On the Another proof of Liapounov's theorem in the Central Limit Theorem

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中心極限定理에 있어서 Liapounov 定理의 別證에 대하여

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Summary

In this paper, we shall study the another proof that the normalized sum converges in distribution to a random variable that is normal with mean and variance 1 by comparing the expectation of test function.

1. Introduction

Let $X_1, X_2, \cdots$ be independent random variables ($r.v's$) with each X_j having finite mean μ_j and finite variance δ_j^2. Let $S_n=\sum_{j=1}^{n} X_j$, $n=1, 2, \cdots$; then $E(S_n)=\sum_{j=1}^{n}\mu_j$, Var $S_n=s_n^2=\sum_{j=1}^{n}\delta_j^2$.

We consider normalized sum $S_n^*=s_n^{-1}(S_n-E(S_n))$ which has mean 0 and variance 1 assuming that $s_n>0$ for sufficiently large n. If X^* is a r.v. having the normal distribution with mean 0 and variance 1, N(0, 1), so that the distribution function (df) of X^* is

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} exp\left(-\frac{1}{2}t^2\right)dt. \quad \cdots\cdots\cdots\cdots(1)$$

Let F_j be the df of X_j, f_j be the Characteristic function (Ch.f) of X_j, F_n^* be the df of S_n^* and $g_n(t)$ be the Ch.f of r.v. S_n^*. We shall investigate the conditions, especially Liapounov's, under which S_n^* converges in distribution to X^* and the method of proofs of the theorem under such restrictions.

2. Converges to a Normal Distribution

I. Let F be a df of r.v. X and $E(X)=\mu$, if $\int_{-\infty}^{\infty}|X|^n dF(x)$ exists and is finite, then the Ch.f $f(t)$ of F may be written as

$$f(t)=1+\sum_{j=1}^{n-1}\mu_j' \frac{(it)^j}{j!}+R_n(t), \quad \cdots\cdots\cdots\cdots\cdots\cdots(2)$$

where

$$R_n(t)=t^n\int_0^1 \frac{(1-u)^{n-1}}{(n-1)!}f^{(n)}(tu)du \quad \cdots\cdots\cdots (3)$$

and

$$f^{(j)}(t)=\int_{-\infty}^{\infty}(ix)^j e^{itx}\, dF(x),$$

$$f^{(j)}(0)=i^j\int_{-\infty}^{\infty}x^j dF(x)=i^j\mu_j$$

Also

$$R_n(t) = \frac{(it)^n}{n!}\int_{-\infty}^{\infty} x^n dF(x) + o(t^n), \quad \cdots\cdots\cdots\cdots (4)$$

where $o(t^n)$ denotes a function of t such that

$$\frac{o(t^n)}{t^n} \longrightarrow 0 \text{ as } t \to \infty.$$

Furthermore

$$|R_n(t)| = \theta\frac{|t^n|}{n!}\int_{-\infty}^{\infty} |x|^n \, dF(x), \quad |\theta| \leqslant 1 \quad \cdots (5)$$

for every real t.

Pf. The exponential function satisfies

$$e^v = 1 + \sum_{j=1}^{n-1} \frac{v^j}{j!} + r_n(v),$$

where

$$r_n(v) = \frac{v^n}{(n-1)!}\int_0^1 (1-u)^{n-1} \frac{d^n}{du^n}(e^{uv})du,$$

or

$$r_n(v) = \frac{v^n}{n!} + o(v^n).$$

If we place v by it x and take expected values, we obtain (3) and (4). Also (5) is obtained by applying) the Maclaurin expansion theorem of calculus to $f(t)$.

Ⅱ. Liapounov's inequality

Lat X and Y be $r.v's$, $0<p<\infty$ and $p^{-1}+q^{-1}=1$, Hölder's inequality

$$E(|XY|) \leqslant \{E(|X|^p)\}^{1/p}. \ \{E(|Y|^q)\}^{1/q} \cdots\cdots (6)$$

If $Y \equiv 1$ in (6) we obtain

$$E(|X|) \leqslant \{E(|X|^p)\}^{1/p}$$

Replacing $|X|$ by $|X|^r$, where $1<r<p$ and writing $s=pr$ we obtain the Liapounov's ineqnality ;

$$E(|X|^r)^{1/r} \leqslant E(|X|^s)^{1/s}, \ 1<r<s<\infty. \quad \cdots\cdots (7)$$

Theorem. Let $S_n = \sum_{j=1}^{n} X_j$ $n=1, 2\cdots$, where the X_j are independent $r.v's$ with μ_j and σ_j^2, they are finite. Let $S_n^* = s_n^{-1}(S_n - E(S_n))$, where $s_n^2 = \sum_{j=1}^{n} \sigma_j^2$ and let F_j be the df of X_j. If for every $\epsilon > 0$,

$$\frac{1}{s_n^2}\sum_{j=1}^{n}\int_{\{x\,:\,|x-\mu_j| \geqslant \epsilon s^n\}} (x-\mu_j)^2 dF_j(x) \longrightarrow 0$$

$$as \ n \longrightarrow \infty. \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (8)$$

then S_n^* converges in distribution to X^*. This theorem implies that $S^* \xrightarrow{d} X^*$ under any one of the following conditions.

1) The uniformly bounded case.

Assume $|X_j| \leqslant M$ for all j, and $s_n \longrightarrow \infty$.

Then

$$\int_{\{x\,:\,|x-\mu_j| \geqslant \epsilon s_n\}} (x-\mu_j)^2 \, dF_j(x)$$
$$= E[(X_j-\mu_j)^2 \, I\{|X_j-\mu_j| \geqslant \epsilon s_n\}]$$
$$\leqslant \frac{(2M)^2 \sigma_j^2}{\epsilon^2 \, s_n^2} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (6)$$

by Chebyshev's inequality.

Thus

$$\frac{1}{s_n^2}\sum_{j=1}^{n}\int_{\{x\,:\,|x-\mu_j| \geqslant \epsilon s_n\}} (x-\mu_j)^2 dF_j(x) \leqslant \frac{(2M)^2}{\epsilon^2 \, s_n^2} \longrightarrow 0$$

2) The identically distributed case.

Assume that the X_j are inddpendent identically distributed $r.v's$ with finite μ and finite $\sigma^2 > 0$, then

$$(8) = \frac{1}{n\sigma^2}\sum_{j=1}^{n}\int_{\{x\,:\,|x-\mu| \geqslant \epsilon\sqrt{n}\}} (x-\mu)^2 \, dF(x$$

$$=\frac{1}{\delta^2}\int_{\{x:|x-\mu|\geq\epsilon\sqrt{n}\}}(x-\mu)^2dF(x)\to 0, \quad\cdots\cdots\cdots(10)$$

since σ^2 is finite and $\{x:|x-\mu|\geq\epsilon\delta\sqrt{n}\}\downarrow\phi$ as $n\to\infty$.

3) The Bernoulli case.

Let S_n be the number of success in n Bernoulli trials, with probability of success p on a given trial. We may write

$$S_n=X_1+X_2+\cdots+X_n,$$

where the X_j are independent and $P(X_j=1)=p$, $P(X_j=0)=q=1-p$. We may take X_j as the indicator of a success on trial j, thus case 2 applies with $\mu=E(X_j)=p$, $\sigma^2=E(X_j^2)-\{E(X_j)\}^2=p(1-p)$, $E(S_n)=n\mu=np$, $s_n^2=n\sigma^2=np(1-p)$. Thus

$$S_n^*=\frac{S_n-np}{(npq)^{1/2}} \quad\cdots\cdots\cdots(11)$$

and

$$S_n^*\xrightarrow{d}X^*, \text{ that is, } P(S_n^*\leq x)\to\Phi(x)$$

for all x.

4) Liapounov's condition.

Assume that

$$\frac{1}{s_n^{2+\delta}}\sum_{j=1}^{n}E\left[|X_j-\mu_j|^{2+\delta}\right]\to 0 \quad\cdots\cdots\cdots(12)$$

for some $\delta>0$.

where $E[|X_j-\mu_j|^{2+\delta}]$ exist.
Then

$$E\left[|X_j-\mu_j|^{2+\delta}\right]=\int_{-\infty}^{\infty}|x-\mu_j|^{2+\delta}\,dF_j(x)$$

$$\geq\int_{\{x:\,||x-\mu_j|\geq\epsilon s_n\}}|x-\mu_j|^{\delta}\,|x-\mu_j|^2\,dF_j(x)$$

$$\geq\epsilon^{\delta}s_n^{\delta}\int_{\{x:|x-\mu_j|\geq\epsilon s_n\}}(x-\mu_j)^2dF_j(x)$$

Thus

$$(8)\leq\frac{1}{s_n^2}\sum_{j=1}^{n}\frac{E[|X_j-\mu_j|^{2+\delta}]}{\epsilon^{\delta}\,s_n^{\delta}}$$

$$=\frac{\sum_{j=1}^{n}E[|X_j-\mu_j|^{2+\delta}]}{\epsilon^{\delta}\,s_n^{2+\delta}}\to 0.$$

Now we shall show that Theorem holds under case 4). Let $E(|X_j|^3)=\gamma_j$ and $\Gamma_n=\sum_{j=1}^{n}\gamma_j=\sum_{j=1}^{n}E(|X_j|^3)$. To prove Theorem in (12) we need only show that $g_n(t)\to e^{-\frac{t^2}{2}}$ or $\log g_n(t)\to -\frac{t^2}{2}$ as $n\to\infty$. Furthermore we may assume without loss of generality that all $\mu_j=0$ and that the r.v's X_j all have finite third moments, that is, $\delta=1$.

Proof 1.

The condition (12) in case 4) is written as

$$\Gamma_n/s_n^3\to 0. \quad\cdots\cdots\cdots(13)$$

By I, the Ch.f $f_j(t)$ of X_j has the expansion;

$$f_j(t)=1-\frac{1}{2}\sigma_j^2t^2+\frac{1}{6}E(X_j^3)(it)^3+o[E(X_j^3)t^3] \quad\cdots\cdots\cdots(14)$$

The Ch.f $g(t)$ of the r.v. S is given by

$$g_n(t)=\prod_{j=1}^{n}f_j\left(\frac{t}{s_n}\right)=\prod_{j=1}^{n}\left\{1-\frac{\sigma_j^2}{2s_n^2}t^2+\frac{E(X_j^3)}{6s_n^3}(it)^3+0\left[\frac{E(X_j^3)t^3}{s_n^3}\right]\right\}. \quad\cdots\cdots\cdots(15)$$

Writing

$$\eta_j=-\frac{\sigma_j^2}{2s_n^2}t^2+\frac{E(X_j^3)}{6s_n^3}(it^3)+0\left[\frac{E(X_j^3)t^3}{s_n^3}\right]$$

we note that :

$$\sum_{j=1}^{n}\eta_j=-\frac{t^2}{2}+\frac{1}{6s_n^3}(it)^3\sum_{j=1}^{n}E(X_j^3)$$

$$+\sum_{j=1}^{n}0\left(\frac{E(X_j^3)t^3}{s_n^3}\right)$$

step 1 : $\eta_j\longrightarrow 0$ for all j, uniformly in j as $n\rightarrow\infty$.
Clearly the o-term $\longrightarrow 0$ as $n\longrightarrow\infty$.
Recalling(7)

$$\{E(|X_j|^2)\}^{1/2}=\sigma_j\leqslant\{E(|X_j|^3)\}^{1/3},$$

we obtain

$$0\leqslant\frac{\sigma_j^2}{2s_n^2}t^2\leqslant\frac{1}{2}[\gamma_j]^{2/3}\frac{t^2}{s_n^2}\leqslant\frac{1}{2}[\Gamma_n]^{2/3}\cdot\frac{t^2}{s_n^2}$$

The first and trird inequalities are obvious and, by (13), the upper bound on the right hand side tends to zero as $n\longrightarrow\infty$, $i.e$;

$$\frac{1}{2}\cdot\frac{\sigma_j^2}{s_n^2}\cdot t^2\longrightarrow 0,\quad \text{uniformly in } j \text{ as } n\longrightarrow\infty.$$

Finally

$$0\leqslant\left|\frac{1}{6}\cdot\frac{E(X_j^3)}{s_n^3}(it)^3\right|\leqslant\frac{1}{6}\cdot\frac{\gamma_j|t|^3}{s_n^3}$$

$$\leqslant\frac{1}{6}\frac{|t|^3}{s_n^3}\Gamma_n.$$

By (13) the quantity on the right tends to zero as $n\rightarrow\infty$, $i.e$;

$$\frac{1}{6}\cdot\frac{E(X_j^3)}{s_n^3}(it)^3\longrightarrow 0,$$

uniformly in j as $n\longrightarrow\infty$.

step 2 : Given $\epsilon>0$, $0<\epsilon<\frac{1}{2}$, we can find $N(t,\ \epsilon)$ such that for $n>N(t,\ \epsilon)$, $|\eta_j|<\epsilon$ for all $j\leqslant n$.
Using the logarithmic expansion we obtain

$$log(1+\eta_j)=\sum_{r=1}^{\infty}(-1)^{r-1}\cdot\frac{1}{\gamma}\eta_j^r$$

$$=\eta_j+\eta_j^2\sum_{r=2}^{\infty}(-1)^{r+1}\frac{1}{\gamma}\eta_j^{r-2}=\eta_j+\eta_j^2\psi(\eta_j), \qquad \cdots\cdots(16)$$

where

$$\psi(\eta_j)=\sum_{r=2}^{\infty}(-1)^{r+1}\frac{1}{\gamma}\eta_j^{r-2}$$

However

$$|\psi(\eta_j)|\leqslant\sum_{r'=0}^{\infty}\frac{1}{2+\gamma'}|\eta_j|^{r'}\leqslant\frac{1}{2}[1-|\eta_j|]^{-1}$$

$$<\frac{1}{2}\cdot\frac{1}{1-\epsilon}<1, \qquad \cdots\cdots(17)$$

since $(2+r')^{-1}<\frac{1}{2}$ for all $r'\geqslant 1$ and since η_j tends to zero uniformly in j. The logarithm of the Ch. f $g_n(t)$ of S_n^* may be written as follows ;

$$log\ g_n(t)=\sum_{j=1}^{n}log(1+\eta_j)=\sum_{j=1}^{n}\eta_j+\sum_{j=1}^{n}\eta_j^2\psi(\eta_j)$$

$$=-\frac{t^2}{2}+\frac{1}{6s_n^3}(it)^3\sum_{j=1}^{n}E(X_j^3)+\sum_{j=1}^{n}0\left(\frac{E(X_j^3)t^3}{s_n^3}\right)$$

$$+\sum_{j=1}^{n}\eta_j^2\psi(\eta_j) \qquad \cdots\cdots(18)$$

step 3 : $\sum_{j=1}^{n}\eta_j^2\psi(\eta_j)\longrightarrow 0$ as $n\longrightarrow\infty$

By (17), $\left|\sum_{j=1}^{n}\eta_j^2\psi(\eta_j)\right|\leqslant\sum_{j=1}^{n}\eta_j^2\left|\psi(\eta_j)\right|\leqslant\sum_{j=1}^{n}\eta_j^2$,

Now, let show $\sum_{j=1}^{n}\eta_j^2\longrightarrow 0$ as $n\longrightarrow\infty$.

$$\left|\sum_{j=1}^{n}\eta_j^2\right|\leqslant①\frac{|t|^4}{4s_n^4}\sum_{j=1}^{n}\sigma_j^4+②\frac{|t|^6}{36s_n^6}\sum_{j=1}^{n}[\gamma_j]^2$$

$$+ ③\sum_{j=1}^{n}\left\{0\left(\frac{t^3}{s_n^3}E(X_j^3)\right)\right\}^2 + ④\frac{|t|^5}{6s_n^5}\sum_{j=1}^{n}\sigma_j^2\gamma_j$$

$$+ ⑤\frac{|t|^2}{s_n^2}\sum_{j=1}^{n}\sigma_j^2\left|0\left(\frac{E(X_j^3)t^3}{s_n^3}\right)\right|$$

$$+ ⑥\frac{|t|^3}{3s_n^3}\sum_{j=1}^{n}\gamma_j\left|0\left(\frac{E(X_j^3)t^3}{s_n^3}\right)\right|$$

Appealing twice to (7) we have

$$\frac{\sum_{j=1}^{n}\sigma_j^4}{s_n^4} \leqslant \frac{\sum_{j=1}^{n}[\gamma_j]^{4/3}}{s_n^4} \leqslant \frac{\Gamma_n}{s_n^3}$$

and the upper bound tends to zero by (13), this taket care of ①.

Next, by expanding the square on the right that

$$\frac{\sum_{j=1}^{n}[\gamma_j]^2}{s_n^6} \leqslant \left(\frac{\Gamma_n}{s_n^3}\right)^2$$

add upper bound of ② tends to zero.

Also ③ tends to zero because of the $o(s)$, so do ⑤ and ⑥. Finally ④ satisfies

$$|t|^5\frac{\sum_{j=1}^{n}\sigma_j^2\gamma_j}{s_n^5} \leqslant \frac{\Gamma_n}{s_n^3}\cdot |t|^5$$

since $\sigma_j^2 \leqslant s_n^2$, $j=1, 2, \cdots, n$.

The upper bounds on the right go to zero as $n \longrightarrow \infty$ by (13), hence completes the proof of step3.

step 4 : $\sum_{j=1}^{n}\eta_j^2 + \frac{t^2}{2} \to 0$ as $n \longrightarrow \infty$

We have ;

$$\left|\sum_{j=1}^{n}\eta_j^2 + \frac{t^2}{2}\right| \leqslant \frac{|t|^3}{6s_n^3}\Gamma_n + \sum_{j=1}^{n}0\left(\frac{E(X_j^3)t^3}{s_n^3}\right)$$

and both on the right tend to zero as $n \to \infty$. The first by (13) and the second because of the small o.

We conculude that

$$\log g_n(t) \longrightarrow -\frac{t^2}{2}$$

for every t and so complete the proof of Theorem.

Let proceed to another proof of Theorem by the idea that is to approximate the sum S_n successively by replacing one X at a time with a normal $r.v.$ Y.

Proof 2.

Let $\{Y_j\ ;\ j \geqslant 1\}$ be $r.v's$ having $N(0, \sigma_j^2)$, thus Y_j has the same mean and variance as the corresponding X_j ; let all the $X's$ and $Y's$ be totally independent.

Now put

$$Z_j = Y_1 + Y_2 \cdots + Y_{j-1} + X_{j+1} + \cdots + X_n, \quad 1 \leqslant j \leqslant n,$$

with the convension that

$$Z_1 = X_2 + \cdots + X_n, \quad Z_n = Y_1 + \cdots + Y_{n-1}.$$

We now write

$$f(X_1+X_2+\cdots+X_n) - f(Y_1+\cdots+Y_n) = \sum_{j=1}^{n}\left\{f(X_j+Z_j) - f(Y_j+Z_j)\right\}.$$

Let estimate the difference for a suitable class of function f to compare the distribution of $(X_j + Z_j)/s_n$ with that of $(Y_j+Z_j)/s_n$. $i.e$;

$$E\left\{f\left(\frac{X_1+\cdots+X_n}{s_n}\right)\right\} - E\left\{f\left(\frac{Y_1+\cdots+Y_n}{s_n}\right)\right\} = \sum_{j=1}^{n}\left[E\left\{f\left(\frac{X_j+Z_j}{s_n}\right)\right\} - E\left\{f\left(\frac{Y_j+Z_j}{s_n}\right)\right\}\right]. \quad (19)$$

On the other hand, if we take f in C^3, the class of bounded continuous function with three bounded continuous derivatives, it suffices to show that

$E\{f(S_n^*)\} \longrightarrow E\{f(X^*)\}$.

Now by Taylor's Th, we have for every x and y;

$$\left| f(x+y) - \left[f(x) + f'(x)y + \frac{f''(x)}{2}y^2 \right] \right| \leq \frac{M|y|^3}{6}$$

where $M = \sup_{x \in R^1} |f^{(3)}(x)|$.

Hence if ξ and η are independent $r.v's$ such that $E\{|\eta|^3\} < \infty$, by substitution and integration,

$$|E\{f(\xi+\eta)\} - E\{f(\xi)\} - E\{f'(\xi)\}E\{\eta\} - \frac{1}{2}E\{f''(\xi)E\{\eta^2\}\}| \leq \frac{M}{6}E\{|\eta^3|\} \quad \cdots\cdots\cdots(20)$$

since the $r.v's$ $f(\xi)$, $f'(\xi)$ and $f''(\xi)$ are bounded hence integrable.

If ζ is another $r.v.$ independent of ξ and having the same mean and variance as η, and $E\{|\zeta|^3\} < 8$, we obtain by replacing η with ζ and taking the difference;

$$|E\{f(\xi+\eta)\} - E\{f(\xi+\zeta)\}| \leq \frac{M}{6}E\{|\eta|^3 + |\zeta|^3\}. \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(21)$$

Appling to the right side of (19) with $\zeta = \frac{Z_j}{s_n}$, $\eta = \frac{X_j}{s_n}$, $\zeta = \frac{X_j}{s_n}$ and the bounds on the right-hand side of (21) then add up to

$$\frac{M}{6}\sum_{j=1}^{n}\left\{ \frac{\gamma_j}{s_n^3} + \frac{c\sigma_j^3}{s_n^3} \right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(22)$$

where $c = \sqrt{8/\pi}$ since the absolute third moment of $N(0, \sigma^2)$ is equal to $c\sigma_j^3$. By (7), $\sigma_j^3 \leq \gamma_j$, so that the quantity in (22) is $o(\Gamma_n/s_n^3)$. We have thus obtained the following estimate;

$$^{\forall}f \in C^3;\ |E\{f(S_n^*) - E\{f(X^*)\}| \leq o\left(\frac{\Gamma_n}{s_n^3}\right)$$

and under (13) this converges to zero as $n \to \infty$. Hence

$$E\{f(S_n^*)\} \longrightarrow E\{f(X^*)\} \text{ as } n \to \infty,$$

Example:

Let $\epsilon > 0$ and $f_\epsilon \in C^3$

$$f_\epsilon(x) = \begin{cases} 1 & if\, x \leq 0 \\ [1-(x\epsilon^{-1})^4]^4 & if\, 0 \leq x \leq \epsilon \\ 0 & if\, x \geq \epsilon \end{cases}$$

Then

$$\Phi(-x+\epsilon) \geq \int_{-\infty}^{\infty} f_\epsilon(x+y)\, d\Phi(y) \geq \Phi(-x), \quad x \in R.$$

Similarly

$$F_n^*(-x+\epsilon) \geq \int_{-\infty}^{\infty} f_\epsilon(x+y) dF_n^*(y) \geq F_n^*(-x), \quad x \in R,$$

since $\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x+y)\, dF_n^*(y) = \int_{-\infty}^{\infty} f(x+y) d\Phi(y)$

for any x (uniformly in x), $f \in C^3$ (Ref [6])

We conclude that

$$\overline{\lim_{n\to\infty}}\, F_n^*(-x) \geq \lim_{n\to\infty}\int_{-\infty}^{\infty} f_\epsilon(x+y)\, dF_n^*(y) = \int_{-\infty}^{\infty} f(x+y)\, d\Phi(y) \leq \Phi(-x+\epsilon)$$

and

$$\underline{\lim_{n\to\infty}} F_n^*(-x+\epsilon) \geq \lim_{n\to\infty}\int_{-\infty}^{\infty} f_\epsilon(x+y)\, dF_n^*(y) = \int_{-\infty}^{\infty} f_\epsilon(x+y)\, d\Phi(y) \geq \Phi(-x) \quad \cdots\cdots\cdots\cdots(23)$$

for all real x and $\epsilon > 0$.

By (23),

$$\underline{\lim_{n\to\infty}}\, F_n^*(-x) \geq \Phi(-x-\epsilon),$$

and we get

$$\Phi(-x-\epsilon)\leqslant\underline{\lim}_{n\to\infty} F_n^*(-x)\leqslant\overline{\lim}_{n\to\infty} F_n^*(-x)$$
$$\leqslant\Phi(-x+\epsilon)$$

for all $x\epsilon R$ and $\epsilon>0$. Since ϵ is arbitrary,

$$\lim_{n\to\infty} F_n^*((-x)=\Phi(-x),\quad x\epsilon R.$$

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〈國 文 抄 錄〉

中心 極限定理에 있어서 Liapounov 定理의 別證에 對하여

本 論文에서는 Liapounov의 條件下에서 中心極限定理가 成立한다는 사실을 C^3의 범위안에서 test function을 비교함에 의하여 고찰하고 實例로서 이를 確認하였다.