, we obtain the following corollary.

**Corollary 6.9.** *Every law of the lattice of all formations is fulfilled in the lattice $\Omega F_n^{\varphi}$ of all $n$-multiply $\Omega$-foliated formations with direction $\varphi$, such that $\varphi_0 \leqslant \varphi$.*

**Corollary 6.10.** *Every law of the lattice of all formations of finite multirings is fulfilled in the lattice of all $n$-multiply local formations of finite multirings.*

**Corollary 6.11** (Theorem 2 [34])**.** *Let $\mathfrak{M}$ be the class of all multioperator $T$-groups satisfying the minimality and maximality conditions for multioperator $T$-subgroups. Then the lattice $\tau\Omega_1 F_n^{\varphi}$ (of all $\tau$-closed $n$-multiply $\Omega_1$-foliated $\mathfrak{M}$-formations) is modular for any nonnegative integer $n$ and any direction $\varphi$, such that $\varphi_0 \leqslant \varphi$.*

*Proof.* By [131, Theorem 4.5) for $n = 0$], the lattice $\Omega_1 F_0^{\varphi}$ is modular. By Lemma 7 from [34] for any $\mathfrak{F}$, $\mathfrak{H} \in \tau\Omega_1 F_0^{\varphi}$, we have

$$\mathfrak{F} \vee^{\tau}_{\Omega_1 F_0^{\varphi}} \mathfrak{H} = \tau\Omega_1 F_0^{\varphi}(\mathfrak{F} \cup \mathfrak{H}) = \Omega_1 F_0^{\varphi}(\mathfrak{F} \cup \mathfrak{H}).$$

Thus, the lattice $\tau\Omega_1 F_0^{\varphi}$ is modular. Then applying Theorem 6.5, we conclude that the lattice $\tau\Omega_1 F_n^{\varphi}$ is modular for any direction $\varphi$, $\varphi_0 \leqslant \varphi$. □

Let $\mathfrak{F}$ and $\mathfrak{H}$ be a $\tau\Omega_1 F_n^{\varphi}$-formations such that $\mathfrak{H} \subseteq \mathfrak{F}$. We write $\mathfrak{F}/^{\tau}_{\Omega_1 F_n^{\varphi}}\mathfrak{H}$ to denote the lattice of all $\tau\Omega_1 F_n^{\varphi}$-formations $\mathfrak{Y}$ such that $\mathfrak{H} \subseteq \mathfrak{Y} \subseteq \mathfrak{F}$. As an immediate corollary from the lattice property of being modular, we have the corollary.

**Corollary 6.12** (Lemma 3.4 [122]). *For any two $\tau$-closed $n$-multiply $\Omega_1$-foliated $\mathfrak{M}$-formations $\mathfrak{H}$ and $\mathfrak{F}$ (with direction $\varphi$, such that $\varphi_0 \leqslant \varphi$) the lattices*

$$(\mathfrak{H} \vee^{\tau}_{\Omega_1 F_n^{\varphi}} \mathfrak{F})/^{\tau}_{\Omega_1 F_n^{\varphi}}\mathfrak{H} \text{ and } \mathfrak{F}/^{\tau}_{\Omega_1 F_n^{\varphi}}(\mathfrak{F} \cap \mathfrak{H})$$

*are isomorphic.*

## 6.2 Frattini subformations of foliated formations of $T$-groups

Let $\mathfrak{F}$ and $\mathfrak{H}$ be $\tau$-closed $n$-multiply $\Omega_1$-foliated $\mathfrak{M}$-formations with direction $\varphi$ such that $\varphi_0 \leqslant \varphi$, and $\mathfrak{H} \subseteq \mathfrak{F}$. We write $\mathfrak{F}/^{\tau}_{\Omega_1 F_n^{\varphi}}\mathfrak{H}$ to denote the lattice of all $\tau$-closed $n$-multiply $\Omega_1$-foliated $\mathfrak{M}$-formations of multioperator $T$-groups (with direction $\varphi$ such that $\varphi_0 \leqslant \varphi$) such that $\mathfrak{H} \subseteq \mathfrak{Y} \subseteq \mathfrak{F}$.

**Definition 6.13.** If $\mathfrak{Y} \subset \mathfrak{F}$ and the lattice $\mathfrak{F}/^{\tau}_{\Omega_1 F_n^{\varphi}}\mathfrak{Y}$ consists of only two elements then $\mathfrak{Y}$ is called a *maximal* $\tau$-closed $n$-multiply $\Omega_1$-foliated $\mathfrak{M}$-formation (with direction $\varphi$ such that $\varphi_0 \leqslant \varphi$) of $\mathfrak{F}$. Denote the intersection of all such subformations of $\mathfrak{F}$ by $\Phi^{\tau}_{\Omega_1 F_n^{\varphi}}(\mathfrak{F})$, and call it the *Frattini subformation* of $\mathfrak{F}$ (we assume that $\Phi^{\tau}_{\Omega_1 F_n^{\varphi}}(\mathfrak{F}) = \mathfrak{F}$ if there are no such subformations).

We write $f \leq h$, if $f(A) \subseteq h(A)$ whenever for all $A \in \Omega_1 \cup \{\Omega_1'\}$.

**Lemma 6.14** (Lemma 3.1 [122]). *Let $\{\mathfrak{F}_i \mid i \in I\}$ be a chain of $\tau\Omega_1 F_n^{\varphi}$-formations of multioperator $T$-groups, where $n \geq 1$, and let $\{f_i \mid i \in I\}$ be a chain of $\tau\Omega_1 F_{n-1}^{\varphi}$-satellites such that $\mathfrak{F}_i = \Omega_1 F(f_i, \varphi)$ and $f_i \leq f_j$ iff $\mathfrak{F}_i \subseteq \mathfrak{F}_j$ for all $i, j \in I$. Then*

$$\mathfrak{F} = \bigcup_{i \in I} \mathfrak{F}_i = \Omega_1 F(f, \varphi),$$

*where $f(A) = \bigcup_{i \in I} f_i(A)$ for any $A \in \Omega_1 \cup \{\Omega_1'\}$.*

**Lemma 6.15** (Kuratowski–Zorn). *Let a partially ordered set $\Lambda$ has the property that each chain in $\Lambda$ has an upper bound in $\Lambda$. Then the set $\Lambda$ contains at least one maximal element.*

**Definition 6.16** ([122]). An $\mathfrak{M}$-group $G$ is said to be a $\tau\Omega_1 F_n^{\varphi}$*-nongenerator* of the formation $\mathfrak{F}$

$$\text{if } \mathfrak{F} = \tau\Omega_1 F_n(\mathfrak{Y} \cup \{G\}, \varphi) \text{ always implies that } \mathfrak{F} = \tau\Omega_1 F_n(\mathfrak{Y}, \varphi),$$

where $\mathfrak{Y} \neq \emptyset$ is a set of $\mathfrak{M}$-groups.

**Lemma 6.17** (Lemma 3.3 [122]). *Let $G$ be an $\mathfrak{M}$-group and $\mathfrak{Y} \neq \emptyset$ be a set of $\mathfrak{M}$-groups. Then the formation*

$$\mathfrak{F} = \tau\Omega_1 F_n(\mathfrak{Y} \cup \{G\}, \varphi)$$

*contains a maximal $\tau$-closed $n$-multiply $\Omega_1$-foliated $\mathfrak{M}$-subformation (with direction $\varphi$ such that $\varphi_0 \leqslant \varphi$) containing $\tau\Omega_1 F_n(\mathfrak{Y}, \varphi) \neq \mathfrak{F}$ for any nonnegative integer $n$.*

*Proof.* Let $\Lambda$ be a partially ordered set of all $\tau$-closed $n$-multiply $\Omega_1$-foliated $\mathfrak{M}$-subformations (with direction $\varphi$ such that $\varphi_0 \leqslant \varphi$) of $\mathfrak{F}$ that contains

$$\tau\Omega_1 F_n(\mathfrak{Y}, \varphi)$$

but does not contain $G$, and let $\{\mathfrak{F}_i \mid i \in I\}$ be a chain from $\Lambda$.

Put $\mathfrak{H} = \bigcup_{i \in I} \mathfrak{F}_i$. Lemma 6.14 implies that $\mathfrak{H}$ is a $\tau$-closed $n$-multiply $\Omega_1$-foliated $\mathfrak{M}$-formation with direction $\varphi$ such that $\varphi_0 \leqslant \varphi$.

We see that $\tau\Omega_1 F_n(\mathfrak{Y}, \varphi) \subseteq \mathfrak{H}$ and $G \notin \mathfrak{H}$. Applying Lemma 6.15, we observe that every $\mathfrak{Y}$ in $\Lambda$ is contained in some maximal element from $\Lambda$. We shall show that any such formation $\mathfrak{Y}$ is a maximal $\tau$-closed $n$-multiply $\Omega_1$-foliated $\mathfrak{M}$-subformation (with direction $\varphi$ such that $\varphi_0 \leqslant \varphi$) of $\mathfrak{F}$.

Suppose that $\mathfrak{L}$ is a $\tau$-closed $n$-multiply $\Omega_1$-foliated $\mathfrak{M}$-formation (with direction $\varphi$ such that $\varphi_0 \leqslant \varphi$) with $\mathfrak{Y} \subset \mathfrak{L} \subset \mathfrak{F}$. We obtain $G \notin \mathfrak{L}$ since

$$\mathfrak{Y} \subseteq \tau\Omega_1 F_n(\mathfrak{Y}, \varphi) \subseteq \mathfrak{Y} \subset \mathfrak{L}.$$

Thus, $\mathfrak{L} \in \Lambda$. We have a contradiction. □

**Theorem 6.18** (Theorem 4.1 [122]). *Let $\mathfrak{F}$ be a nonempty $\tau\Omega_1 F_n^{\varphi}$-formation such that $\mathfrak{F} \neq (1)$, where $n$ is a positive integer. Then $\Phi^{\tau}_{\Omega_1 F_n^{\varphi}}(\mathfrak{F})$ consists of all $\tau\Omega_1 F_n^{\varphi}$-nongenerators of $\mathfrak{F}$.*

*Proof.* Let $G$ be a $\tau\Omega_1 F_n^{\varphi}$-nongenerator, and $\mathfrak{L}$ be a maximal $\tau\Omega_1 F_n^{\varphi}$-subformation of $\mathfrak{F}$. Assume that $G \notin \mathfrak{L}$. Then we have

$$\tau\Omega_1 F_n(\mathfrak{L} \cup \{G\}, \varphi) = \mathfrak{F} = \tau\Omega_1 F_n(\mathfrak{L}, \varphi) = \mathfrak{L},$$

which is a contradiction. Thus $G \in \mathfrak{L}$.

Let $\mathfrak{Y}$ be a nonempty set of $\mathfrak{M}$-groups contained in $\mathfrak{F}$ and $\Phi^{\tau}_{\Omega_1 F_n^{\varphi}}(\mathfrak{F})$. Assume that

$$\tau\Omega_1 F_n(\mathfrak{Y} \cup \{G\}, \varphi) = \mathfrak{F} \neq \tau\Omega_1 F_n(\mathfrak{Y}, \varphi).$$

By Lemma 6.17, $\mathfrak{F}$ contains a maximal $\tau\Omega_1 F_n^{\varphi}$-subformation $\mathfrak{Y}$ such that

$$\tau\Omega_1 F_n(\mathfrak{Y}, \varphi) \subseteq \mathfrak{Y}.$$

Thus, since $G \in \Phi^{\tau}_{\Omega_1 F_n^{\varphi}}(\mathfrak{F})$, we have $\mathfrak{Y} = \mathfrak{F}$. This contradicts the choices of $\mathfrak{Y}$. Consequently, $\mathfrak{F} = \tau\Omega_1 F_n(\mathfrak{Y}, \varphi)$. $\square$

**Theorem 6.19** (Theorem 4.2 [122]). *Let $\mathfrak{F}_1$, $\mathfrak{F}_2$ be nonempty $\tau\Omega_1 F_n^{\varphi}$-formations for a nonnegative integer $n$. If $\mathfrak{F}_1 \subseteq \mathfrak{F}_2 \neq (1)$ then*

$$\Phi^{\tau}_{\Omega_1 F_n^{\varphi}}(\mathfrak{F}_1) \subseteq \Phi^{\tau}_{\Omega_1 F_n^{\varphi}}(\mathfrak{F}_2).$$

*Proof.* Suppose that $\Phi^{\tau}_{\Omega_1 F_n^{\varphi}}(\mathfrak{F}_1) \not\subseteq \Phi^{\tau}_{\Omega_1 F_n^{\varphi}}(\mathfrak{F}_2)$. Let $\mathfrak{Y}$ be a maximal $\tau\Omega_1 F_n^{\varphi}$-subformation of $\mathfrak{F}_2$ such that $\Phi^{\tau}_{\Omega_1 F_n^{\varphi}}(\mathfrak{F}_1) \not\subseteq \mathfrak{Y}$. Thus $\mathfrak{F}_1 \not\subseteq \mathfrak{Y}$.

By Lemma 6.12, $\mathfrak{F}_2/^{\tau}_{\Omega_1 F_n^{\varphi}}\mathfrak{Y} = (\mathfrak{Y} \bigvee^{\tau}_{\Omega_1 F_n^{\varphi}} \mathfrak{F}_1)/^{\tau}_{\Omega_1 F_n^{\varphi}}\mathfrak{Y} \simeq \mathfrak{F}_1/^{\tau}_{\Omega_1 F_n^{\varphi}}(\mathfrak{F}_1 \cap \mathfrak{Y})$.

The lattice $\mathfrak{F}_2/^{\tau}_{\Omega_1 F_n^{\varphi}}\mathfrak{Y}$ consists of only two elements. Then $\mathfrak{F}_1 \cap \mathfrak{Y}$ is the maximal $\tau\Omega_1 F_n^{\varphi}$-subformation of $\mathfrak{F}_1$. Hence $\Phi^{\tau}_{\Omega_1 F_n^{\varphi}}(\mathfrak{F}_1) \subseteq \mathfrak{Y}$. We obtain a contradiction. Consequently, $\Phi^{\tau}_{\Omega_1 F_n^{\varphi}}(\mathfrak{F}_1) \subseteq \Phi^{\tau}_{\Omega_1 F_n^{\varphi}}(\mathfrak{F}_2)$, as asserted. □

**Corollary 6.20** ([122])**.** *Let $\mathfrak{F} \neq (1)$ be a nonempty $\tau$-closed $n$-multiply $\Omega_1$-free $\mathfrak{M}$-formation, and $n$ be a positive integer. Then the following holds:*

(1) *$\Phi^{\tau}_{\Omega_1 F_n^{\varphi_0}}(\mathfrak{F})$ consists of all $\tau\Omega_1 F_n^{\varphi_0}$-nongenerators of $\mathfrak{F}$.*

(2) *Let $\mathfrak{Y}$ be a $\tau$-closed $n$-multiply $\Omega_1$-free $\mathfrak{M}$-formation such that $\mathfrak{Y} \subseteq \mathfrak{F}$. Then $\Phi^{\tau}_{\Omega_1 F_n^{\varphi_0}}(\mathfrak{Y}) \subseteq \Phi^{\tau}_{\Omega_1 F_n^{\varphi_0}}(\mathfrak{F})$.*

**Corollary 6.21** ([122])**.** *Let $\mathfrak{F} \neq (1)$ be a nonempty $\tau$-closed $n$-multiply $\Omega_1$-bicanonical $\mathfrak{M}$-formation, and $n$ be a positive integer. Then the following holds:*

(1) *$\Phi^{\tau}_{\Omega_1 F_n^{\varphi_2}}(\mathfrak{F})$ consists of all $\tau\Omega_1 F_n^{\varphi_2}$-nongenerators of $\mathfrak{F}$.*

(2) *Let $\mathfrak{Y}$ be a $\tau$-closed $n$-multiply $\Omega_1$-bicanonical $\mathfrak{M}$-formation such that $\mathfrak{Y} \subseteq \mathfrak{F}$. Then $\Phi^{\tau}_{\Omega_1 F_n^{\varphi_2}}(\mathfrak{Y}) \subseteq \Phi^{\tau}_{\Omega_1 F_n^{\varphi_2}}(\mathfrak{F})$.*

**Corollary 6.22** (Theorems 3.1 and 3.2 [119])**.** *Let $\mathfrak{F} \neq (1)$ be a nonempty $\tau$-closed $n$-multiply $\omega$-composition formation of finite groups, where $n$ is a positive integer. Then the following holds:*

(1) *$\Phi^{\tau}_{\omega_n}(\mathfrak{F})$ consists of all $c^{\tau}_{\omega_n}$-nongenerators of $\mathfrak{F}$.*

(2) *Let $\mathfrak{Y}$ be a $\tau$-closed $n$-multiply $\omega$-composition formation of finite groups such that $\mathfrak{Y} \subseteq \mathfrak{F}$. Then $\Phi^{\tau}_{\omega_n}(\mathfrak{Y}) \subseteq \Phi^{\tau}_{\omega_n}(\mathfrak{F})$.*

**Corollary 6.23** ([114]). *Let $\mathfrak{F} \neq (1)$ be a nonempty $n$-multiply $\mathfrak{L}$-composition formation of finite groups, where $n$ is a positive integer. Then the following holds:*

(1) $\Phi_n^{\mathfrak{L}}(\mathfrak{F})$ *consists of all* $c_n^{\mathfrak{L}}$*-nongenerators of* $\mathfrak{F}$.

(2) *Let $\mathfrak{Y}$ be a $\tau$-closed $n$-multiply $\omega$-composition formation of finite groups such that $\mathfrak{Y} \subseteq \mathfrak{F}$. Then* $\Phi_n^{\mathfrak{L}}(\mathfrak{Y}) \subseteq \Phi_n^{\mathfrak{L}}(\mathfrak{F})$.

## Conclusion

Let $\mathfrak{M}$ be the class of all multioperator $T$-groups satisfying the minimality and maximality conditions for $T$-subgroups, and let $n$ be a positive integer. In the present chapter, it is proved that every law of the lattice of all $\tau$-closed $\mathfrak{M}$-formations is fulfilled in the lattice of all $\tau$-closed $n$-multiply $\Omega_1$-foliated $\mathfrak{M}$-formations with direction $\varphi$, such that $\varphi_0 \leqslant \varphi$. Let $\mathfrak{F}$ and $\mathfrak{H}$ be $\tau$-closed $n$-multiply $\Omega_1$-foliated $\mathfrak{M}$-formations with direction $\varphi$ such that $\varphi_0 \leqslant \varphi$, and $\mathfrak{H} \subseteq \mathfrak{F}$. If $\mathfrak{X} \subset \mathfrak{F}$ and the lattice $\mathfrak{F}/^{\tau}_{\Omega_1 F_n^{\varphi}}\mathfrak{X}$ consists of only two elements then $\mathfrak{X}$ is called a *maximal* $\tau$-closed $n$-multiply $\Omega_1$-foliated $\mathfrak{M}$-formation of $\mathfrak{F}$. Some properties of the intersection of these formations are studied. The results have been published in the papers [120, 122].

# Chapter 7

# Possible Future Directions

A *monoid* is an algebraic structure with a single associative binary operation and an identity element. A *group* can be defined as a monoid such that each element of this monoid possesses an inverse element.

Languages are subsets of a certain type of monoid, the free monoid over an alphabet. Regular languages are precisely the behaviors of finite automata. A language is regular if its syntactic monoid is a finite monoid, and a regular language is a group language if its syntactic monoid is a finite group.

A *variety* of finite monoids is a class of finite monoids closed under taking submonoids, quotients and finite direct products. Formations of finite monoids extend the notion of a variety of finite monoids, and the weaker closure conditions for formations lead to more possibilities than for varieties, and more general classes can be studied; see [13].

The Eilenberg theorem establishes that there exists a bijection between the set of all varieties of regular languages and the set of all varieties of finite monoids [38]. An analogous result holds for formations [13], i.e, there is a one-to-one correspondence between formations of finite monoids and formations of languages. This fact rises up the motivation to study formations in tasks of abstract machines and automata, which commonly appear in the theory of computation, compiler construction, artificial intelligence, parsing, formal verification and other aspects of theoretical computer science.

## 7.1 Formations of formal languages

A formation $\mathfrak{F}$ of groups is local (or saturated) if $G/\Phi(G) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$. A local satellite of $\mathfrak{F}$ is a function with domain $\mathbb{P}$ whose images are formations of finite groups. If the values of this function are themselves local formations, then this circumstance leads to the definition of multiply local formation.

Ballester-Bolinches, Pin, and Soler-Escrivà [14] developed a general method to describe the languages corresponding to local formations. In present work, it is show that the mentioned result is applicable to the languages corresponding to $n$-multiply local and totally local formations, which find deep applications in the study of finite groups.

*Thus we are equipped now with a powerful tool to translate efficiently any result of the theory of multiply local formations of finite groups to formal languages!*

A formation of groups $\mathfrak{F}$ is said to be *solvably saturated* if it contains each group $G$ with $G/\Phi(N) \in \mathfrak{F}$ for some solvable normal subgroup $N$ of $G$. Any saturated formation is solvably saturated, but not all the properties of saturated formations can be translated directly for solvably saturated formations.

**Problem 7.1.** *Describe the languages corresponding to solvably saturated formations of finite groups.*

The motivation to study $\sigma$-local formations rises from the result of Chi, Safonov and Skiba [29] which deals with so-called $\Sigma_t$-closed formations. In the present work it is shown that every law of the lattice of all formations is fulfilled in the lattice of all $n$-multiply $\sigma$-local formations of finite groups. This implies immediately that the lattice of all $n$-multiply $\sigma$-local formations of finite groups is modular but not distributive for any nonnegative integer $n$.

**Problem 7.2.** *Describe the languages corresponding to $n$-multiply $\sigma$-local and totally $\sigma$-local formations.*

Baer-local formations form a broader than local formations family of classes. By Baer's theorem, those formations are precisely solvably saturated formations; see p. 373 in [37]. Baer-$\sigma$-local formations, introduced recently in [91], generalize $\sigma$-local and Baer-local formations at the same time.

Let $G$ be a group, and $\mathfrak{F}$ be a formation of groups. The symbol $R_\sigma(G)$ denotes the product of all normal $\sigma$-solvable subgroups of $G$, and $F_{\{g\sigma_i\}}(G)$ denotes the product of all normal generalized $\{\sigma_i\}$-nilpotent subgroups of $G$. Set $\sigma^+(\mathfrak{F}) = \cup_{G\in\mathfrak{F}}\sigma^+(G)$, where

$$\sigma^+(G) = \{\sigma_i \mid G \text{ has a chief factor } H/K \text{ such that } \sigma(H/K) = \sigma_i\},$$

**Definition 7.1** ([91]). Following Skiba, we call any function $f$ of the form

$$f : \sigma \cup \{\varnothing\} \to \{\text{formations of groups}\},$$

where $f(\varnothing) \neq \emptyset$, a *generalized formation $\sigma$-function*, and put

$$BLF_\sigma(f) = (G \mid G/R_\sigma(G) \in f(\varnothing) \text{ and } F_{\{g\sigma_i\}}(G) \in f(\sigma_i) \text{ for all } \sigma_i \in \sigma^+(G)).$$

If $\mathfrak{F} = BLF_\sigma(f)$ for some generalized formation $\sigma$-function $f$, then $\mathfrak{F}$ is called *Baer-$\sigma$-local*, and $f$ is a *generalized $\sigma$-local definition* of $\mathfrak{F}$. The symbol $Supp(f)$ denotes the support of $f$, i.e., the set of all $\sigma_i$ such that $f(\sigma_i) \neq \emptyset$.

**Problem 7.3.** *Describe the languages corresponding to Baer-$\sigma$-local formations of finite groups.*

## 7.2 Classes of fuzzy languages

In 1934, at the eight Congress of Scandinavian Mathematics, Marty [72] introduces a concept of algebraic hyperstructure, which naturally generalizes classical algebraic structures such as groups and rings. As mentioned in [33], the first example of hypergroups, which motivates the introduction of this structure, is the quotioent of a finite group by arbitrary (not necessarily normal) subgroup, i.e., if the subgroup is not normal, then the quotient is not a group, but it is always a hypergroup with respect to a certain hyperoperation. Keeping in mind this idea, we can introduce a concept of hyperformation, assuming that subgroups in Definition 2.8 are not necessarily normal.

**Definition 7.2.** A *hyperformation* is a class of hypergroups $\mathfrak{F}$ satisfying the following two conditions:

(1) if $H \in \mathfrak{F}$, then $H/N \in \mathfrak{F}$, and

(2) if $H/N_1$, $H/N_2 \in \mathfrak{F}$, then $H/N_1 \cap N_2 \in \mathfrak{F}$,

for any subhypergroups $N$, $N_1$, $N_2$ of $H$.

It will be interesting to study the relation between classical formations and hyperformations. Fuzzy sets, introduced by Zadeh [141] and Klaua [65], became applied in fields such as pattern recognition, machine learning and data mining [22, 60]. Examples of hypergroups associated with models of biological inheritance were considered recently in [3], and connections between hypergroups and fuzzy sets were discussed in Chapter 5 of the book [33].

In [83, 73], the concept of a fuzzy set was applied to generalize the basic concepts of group theory. The most fundamental result in the theory of classes of finite groups states that any formation is saturated iff it is local [37, Gaschütz–Lubeseder–Schmid].

**Problem 7.4.** *To obtain a generalization of Gaschütz–Lubeseder–Schmid theorem in terms of fuzzy group theory.*

Wee [139] introduced the fuzzy automaton as a model of learning systems. This model is the natural fuzzification of the classical finite automaton: a *fuzzy automaton* is a tuple $\mathcal{A} = (A, X, \mu)$, where $A$ is a finite set of states, $X$ is a finite set of input symbols and $\mu$ is a fuzzy subset of $A \times X \times A$ representing the transition mapping, which can be represented as a collection of matrices with entries from $[0, 1]$. Let $X^*$ be a free monoid, then a *fuzzy language* over an alphabet $X$ is a fuzzy subset of $X^*$. A fuzzy language is *regular* if it is recognizable by a fuzzy automaton. Some applications of fuzzy languages are discussed in [23, 24, 61, 79].

Petković [77] proved a counterpart of Eilenberg's theorem for varieties of fuzzy languages. That motivates us to study formations of fuzzy languages. For instance, the following two problems are of great interest.

**Problem 7.5.** *Prove a counterpart of Eilenberg's theorem for formations of fuzzy languages.*

**Problem 7.6.** *Describe the fuzzy languages corresponding to $\sigma$-local ($n$-multiply $\sigma$-local, totally $\sigma$-local) formations.*

## 7.3 Applications in computer programming and data science

In computer programming, various abstract data types can be considered as monoids. Some applications for functional programming are discussed in the book [31]. Because the operation takes two values of a given type and returns a new value of the same type, it can be chained indefinitely. The associativity of monoid operations ensures that the operation can be parallelized.

Out of the sixteen possible binary Boolean operators, each of the four that has a two sided identity is also commutative and associative, that makes the set $\{False, True\}$ a commutative monoid.

Every group is a monoid. Every abelian group is a commutative monoid.

The elements of any unital ring, with addition or multiplication as the operation, form a monoid. Any complete lattice can be endowed with a meet-and-join monoid structure. The same holds for complete lattices of formations. Boolean algebras have these monoid structures as well.

Formations are a useful tool to study finite rings [32], which find applications in coding theory [20, 102]. In the present work it is shown that the lattice of all formations of finite rings is algebraic and modular.

**Problem 7.7.** *Describe algebraic and modular lattices of local formations of rings.*

In 2013, Twitter open-sourced Algebird [1], a library which provides abstractions for abstract algebra in the Scala programming language to work with *semigroups, monoids, groups* and *rings*. Algebird was designed on Twitter with a target to simplify building aggregation systems like Apache Spark, Scalding, Apache Storm, etc. Many of the data structures included in Algebird have a monoid implementations, making them ideal to use as values in Summingbird aggregations. (Summingbird is a library that lets us write MapReduce programs that look like native Scala or Java collection transformations.) Algebird is extremely helpful in the problems of *Large-Scale Data Analytics*, some real world examples of Algebird at Twitter-scale are discussed in [74]. Thus formations of monoids, groups and rings can be applicable as a tool for *Data Mining*, social media research and knowledge discovery.

**Task 7.1.** *To develop an Algebird based Scala library applying advances of formation theory for Big Data Analytics.*

Finally, we note that Pin and Soler-Escrivà [78] described the two classes of languages recognized by the groups $D_4$ and $Q_8$, and they proved that the formations of languages generated by these two classes are the same, and in the most recent paper [25], the authors describe two sublattices of the lattice of all formations of monoids, and give, for each of them, an isomorphism with a known lattice of varieties of monoids, and study formations containing Clifford monoids.

—$\mathfrak{F}$—

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# 국문초록

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## 대수적 구조의 형성의 격자들

본 논문에서는 대수 구조의 형성에 대한 다양한 격자를 조사 연구하였다. 유한군들의 클래스가 형성(formation)이라는 정의는 클래스의 한 원소인 군 $G$에 대하여 그의 상군 $G/N$을 포함하고, 만약 $G/N_1$과 $G/N_2$가 클래스에 속한다면 $G/N_1 \cap N_2$도 포함하는 조건을 만족하는 클래스이다. 본 논문에서의 주요결과는 다음과 같다.

다양한 국소 형성들에 대응하는 언어들을 설명하였다. $\sigma$는 모든 소수의 집합의 분할이라 하자. 모든 형성에 대한 격자의 모든 법칙은 모든 다양한 $\sigma$-국소형성들에 대한 격자 안에서 만족된다는 것을 증명하였다.

모든 함자 닫힌 합성 형성의 격자는 대수 격자라는 것과 모든 함자 닫힌 형성에 대한 격자의 모든 법체계가 모든 함자 닫힌 곱셈 부분 합성 형성들의 격자와 법체계가 일치한다는 것을 보였다. 또한, 모든 $\mathfrak{X}$-국소형성의 격자는 대수적이고 모듈러 격자라는 것을 증명하였다.

$\mathfrak{M}$을 $T$-부분군의 최소성과 최대성 조건들을 만족하는 모든 복합연산자 $T$-군의 클래스라 하자. 그러면, 모든 함자 닫힌 $\mathfrak{M}$-형성에 대한 격자의 모든 법칙은 모든 함자 닫힌 곱셈 부분 엽층 형성들에 대한 격자 안에서 만족된다는 것을 증명하였다.

**중심어:** 모노이드, 언어, 군, 환, 중복연산자 $T$-군, 퍼지집합, 격자, 형성, 포화형성, 국소형성, 합성형성