A Note on f-structure

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F－構造에 관한 研究

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Summary

Induced an almost complex structure J on M × R^{n-r} is integrable, then the globally framed structure f on M is said to be normal (Def. 2.3).

The f-structure induced on a submanifold of an almost complex manifold is equivalent to $\nabla_c f^a_b$.

1. Introduction

K. Yano 1961, 1963, 1964, 1965 have introduced the notion of an f-structure defined by a tensor f of type (1,1) satisfying $f^3 + f = 0$. Afterward, H. Nakagawa, D.E. Blair, S.I. Goldberg have studied f-structure with complementary frame. The purpose of the present paper is to introduce a manifold with an f-structure and globally frame structure and to study on geometry of manifold with such a structure.

In §1, we introduce the f-structure and study the integrability condition of the structure. An almost complex structure and almost condition structure are well-known examples of f-structure. The existence of f-structure is equivalent to a reduction of the structure group of tangent bundle to u(r) x o(n-r).

In §2, we define the globally framed structure and we find the normality condition $N^1 = 0$ of a globally framed structure (f, ξ_x, η_x).

In §3, we discuss the f-structure induced on a submanifold of an almost complex manifold.

§1. f-structure and integrability condition.

Let there be given, in an n-dimensional differentiable manifold M^n of class c^∞, a non-null tensor of type (1,1) satisfying

(1,1) $\quad f^3 + f = 0.$

We call such a structure an f-structure of rank r, when the rank r of f is constant everywhere, r being necessary even Yano, K. (1961), (1963).

If we put

(1,2) $\quad \ell = -f^2, \quad m = f^2 + 1.$

then we have

(1,3) $\quad \ell + m = 1, \ell^2 = \ell, m^2 = m, \ell m = m\ell = 0,$
$\qquad f\ell = \ell f = f, fm = mf = 0.$

Thus the operators ℓ and m applied to the tangent space at a point of the manifold are complementary projection operators.

If there is given a non-null tensor field f satisfying (1,1), then there exist complementary distributions L and M corresponding to the projection operators ℓ and m respectively.

If the rank of f is equal to r everywhere, than L is r-dimensional and M is (n-r)-dimensional. We call such a structure an f-structure of rank r.

Now, we can introduce a positive definite Riemannian metric such that the vector space of L and the vector space of M are orthogonal to each other, that is

(1,4) $\quad h(\ell X, mY) = 0.$

for any vector fields X and Y on M then we can easily the relation

$$(1,5)\quad h(X,Y) = h(\ell X,\ell Y) + h(mX,mY).$$

If we put

$$(1,6)\quad g(fX,fY) = ½\{h(X,Y) + h(fX,fY) + h(mX,\ell Y)\}.$$

then we have

$$(1,7)\quad g(\ell X,mY) = 0.$$

from (1,2) and (1,3) we get

$$(1,9)\quad g(fX,fY) = g(X,Y) - g(mX,Y).$$

Next, Let λ be an eigenvalue of matrix (f) and X the corresponding eigenvector, that is $fX = \lambda X$.

Transvecting f^2 to the equation we get $\lambda X = -\lambda^3 X$, which shows that the eigenvalues of the matrix (f) are i, -i and 0.

We denote the multiplicities of the roots i and -i by p. The characteristic spaces corresponding to i and -i by V_i and V_{-i} respectively. Then V_i and V_{-i} are orthogonal on the vector space of L and the characteristic space V_o corresponding to the root o is vector space of M.

Hence the tangent space TM_p at each point p of M^n is complicated such that

$$TM_p = V_i \oplus V_{-i} \oplus V_o.$$

We take sufficiently fine open covering $\{U_\alpha\}$ by coordinate neighborhood of M^n and determine a suitable frame in every U_α, Now we take orthogonal frame $\{e_1,\ldots\ldots,e_p,e_{p+1},\ldots\ldots,e_{2p},e_{2p+1},\ldots\ldots,e_n\}$ such that $e_1,\ldots\ldots,e_p$ span the space V_i and $e_{p+1},\ldots\ldots,e_{2p}$ span the space V_{-i} and $e_{2p+1},\ldots\ldots,e_n$ span the space V_o respectively.

Then we have

(1,10)

$$fe_a = ie_a \qquad (a = 1,\ldots\ldots,p),$$
$$fe_{p+a} = -ie_{p+a},$$
$$fe_{2p+h} = 0 \qquad (h = 1,\ldots\ldots,n-2p).$$

We call such a frame $\{e_i\}$ an adapted frame. Then we can easily see that f and g have the following forms with respect to an adapted frame $\{e_i\}$.

(1,11)

$$f = \begin{pmatrix} 0 & -E_a & 0 \\ E_a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad g = \begin{pmatrix} E_a & 0 & 0 \\ 0 & E_a & 0 \\ 0 & 0 & E_a \end{pmatrix}$$

where E_a is the a × a-unt matrix.

We suppose now that there exists in each coordinate neighborhood a coordinate system in which an f-structure f has numerical components (1,11).

In this case, an f-structure f is said to be integrable.

We can easily prove the following

PROPOSITION 1.1. It is necessary and sufficient for an f-structure f to be integrable that $[f,f](X,Y)=0$.

where [f,f] is Nijenhuis tensor of f given by

$$[f,f](X,Y) = [fX,fY] - f[fX,Y] - f[X,fY] + f^2[X,Y].$$

2. Globally framed structure

Let M^n be a manifold with an f-structure of rank r. there exist n-r vector fields ξ_x spanning the distribution L and its dual 1-form η_x, where the indices x, y, z over the range {1,2,......,n-r}. Then we can put

$$(2,1)\quad m = \eta_x \times \xi_x, \qquad \eta_x(\xi_y) = \delta_{xy}$$

The summation convention being employed here and in the sequel Therefore, for any vector field X, we have

$$\ell X = f^2 X, \quad mX = \eta_x(X)\xi_x.$$

from which we get

$$(2;2)\quad f^2 = I - \eta_x \times \xi_x,$$

$$(2,3)\quad f\xi_x = 0, \qquad \eta_x \circ f = 0.$$

We assume now that, in a differentiable manifold admitting an f-structure of rank r. there exist globally (n-r)-frame $\{\xi_x\}$ and co-frame $\{\eta_x\}$.

The the set (f, ξ_x, η_x) is called an f-structure with complementary frame or globally framed structure. Next, let M^n be a manifold with a globally framed structure, then the manifold M^n admits a positive definite Riemannian metric g such that

$$(2,4) \qquad g(X, \xi_x) = \eta_x(X).$$

from (1,9) we have

$$(2,5) \qquad g(fX, fY) = g(X, Y) - \eta_x(X)\,\xi_x(Y).$$

for any vector fields X and Y on M.

If we put

$$(2,6) \qquad F(X, Y) = g(X, fY).$$

from (1,8) we get

$$(2,7) \qquad F(X, Y) = -F(Y, X).$$

which shows that F_{ij} is an anti-symmetric tensor.

Next, we shall introduce an almost complex structure J on product manifold $M^n \times \bar{M}^m$.

Let $M^n(f, \xi_x, \eta_x)$ and $\bar{M}^m(\bar{f}, \bar{\xi}_x, \bar{\eta}_x)$ be two globally framed manifolds of dimensions n,m and ranks r,s, respectively.

For any vector field $X_p \in TM^n_p$ and $\bar{X}_p \in T\bar{M}^m_p$.

We define a linear map of tangent space $T(M \times \bar{M})_{(p,\bar{p})}$ onto itself by

$$(2,8) \quad J(X, \bar{X}) = (fX - \bar{\eta}_x(\bar{X})\xi_x,\ \bar{f}\bar{X} + \eta_x(X)\bar{\xi}_x).$$

clearly we get

$$(2,9) \qquad J^2 = -(I, \bar{I}).$$

which shows that J is an almost complex structure.

Thus we have

PROPOSITION 2.1. Let $M(f, \xi_x, \eta_x)$ and $\bar{M}(\bar{f}, \xi_x, \bar{\eta}_x)$ be two globally framed manifolds. Then the product manifold $M \times \bar{M}$ has an almost complex structure defined by (2,8).

Now, since R^{n-r} has a trivial globally framed structure $(f, d/_{dt}x, dt^x)$, (t^x) being the coordinate in R^{n-r} we can introduce an almost product structure J on product manifold $M \times R^{n-r}$ as follows:

$$(2,10) \qquad J(X, \lambda^x d/_{dt}x) = (fX - \lambda^x \xi_x,\ \eta_x(X) d/_{dt}x).$$

Then we have

$$(2,11) \qquad J^2 = -I.$$

Thus we have

PROPOSITION 2.2 Let M be a globally framed manifold of rank r. Then the product manifold $M \times R^{n-r}$ has an almost complex structure J defined by (2,10).

DEFINITION 2.3. If the induced almost complex structure J on $M \times R^{n-r}$ is integrable, then the globally framed structure f on M is said to be normal.

Denoting by $N^h{}_{ji}$ the components of the Nijenhuis tensor $[J,J](X, Y)$, $N^h{}_{ji}$ is given by

$$(2,12) \quad N^h{}_{ij} = J^k_j \partial_k J^h_i - J^k_i \partial_k J^h_j - J^h_k(\partial_j J^k_i - \partial_i J^k_j)$$

where i,j,k,......run over the range {1,2,......,2n-r}.

Considering the Nijenhuis tensor [J, J] of J. They computed [J, J] (X+0, Y+0), [J, J] $(X+0, 0+d/_{dt})$ and [J, J] $(0+d/_{dt}x, 0+d/_{dt}y)$, which rise the five tensors N^1, N^2, N^3, N^4, N^5 given by

(2,13)

$$\begin{aligned}
N^1(X,Y) &= N^a{}_{bc} = [f, f](X, Y) + d\eta_x(X, Y)\,\xi_x, \\
N^2(X,Y) &= N^x{}_{bc} = (L_{fx}\,\eta_x)(Y) - (L_{fy}\,\eta_x)(X), \\
N^3(X,Y) &= N^a{}_{bx} = (L\xi_x\, f)(X), \\
N^4(X,U) &= N^x{}_{by} = -(L\xi_x\,\eta_y)(X), \\
N^5(U,V) &= N^a{}_{xy} = L\xi_x\,\xi_y,
\end{aligned}$$

for any vector field X and Y on M, U and V or R^{n-r}, where L_X denotes the Lie derivative with respect to X, the result is that J is integrable if and only if $N^1 = 0$.

PROPOSITION 2.4. It is necessary and sufficient for a globally framed structure (f, ξ_x, η_x) to be normal that the tensor $N^1 = 0$, that is

(2,14)

$$N^1(X,Y) = [f, f](X, Y) + d\eta_x(X, Y)\,\xi_x = 0.$$

moreover we can prove that if $N^1 = 0$, then $N^2 = N^3 = N^4 = N^5 = 0$.

§3. **f-submanifolds in an almost complex manifold**

Let W be an N-dimensional differentiable manifold of class C^∞ with an almost complex structure J, that is $J^2 = -(I, \bar{I})$.

Let there be given an n-dimensional submanifold V differentiably immersed in W, and denote by $T_p(V)$ the tangent space of at a point p of V and $r = \dim V_p^H$.

There be given a f-submanifold V in an almost complex space W. Then there exists a subspace N_p of $T_p^H(V)$ at each point of V such that

$$J(N_p) \subset T_p^H(V), \qquad T_p^H(V) = N_p(V) \oplus N_p .$$

In the tangent space $T_p(W)$ of the enveloping space W at a point p, there exists a subspace $\bar{N}_p$ such that

$$J(\bar{N}_p) = \bar{N}_p, \qquad T_p(W) = T_p^H(V) \oplus N_p .$$

The subspaces N_p and $\bar{N}_p$ are respectively (n-r)-dimensional and (N-2n+r)-dimensional. Therefore, there exist along V two fields of subspaces N_p and $\bar{N}_p$.

If we put

$$N(V) = \bigcup_{p\in V} N_p, \qquad \bar{N}(V) = \bigcup_{p\in V} \bar{N}_p .$$

Then N(V) and $\bar{N}$(V) are vector bundles over V. Letting N(V) and $\bar{N}$(V) be fixed, we call the set {V, N(V), $\bar{N}$(V)} in an almost complex space W and its base submanifold V be expressed in local coordinates (X^h) in W, by parametric equiation $X^h = X^h(u^a)$,

where (u^a) is a local coordinate system in V.

If we put

$$X_a^h = \partial_a X^h, \qquad \partial_a = \partial/\partial u^a$$

Then X_a^h are local tangent vector fields on V and span the tangent space $T_p(V)$ of V at each point p of V.

there exist n-r local vector fields C_y^h and N-2n+r local vector fields D_β^h along V which span respectively N_p and $\bar{N}_p$ at each point p of V, we put now

$$\begin{pmatrix} X_a^{h*1)} \\ C_x^h \\ D_\alpha^h \end{pmatrix}^{-1} = (X_i^b, C_i^y, D_i^\lambda)$$

Taking account of the fact that

$$J(N_p) \subset T_p(V), \qquad J(\bar{N}_p) = \bar{N}_p$$

We can put

$$\begin{aligned} (3,1) \qquad J_i^h X_b^i &= f_b^a X_a^h + f_b^x C_x^h \\ J_i^h C_y^i &= -f_y^a X_a^h \\ J_i^h D_\beta^i &= f_\beta^\alpha D_\alpha^h \end{aligned}$$

where J_i^h are the components of the almost complex structure J in W.

If we take account of $J^2 = -I$, we find easily

$$\begin{aligned} (3,2) \qquad & f_b^c f_c^a = -\delta_b^a + f_b^x f_x^a, \quad f_b^c f_c^x = 0, \quad f_y^c f_c^a = 0, \\ & f_y^c f_c^x = \delta_y^x, \quad f_\beta^r f_r^\alpha = -\delta_\beta^\alpha . \end{aligned}$$

which imply

$$f^3 + f = 0.$$

The f being tensor field of type (1,1) defined in V by the components f_b^a.

Thus f_b^a is an f-structure in V which is called the induced f-structure of the given f-surface {V, N(V), $\bar{N}$(V)}. There exist in V, n-r local vector fields f_y^a and n-r local covector fields f_b^x.

Let there be given a symmetric linear connection $\Gamma_j{}^h{}_i$ in the enveloping space W, If we put

$$\Gamma_c{}^a{}_b = (\partial_c X_b^h + X_c^j X_b^i \Gamma_j{}^h{}_i) X_h^{a*2)}$$

Then $\Gamma_c{}^a{}_b$ define a symmetric linear connection w in the base submanifold V, which is called the induced connection in V, If we put

$$\Gamma_c{}^x{}_y = (\partial_c C_y^h + X_c^j C_y^i \Gamma_j{}^h{}_i) C_h^x .$$

Then $\Gamma_c{}^x{}_y$ define a connection in the vector bundle N(V) and it is called the induced connection in N(V). We define the so-called Van der Waerden-Bortolotti covariant derivatives along V of X_b^h, C_y^h and D_β^h by

(3,3)
$$\nabla_c X_b^h = \partial_c X_b^h + X_c^j X_b^i \Gamma_j{}^h{}_i - X_a^h \Gamma_c{}^a{}_b ,$$
$$\nabla_c C_y^h = \partial_c C_y^h + X_c^j C_y^i \Gamma_j{}^h{}_i - C_x^h \Gamma_c{}^x{}_y ,$$
$$\nabla_c D_\beta^h = \partial_c D_\beta^h + X_c^j D_\beta^i \Gamma_j{}^h{}_i - D_\alpha^h \Gamma_c{}^\alpha{}_\beta .$$

respectively. Then $\nabla_c X_b^h$, $\nabla_c C_y^h$ and $\nabla_c D_\beta^h$ belong respectively to $N_p + \bar{N}_p$, $T_p(V) + \bar{N}_p$ and $T_p(V) + N_p$ at each point p of V, Thus we put

(3,4)
$$\nabla_c X_b^h = h_{cb}{}^x C_x^h + h_{cb}{}^\alpha D_\alpha^h ,$$
$$\nabla_c C_y^h = - h_c{}^a{}_y X_a^h + h_{cy}{}^\alpha D_\alpha^h ,$$
$$\nabla_c D_\beta^h = - h_c{}^a{}_\beta X_a^h - h_c{}^x{}_\beta C_x^h .$$

where h's are so-called second fundamental tensors of the given f-surface V. It is easily erified that

$$h_{cb}{}^x = h_{bc}{}^x \quad {}^{*4)}, \qquad h_{cb}{}^\alpha = h_{bc}{}^\alpha \quad {}^{*3)}$$

If we differentiate covariantly the both sides of (3,1) and take account of (3,3), we have

(3,5)
$$\nabla_c f_b^a + h_{cb}{}^y f_y^a - h_c{}^a{}_y f_b^y = 0$$
$$\nabla_c f_b^x + h_{ce}{}^x f_b^e = 0$$
$$\nabla_c f_y^a + h_c{}^e{}_y f_e^a = 0$$
$$\nabla_c f_\beta^\alpha = 0 .$$

and

(3,6)
$$h_c{}^e{}_y f_e^x - h_{ce}{}^x f_y^e = 0$$
$$h_{ce}{}^\alpha f_y^e - h_{cy}{}^\beta f_\beta^\alpha = 0$$
$$h_c{}^e{}_\beta f_e^a h_c{}^a{}_r f_\beta^r - h_c{}^y{}_\beta f_y^a = 0$$
$$h_{c\beta}{}^e f_e^x - h_c{}^x{}_r f_\beta^r = 0$$

where the covariant derivatives $\nabla_c f_b^a$, $\nabla_c f_b^x$, $\nabla_c f_y^a$ and $C_c f_\beta^\alpha$ are defined respectively by

$$\nabla_c f_b^a = \partial_c f_b^a + \Gamma_c{}^a{}_e f_b^e - \Gamma_c{}^d{}_b f_d^a ,$$
$$\nabla_c f_b^x = \partial_c f_b^x + \Gamma_c{}^x{}_y f_b^y - \Gamma_c{}^d{}_b f_d^x ,$$
$$\nabla_c f_y^a = \partial_c f_y^a + \Gamma_c{}^a{}_e f_y^e - \Gamma_c{}^x{}_y f_x^a ,$$
$$\nabla_c f_\beta^\alpha = \partial_c f_\beta^\alpha + \Gamma_c{}^\alpha{}_r f_\beta^r - \Gamma_c{}^r{}_\beta f_r^\alpha .$$

on the other hand, the Nijenhuis tensor $N_{ji}{}^h$ vanishes identically, that is $N_{ji}{}^h = 0$, which is equivalent to the condition:

(3,7)
$$S_{cb}{}^a = f_c^x (h_b{}^e{}_x f_e^a - f_b^e h_e{}^a{}_x) - f_b^x (h_c{}^e{}_x f_e^a - f_c^e h_e{}^a{}_x),$$
$$S_{cb}{}^x = f_c^y f_y^e h_{be}{}^x - f_b^y f_y{}^e h_{ce}{}^x ,$$
$$S_{cy}{}^a = -(h_c{}^a{}_y + h_e{}^d{}_y f_c^e f_d^a) + f_c^z (h_e{}^a{}_z f_y^e) ,$$
$$S_{cy}{}^x = - f_c^e f_y^d h_{ed}{}^x ,$$
$$S_{xy}{}^a = 0 .$$

where tensor S's are defined by

(3,8)
$$S_{cb}{}^a = N_{cb}{}^a + (\nabla_c f_b^x - \nabla_b f_c^x) f_x^a ,$$
$$S_{cb}{}^x = f_c^e (\nabla_e f_b^x - \nabla_b f_e^x) - f_b^e (\nabla_e f_c^x - \nabla_c f_e^x),$$
$$S_{cy}{}^a = f_y^e (\nabla_e f_c^a - f_c^x \nabla_e f_y^a + f_c^e \nabla_e f_y^a ,$$
$$S_{cy}{}^x = f_y^e (\nabla_e f_y^x - \nabla_c f_e^x) ,$$
$$S_{xy}{}^a = f_x^e \nabla_e f_y^a - f_y^e \nabla_e f_x^a .$$

$N_{cb}{}^a$ being the Nijenhuis tensor of the induced f-structure f in V. The first tensor $S_{cb}{}^a$ appearing above is nothing but the tensor S appearing in PROPOSITION 2.2. stated in §2. It is easily verified that all of the tensors S's vanishes identically if and only if the tensor S vanishes Sasaki, S (1960), Yano, K (1965). Now we have following;

THEOREM 3.1. For an f-submanifold of a complex space, the following the following three conditions are equivalent to each other;

1) $\nabla_c f_b^x = 0$,

2) $\nabla_c f_b^x = 0$ and $\nabla_c f_y^a = 0$

3) $h_{cb}{}^x = f_c^a f_b^y \lambda_{zy}{}^x$ and $h_c{}^a{}_y = f_c^z f_x^a \lambda_{zy}{}^x$

where $\lambda_{zy}{}^x$ being a certain tensor field such that $\lambda_{zy}{}^x = \lambda_{yz}{}^x$.

When one of three condition is satisfied, the induced f-structure f_b^a is integrable and $S_{cb}{}^a = 0$. Yano, K (1965).

PROOF)

1) ⇒ 2) by hypothesis $h_{cb}{}^y f_y^a - h_c{}^a{}_y f_b^y = 0$

Transvecting f_a^x to the both sides of (3,5) and taking use of (3,2) we get

$$h_{cb}{}^y f_y^a f_a^x - h_c{}^a{}_y f_b^y f_a^x = 0$$

$$h_{cb}{}^y \delta_y^x - h_c{}^a{}_y f_b^y f_a^x = 0$$

$$h_{cb}{}^x - h_c{}^a{}_y f_b^y f_a^x = 0 \qquad *)$$

Transvecting f_a^b to the both sides of *) and taking use of (3,2) we get

$$h_{cb}{}^x f_a^b - h_c{}^a{}_y f_b^y f_a^x f_a^b = 0$$

$$h_{cb}{}^x f_a^b = 0$$

$$\nabla_c f_b^x = 0 \qquad 1)$$

and transvecting f_x^b to the both sides of (3,5) and taking use of (3,2) we get

$$h_{cb}{}^y f_y^a f_x^b - h_c{}^a{}_y f_b^y f_x^b = 0$$

$$h_{cb}{}^y f_y^a f_x^b - h_c{}^a{}_y \delta_x^y = 0$$

$$h_{cb}{}^y f_y^a f_x^b - h_c{}^a{}_x = 0 \qquad **)$$

Transvecting f_a^b to the both sides of **) and taking use of (3,2) we get

$$h_{cb}{}^y f_y^a f_x^b - h_c{}^a{}_x f_a^b = 0$$

$$-h_c{}^a{}_x f_a^b = 0$$

$$\nabla_c f_x^b = 0$$

that is $\nabla_c f_y^a = 0$

from 1) and 2), 1) ⇒ 2) is proved.

2) ⇒ 3), by hypothesis $h_{ce}{}^x f_b^e = 0$

Transvecting f_e^c to the both sides of (3,5) and taking use of (3,2) we get

Transvecting g to the both sides of ***)

$$h_{ce}{}^x f_b^e f_c^e = 0$$

$$h_{ce}{}^x(-\delta_b^e + f_b^y f_y^e) = 0$$

$$-h_{be}{}^x + h_{ce}{}^x f_b^y f_y^e = 0$$

$$h_{be}{}^x = h_{ce}{}^x f_b^y f_y^e \qquad ***)$$

Transvecting g to the both sides of ***)

$$h_{be}{}^x g_{ce} g^{yz} = h_{ce}{}^x f_b^y f_y^e g_{ce} g^{yz}$$

$$h_{be}{}^x g_{ce} g^{yz} = h_{ce}{}^x f_b^y f_c^z$$

$$h_{be}{}^x = f_b^y f_c^z h_{ce}{}^x g^{ce} g_{yz}$$

$$h_{be}{}^x = f_b^y f_c^z \lambda_{zy}{}^x \qquad 3)$$

where $\lambda_{zy}{}^x = h_{ce}{}^x g^{ce} g_{yz}$.

and transvecting f_a^b to the both sides of (3,5) and taking use of (3,2) we get

$$h_{c\ y}^{\ e} f_e^a f_a^b = 0$$

$$h_{c\ y}^{\ e} (-\delta_e^b + f_e^z f_z^b) = 0$$

$$-h_{c\ y}^{\ b} = h_{c\ y}^{\ e} f_e^z f_z^b$$

Transvecting g to the both sides of ****)

$$h_{c\ y}^{\ b} g_{ce} g^{xz} = f_e^z f_z^b h_{c\ y}^{\ e} g_{ce} g^{xz}$$

$$h_{e\ y}^{\ b} g^{xz} = f_e^z f_z^b h_{c\ y}^{\ e} g_{ce} g^{xz}$$

$$h_{e\ y}^{\ b} = f_e^z f_z^b h_{c\ y}^{\ e} g_{ce} g^{xz}$$

$$h_{e\ y}^{\ b} = f_e^z f_z^b \lambda_{zy}^{\ x} \qquad 4)$$

where $\lambda_{zy}^{\ x} = h_{c\ y}^{\ e} g_{ce} g^{xz}$
from 3 and 4), 2) ⇒ 3) is proved.

3) ⇒ 1) by hypothesis

$$h_{cb}^{\ \ x} f_x^a - f_c^z f_b^y f_x^a \lambda_{zy}^{\ x} = 0 \qquad \#)$$

Transvecting f_x^a to the both sides of #) and taking use of the hypothesis

$$h_{cb}^{\ \ x} f_x^a - f_c^z f_b^y \lambda_{zy}^{\ x} = 0$$

$$h_{cb}^{\ \ x} f_x^a - h_{c\ y}^{\ a} f_b^y = 0 \qquad \#\#)$$

from ##) and taking use of (3,5)

$$\nabla_c f_b^a = 0, \qquad 3) \Rightarrow 1) \text{ is proved.}$$

This proves the theorem.

※ 1) The indices h, i, j, ……… run over runge { 1, 2, ……… ,N }.
2) The indices a, b, c, ……… run over range { 1, 2, ……… ,n }.
3) The indices α, β, γ ……… run over range { 2n－r+1,…… ,N }.
4) The indices x, y, z ……… run over range { n+1, …… ,2n－r }.

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一 國 文 抄 錄 一

W. Yano 는 $f^3 + f = 0$ 을 만족하는 (1, 1) 형의 텐사 f 에 의하여 정의되는 f－구조를 소개 하였다. 구후 H.Nakagawa, D.E.Blair, S.I.Goldberg 는 보조표구를 갖는 f－구조를 연구했다.

이 논문의 목적은 f－구조와 대역적 표구 구조를 갖는 다양체를 소개하고, 그 기하학을 연구하는데 있다.

§1. 에서 우리는 f－구조를 정의하고, 그것의 적분 가능조건을 구한다. 개복소 구조의 개접촉 구조는 잘 알려진 f－구조의 예이다. f－구조의 존재성은 접번들의 구조군의 u(r) × o(n－r) 되는 것과 동치이다.

§2. 에서 우리는 보조 표구를 갖는 f－구조를 소개하고, 이것의 정규성을 정의하고, 정규조건이 $N^1 = 0$ 임을 밝힌다.

§3. 에서 우리는 개복소 다양체의 부분 다양체 상에 유도되는 f－구조를 조사하고 $\nabla_c f_b^a$ 와 동치가 되는 조건을 연구한다.