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FIXED POINT THEOREMS ON CONE METRIC SPACES WITH c-DISTANCE

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FIXED POINT THEOREMS ON CONE METRIC SPACES WITH c-DISTANCE

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< **Abstact**>

Fixed Point Theorems On Cone Metric Spaces With $c-$distance

Hong Joon Choi

Let X be an arbitrary nonempty set and $f : X \to X$ be a mapping. A fixed point for f is a point $x \in X$ such that $fx = x$. Fixed point theory is one of the most powerful and fruitful tools of modern mathematics and may be considered a core subject of nonlinear analysis started by Banach in 1922. After that a considerable amount of research work for the development of fixed point theory have been executed by several authors.

Banach has proved the fixed point theorem for a single-valued mapping in the setting of a complete metric space known as the Banach contraction principle.

Huang and Zhang([5]) introduced the cone metric space which is more general than the concept of a metric space and obtained some fixed point theorems in that space. After that, a series of articles have been dedicated to the improvement of fixed point theory. Also Cho et al.([3]) introduced the $c-$distance in a cone metric space and obtained some fixed point results.

The idea of common fixed point was initially given by Jungck([6]), and Wang and Guo([12]) proved common fixed points results for two self mappings in a cone metric space under $c-$distance.

In this paper, we obtain sufficient conditions for existence of a unique coincidence point and a common fixed point for a pair of self mappings and give an example satisfying the sufficient conditions of our result. Also we obtain sufficient conditions for existence of fixed points for a nondecreasing continuous mapping on a partially ordered set satisfying contractive conditions in a cone metric space using $c-$distance.

1 Introduction

Let X be an arbitrary nonempty set and $f : X \to X$ be a mapping. A fixed point for f is a point $x \in X$ such that $fx = x$. Fixed point theory is one of the most powerful and fruitful tools of modern mathematics and may be considered a core subject of nonlinear analysis started by Banach in 1922. After that a considerable amount of research work for the development of fixed point theory have been executed by several authors.

Banach has proved the fixed point theorem for a single-valued mapping in the setting of a complete metric space known as the Banach contraction principle. The famous Banach contraction principle states that if (X, d) is a complete metric space and $f : X \to X$ is a contraction mapping (i.e., $d(fx, fy) \leq cd(x, y)$ for all $x, y \in X$, where c is a nonnegative number such that $c \in [0, 1)$), then f has a unique fixed point. As a classical example, it is well known that every continuous function $f : [0, 1] \to [0, 1]$ has a fixed point and Brouwer generalized it like this: If $f : D^n \to D^n$ is continuous where $D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, then f has a fixed point. This contraction principle has further several generalizations in metric spaces as well as in cone metric spaces.

Huang and Zhang([5]) introduced the cone metric space which is more general than the concept of a metric space and obtained some fixed point theorems in that space. After that, a series of articles have been dedicated to the improvement of fixed point theory. In most of those articles, the authors used normality property of cones in their results. Also Cho et al.([3]) introduced the $c-$distance in a cone metric space which is a cone version of the $w-$distance of Kada et al.([7]) and obtained some fixed point results.

The idea of common fixed point was initially given by Jungck([6]). Afterwards, many generalizations of this common fixed point result under a variety of settings were obtained by several mathematicians. In 2011, Wang and Guo([12]) proved common fixed points results for two self mappings in a cone metric space under $c-$distance.

In this paper, we obtain sufficient conditions for existence of a unique coincidence point and a common fixed point for a pair of self mappings as well as fixed points for a nondecreasing continuous mapping on a partially ordered set satisfying contractive conditions in a cone metric space using $c-$distance.

In this section we need to recall some basic notations, definitions, and necessary results from existing literature. Let E be a real Banach space and θ denote the zero element in E. A *cone* P is a subset of E such that

(i) P is closed, nonempty and $P \neq \{\theta\}$;

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \;\Rightarrow\; ax + by \in P$;

(iii) $P \cap (-P) = \{\theta\}$ i.e, $x \in P$ and $-x \in P$ imply $x = \theta$.

For any cone $P \subseteq E$, the partial ordering $\preceq$ with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. The notation of $\prec$ stands for $x \preceq y$ but $x \neq y$. Also, we used $x \ll y$ to indicate that $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. A cone P is called *normal* if there exists a number K such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \quad \text{implies} \quad \|x\| \leq K\|y\|. \tag{1.1}$$

Equivalently, the cone P is normal if

$$x_n \preceq y_n \preceq z_n \text{ and } \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \text{ imply } \lim_{n\to\infty} y_n = x \tag{1.2}$$

The least positive number K satisfying condition (2.1) is called the *normal constant* of P.

Example 1.1 *([4]) Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0\}$. This cone is nonnormal. For example, consider $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$. Then $\theta \preceq x_n \preceq y_n$ and $y_n \to \theta$ as $n \to \infty$. but*

$$\|x_n\| = \max_{t\in[0,1]} |\frac{t^n}{n}| + \max_{t\in[0,1]} |t^{n-1}| = \frac{1}{n} + 1 > 1.$$

Hence x_n does not converge to zero and hence P is a nonnormal cone.

Definition 1.2 *Let X be a nonempty set and let E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone $P \subseteq E$. Suppose the mapping $d : X \times X \to E$ satisfies the following conditions:*

(1) $\theta \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(3) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space.

Definition 1.3 *Let (X,d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.*

(1) If for every $c \in E$ with $\theta \ll c$, there exists a natural number N such that $d(x_n, x) \ll c$ for all $n > N$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and the point x is the limit of $\{x_n\}$. We denote this by

$$\lim_{n\to\infty} x_n = x \quad or \quad x_n \to x \quad (n \to \infty).$$

(2) If for all $c \in E$ with $\theta \ll c$, there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $m, n > N$, then $\{x_n\}$ is called a Cauchy sequence in X.

(3) A cone metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent.

Here we point to some elementary results.

Lemma 1.4 *([10]) Let E be a real Banach space with a cone P. Then*

(1) If $a \ll b$ and $b \ll c$, then $a \ll c$.

(2) If $a \preceq b$ and $b \ll c$, then $a \ll c$.

Lemma 1.5 *([10]) Let E be a real Banach space with cone P. Then*

(1) If $\theta \ll c$, then there exists $\delta > 0$ such that $\|b\| < \delta$ implies $b \ll c$.

(2) If $\{a_n\}, \{b_n\}$ are sequences in E such that $a_n \to a, b_n \to b$ and $a_n \preceq b_n$ for all $n \geq 1$, then $a \preceq b$.

Proof. (1) Since $\theta \ll c$, we have $c \in \text{int}P$. Hence, we find $\delta > 0$ such that

$$\{b \in E : \|b - c\| < \delta\} \subseteq \text{int}P.$$

If $\|b\| < \delta$, then $\|(c - b) - c\| = \| - b\| = \|b\| < \delta$ and hence $(c - b) \in \text{int}P$.

(2) $a_n \preceq b_n$ implies $b_n - a_n \in P$. Since P is closed and $b_n - a_n \to b - a$, $b - a \in P$. □

Lemma 1.6 *([5]) Let (X, d) be a cone metric space, P a normal cone, $x \in X$ and $\{x_n\}$ a sequence in X. Then*

(1) $\{x_n\}$ converges to x if and only if $d(x_n, x) \to \theta$.

(2) The limit point of every sequence is unique.

(3) Every convergent sequence is a Cauchy sequence.

(4) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$ as $n, m \to \infty$.

(5) If $x_n \to x$ and $y_n \to y$ then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.

Proof. (1) Suppose that $\{x_n\}$ converges to x. For every $\epsilon > 0$, choose $c \in E$ with $\theta \ll c$ and $K\|c\| < \epsilon$. Then there exists a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$. If $n > N$, then $\|d(x_n, x)\| \leq K\|c\| < \epsilon$. This means $d(x_n, x) \to \theta$ $(n \to \infty)$.

Conversely, suppose that $d(x_n, x) \to \theta$ $(n \to \infty)$. For $c \in E$ with $\theta \ll c$, there exists $\delta > 0$ such that $\|x\| < \delta$ implies $c - x \in \text{int}\, P$ by Lemma 1.5(1) . For this δ there is a positive integer N such that $\|d(x_n, x)\| < \delta$ for all $n > N$. So $c - d(x_n, x) \in \text{int}\, P$. This means $d(x_n, x) \ll c$. Therefore $\{x_n\}$ converges to x.

(2) Suppose that $\{x_n\}$ converge to x and y. Then for any $c \in E$ with $\theta \ll c$, there exists a positive integer N such that for all $n > N$, $d(x_n, x) \ll c$ and $d(x_n, y) \ll c$. We have $d(x, y) \preceq d(x_n, x) + d(x_n, y) \preceq 2c$. Hence $\|d(x, y)\| \leq 2K\|c\|$. Since c is arbitrary $d(x, y) = \theta$ and so $x = y$.

(3) Suppose that $\{x_n\}$ converge to x. Then for any $c \in E$ with $\theta \ll c$, there exists a positive integer N such that for all $n, m > N$, $d(x_n, x) \ll \frac{c}{2}$ and $d(x_m, x) \ll \frac{c}{2}$. Hence

$$d(x_n, x_m) \preceq d(x_n, x) + d(x_m, x) \ll c.$$

Therefore $\{x_n\}$ is a Cauchy sequence.

(4) Suppose that $\{x_n\}$ is a Cauchy sequence. For every $\epsilon > 0$, choose $c \in E$ with $\theta \ll c$ and $K\|c\| < \epsilon$. Then there exists a positive integer N such that for all $n, m > N$, $d(x_n, x_m) \ll c$. Thus $n, m > N$ implies $\|d(x_n, x_m)\| \leq K\|c\| < \epsilon$. This means $d(x_n, x_m) \to \theta$ $(n, m \to \infty)$.

Conversely, suppose that $d(x_n, x_m) \to \theta$ $(n, m \to \infty)$. For $c \in E$ with $\theta \ll c$, there exists $\delta > 0$ such that $\|x\| < \delta$ implies $c - x \in \operatorname{int} P$. For this δ there exists a positive integer N, such that for all $n, m > N$, $\|d(x_n, x_m)\| < \delta$. So $c - d(x_n, x_m) \in \operatorname{int} P$. This means $d(x_n, x_m) \ll c$. Therefore $\{x_n\}$ is a Cauchy sequence.

(5) For every $\epsilon > 0$, choose $c \in E$ with $\theta \ll c$ and $\|c\| < \frac{\epsilon}{4K+2}$. From $x_n \to x$ and $y_n \to y$, there exists N such that for all $n > N$, $d(x_n, x) \ll c$ and $d(y_n, y) \ll c$. We have

$$d(x_n, y_n) \preceq d(x_n, x) + d(x, y) + d(y_n, y) \preceq d(x, y) + 2c,$$

$$d(x, y) \preceq d(x_n, x) + d(x_n, y_n) + d(y_n, y) \preceq d(x_n, y_n) + 2c.$$

Hence $\theta \preceq d(x, y) + 2c - d(x_n, y_n) \preceq 4c$ and

$$\|d(x_n, y_n) - d(x, y)\| \leq \|d(x, y) + 2c - d(x_n, y_n)\| + \|2c\| \leq (4K + 2)\|c\| < \epsilon.$$

Therefore $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$. □

Lemma 1.7 *([10]) If E is a real Banach space with cone P. Then*

(1) If $a \preceq \lambda a$ where $a \in P$ and $0 < \lambda < 1$ then $a = \theta$.

(2) If $c \in \operatorname{int} P$, $\theta \preceq a_n$ and $a_n \to \theta$, then there exists a positive integer N such that $a_n \ll c$ for all $n \geq N$.

Proof. (1) The condition $a \preceq \lambda a$ means that $\lambda a - a \in P$ that is, $-(1-\lambda)a \in P$. Since $a \in P$ and $1 - \lambda > 0$, then also $(1 - \lambda)a \in P$. Thus we have

$$(1 - \lambda)a \in P \cap (-P) = \{\theta\}$$

and so $a = 0$.

(2) Let $\theta \ll c$ be given. Choose a symmetric neighborhood V such that $c + V \subseteq P$. Since $a_n \to \theta$, there exists a positive integer n_0 such that $a_n \in V = -V$ for $n > n_0$. This means that $c \pm a_n \in c + V \subseteq P$ for $n > n_0$, that is, $a_n \ll c$. □

Definition 1.8 *Let (X, d) be a cone metric space. Then a mapping $q : X \times X \to E$ is called a $c-$distance on X if the following are satisfied :*

(q1) $\theta \preceq q(x, y)$ for all $x, y \in X$.

(q2) $q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$.

(q3) for all $x \in X$ and all $n \geq 1$, if $q(x, y_n) \preceq u$ for some $u = u_x \in P$, then $q(x, y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$.

(q4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that

$$q(z, x) \ll e \text{ and } q(z, y) \ll e \text{ imply } d(x, y) \ll c.$$

Example 1.9 *([3]) Let (X, d) be a cone metric space and let P be a normal cone. Put*

$$q(x, y) = d(x, y)$$

for all $x, y \in X$. Then q is a $c-$distance.

Proof. (q1) and (q2) are immediate. Lemma 1.6(5) shows that (q3) holds. Let $c \in E$ with $\theta \ll c$ be given and put $e = \frac{c}{2}$. Suppose that $q(z, x) \ll e$ and $q(z, y) \ll e$. Then

$$d(x, y) = q(x, y) \preceq q(x, z) + q(z, y) \ll e + e = c.$$

This shows that q satisfies (q4) and hence q is a $c-$distance. □

Example 1.10 *([3]) Let (X, d) be a cone metric space and let P be a normal cone. Put*

$$q(x, y) = d(u, y)$$

for all $x, y \in X$, where $u \in X$ is constant. Then q is a $c-$distance.

Proof. (q1) and (q3) are immediate. Since

$$d(u, z) \preceq d(u, y) + d(u, z) \ \ i.e., \ \ q(x, z) \preceq q(x, y) + q(y, z),$$

(q2) holds. Let $c \in E$ with $\theta \ll c$ and put $e = \frac{c}{2}$. If $q(z, x) \ll e$ and $q(z, y) \ll e$, then we have

$$\begin{aligned} d(x, y) &\preceq d(x, u) + d(u, y) = d(u, x) + d(u, y) \\ &= q(z, x) + q(z, y) \ll e + e = c. \end{aligned}$$

This shows that q satisfies (q4) and hence q is a $c-$distance. □

Example 1.11 *([3]) Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a $c-$distance.*

Proof. (q1) and (q3) are immediate. From

$$z = q(x, z) \preceq q(x, y) + q(y, z) = y + z,$$

it follows that (q2) holds. From

$$d(x, y) = |x - y| \preceq x + y = q(z, x) + q(z, y),$$

it follows that (q4) holds. Hence q is a $c-$distance. □

Example 1.12 *([4]) Let $X = [0, 1]$. In the example 1.1, a cone metric d on X is defined by $d(x, y)(t) = |x - y|f(t)$ where $f \in P$ is an arbitrary function (e.g., $f(t) = e^t$). Define a mapping $q : X \times X \to E$ by*

$$q(x, y)(t) = y \cdot e^t$$

for all $x, y \in X$. It is easy to see that q is a $c-$distance on X.

Remark 1.13 *(1) $q(x, y) = q(y, x)$ does not necessarily hold for all $x, y \in X$.*
(2) $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

Lemma 1.14 *([3]) Let (X, d)be a cone metric space and let q be a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$be sequences in X and $x, y, z \in X$. Suppose that $\{u_n\}$ is a sequence in P converging to θ. Then the following hold:*
(1) If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $y = z$.
(2) If $q(x_n, y_n) \preceq u_n$and $q(x_n, z) \preceq u_n$, then $\{y_n\}$ converges to z.
(3) If $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.
(4) If $q(y, x_n) \preceq u_n$, then $\{x_n\}$is a Cauchy sequence in X.

Proof. We first prove (2). Let $c \in E$ with $\theta \ll c$. Then there exists $\delta > 0$ such that $c - x \in \text{int}P$ for any $x \in P$ with $\|x\| < \delta$ by Lemma 1.5(1). Since $\{u_n\}$ converges to θ, there exists a positive integer N such that $\|u_n\| < \delta$ for all $n \geq N$ and so $c - u_n \in \text{int}P$, i.e., $u_n \ll c$ for all $n \geq N$. Hence by (q4) with $e = c$, from $q(x_n, y_n) \ll c$ and $q(x_n, z) \ll c$, it follows that $d(y_n, z) \ll c$ for all $n \geq N$. This shows that $\{y_n\}$ converges to z.

From (2) it is obvious that (1) holds.

Now, we prove (3). Let $c \in E$ with $\theta \ll c$ be given. As in the proof of (2), choose $e \in E$ with $\theta \ll e$. Then there exists a positive integer n_0 such that

$$q(x_n, x_{n+1}) \ll e, \quad q(x_n, x_m) \ll e$$

for any $m > n \geq n_0$ and hence $d(x_{n+1}, x_m) \ll c$. This implies that $\{x_n\}$ is a Cauchy sequence in X. As in the proof of (3), we can prove (4). This completes the proof. □

Definition 1.15 *Let T and S be self mappings of a set X. If $y = Tx = Sx$ for some $x \in X$, then x is called a coincidence point of T and S and y is called a point of coincidence of T and S.*

Definition 1.16 *The mappings $T, S : X \to X$ are weakly compatible if for every $x \in X$, the following holds:*

$$T(Sx) = S(Tx) \quad \textit{whenever} \quad Sx = Tx.$$

Definition 1.17 *The mapping $T : X \to X$ is continuous if $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} Tx_n = Tx$.*

2 Common fixed point results on cone metric spaces with normal cone

Theorem 2.1 *([12]) Let (X,d) be a cone metric space. Let P be a normal cone with normal constant K and let q be a $c-$distance on X. Let $f : X \to X$ and $g : X \to X$ be two self mappings such that $f(X) \subseteq g(X)$ and $g(X)$ be a complete subset of X.*

Suppose that there exist nonnegative constants $a_i\,(i = 1, 2, 3, 4)$ are nonegative real numbers with $a_1 + a_2 + a_3 + 2a_4 < 1$ such that the following contractive condition holds for all $x, y \in X$:

$$q(fx, fy) \preceq a_1 q(gx, gy) + a_2 q(gx, fx) + a_3 q(gy, fy) + a_4 q(gx, fy)$$

and that

$$\inf\{\|q(gx, y)\| + \|q(fx, y)\| + \|q(gx, fx)\| : x \in X\} > 0$$

for all $y \in X$ with $y \neq fy$ or $y \neq gy$.

Then f and g have a common fixed point in X.

Theorem 2.2 *Let (X, d) be a cone metric space, P be a normal cone with normal constant K and q be a $c-$distance on X. Let $f, g : X \to X$ be two self mappings such that $f(X) \subseteq g(X)$ and $g(X)$ be a complete subset of X. Suppose that there exist nonnegative constants $a_i \in [0, 1), i = 1, 2, 3, 4, 5$ with $a_1 + 2a_2 + 2a_3 + 3a_4 + a_5 < 1$ such that the following contractive condition holds for all $x, y \in X$:*

$$q(fx, fy) \preceq a_1 q(gx, gy) + a_2 q(gx, fx) + a_3 q(gy, fy) + a_4 q(gx, fy) + a_5 q(gy, fx)$$

and that

$$\inf\{\|q(gx, y)\| + \|q(fx, y)\| + \|q(gx, fx)\| \,:\, x \in X\} > 0$$

for all $y \in X$ with y is not a point of coincidence of f and g. Then f and g have a unique point of coincidence in X.

Moreover if f and g are weakly compatible then f and g have a unique common fixed point in X.

Proof. Let $x_0, x_1 \in X$. Using the fact that $f(X) \subseteq g(X)$, construct $\{x_{2n}\}, \{x_{2n+1}\}$

such that $gx_{2n} = fx_{2n-2}$ and $gx_{2n+1} = fx_{2n-1}$ $(n \in \mathbb{N})$. Then we have

$$\begin{aligned}
q(gx_{2n}, gx_{2n+1}) &= q(fx_{2n-2}, fx_{2n-1}) \\
&\preceq a_1 q(gx_{2n-2}, gx_{2n-1}) + a_2 q(gx_{2n-2}, fx_{2n-2}) + a_3 q(gx_{2n-1}, fx_{2n-1}) \\
&+ a_4 q(gx_{2n-2}, fx_{2n-1}) + a_5 q(gx_{2n-1}, fx_{2n-2}) \\
&= a_1 q(gx_{2n-2}, gx_{2n-1}) + a_2 q(gx_{2n-2}, gx_{2n}) \\
&+ a_3 q(gx_{2n-1}, gx_{2n+1}) + a_4 q(gx_{2n-2}, gx_{2n+1}) + a_5 q(gx_{2n-1}, gx_{2n}) \\
&\preceq a_1 q(gx_{2n-2}, gx_{2n-1}) + a_2 \{q(gx_{2n-2}, gx_{2n-1}) + q(gx_{2n-1}, gx_{2n})\} \\
&+ a_3 \{q(gx_{2n-1}, gx_{2n}) + q(gx_{2n}, gx_{2n+1})\} \\
&+ a_4 \{q(gx_{2n-2}, gx_{2n-1}) + q(gx_{2n-1}, gx_{2n}) + q(gx_{2n}, gx_{2n+1})\} \\
&+ a_5 q(gx_{2n-1}, gx_{2n}).
\end{aligned}$$

Hence

$$\begin{aligned}
q(gx_{2n}, gx_{2n+1}) &\preceq \frac{a_2 + a_3 + a_4 + a_5}{1 - a_3 - a_4} q(gx_{2n-1}, gx_{2n}) \\
&+ \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} q(gx_{2n-2}, gx_{2n-1}).
\end{aligned} \tag{2.1}$$

Similarly,

$$\begin{aligned}
q(gx_{2n-1}, gx_{2n}) &\preceq \frac{a_2 + a_3 + a_4 + a_5}{1 - a_3 - a_4} q(gx_{2n-2}, gx_{2n-1}) \\
&+ \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} q(gx_{2n-3}, gx_{2n-2}).
\end{aligned} \tag{2.2}$$

Clearly $0 \leq \frac{a_2+a_3+a_4+a_5}{1-a_3-a_4} < 1$ and $0 \leq \frac{a_1+a_2+a_4}{1-a_3-a_4} < 1$. Set

$$b_1 = \alpha = \frac{a_2 + a_3 + a_4 + a_5}{1 - a_3 - a_4} \quad \text{and} \quad c_1 = \beta = \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4}.$$

Applying (2.1) and (2.2) and putting $b_2 = c_1 + \alpha b_1 = \beta + \alpha b_1, c_2 = \beta b_1$,

$$\begin{aligned}
q(gx_{2n}, gx_{2n+1}) &\preceq b_1 q(gx_{2n-1}, gx_{2n}) + c_1 q(gx_{2n-2}, gx_{2n-1}) \\
&\preceq b_2 q(gx_{2n-2}, gx_{2n-1}) + c_2 q(gx_{2n-3}, gx_{2n-2}) \\
&\vdots \\
&\preceq b_{2n-1} q(gx_1, gx_2) + c_{2n-1} q(gx_0, gx_1),
\end{aligned} \tag{2.3}$$

where $b_{2n-1} = \beta b_{2n-3} + \alpha b_{2n-2}$ and $c_{2n-1} = \beta b_{2n-2}$. Similarly

$$q(gx_{2n-1}, gx_{2n}) \preceq b_{2n-2} q(gx_1, gx_2) + c_{2n-2} q(gx_0, gx_1) \tag{2.4}$$

where $b_{2n-2} = \beta b_{2n-4} + \alpha b_{2n-3}$ and $c_{2n-2} = \beta b_{2n-3}$. From (2.3) and (2.4),

$$q(gx_{n+1}, gx_{n+2}) \preceq b_n q(gx_1, gx_2) + c_n q(gx_0, gx_1)$$

where $b_n = \beta b_{n-2} + \alpha b_{n-1}$ and $c_n = \beta b_{n-1}$.

Consider

$$b_{n+2} = \alpha b_{n+1} + \beta b_n \quad (0 \le \alpha, \beta < 1, b_1, b_2 \ge 0).$$

Then $b_n \ge 0$ for all $n \in \mathbb{N}$. Its characteristic equation is that $t^2 - \alpha t - \beta = 0$. If $1-\alpha-\beta > 0$ and $1+\alpha-\beta > 0$ then it has two roots t_1, t_2 such that $-1 < t_1 \le 0 \le t_2 < 1$. Also the hypothesis $a_1+2a_2+2a_3+3a_4+a_5 < 1$ implies $1-\alpha-\beta > 0$ and $1+\alpha-\beta > 0$. For such t_1, t_2, $b_n = k_1(t_1)^n + k_2(t_2)^n$ for some $k_1, k_2 \in \mathbb{R}$.

Let $m > n \ge 1$. It follows that

$$\begin{aligned}
q(gx_n, gx_m) &\preceq q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_{n+2}) + \cdots + q(gx_{m-1}, gx_m) \\
&\preceq (b_{n-1} + b_n + \cdots + b_{m-2})q(gx_1, gx_2) + (c_{n-1} + c_n + \cdots + c_{m-2})q(gx_0, gx_1) \\
&\preceq \{k_1(t_1^{n-1} + t_1^n + \cdots + t_1^{m-2}) + k_2(t_2^{n-1} + \cdots + t_2^{m-2})\}q(gx_1, gx_2) \\
&+ \beta\{k_1(t_1^{n-2} + \cdots + t_1^{m-3}) + k_2(t_2^{n-2} + \cdots + t_2^{m-3})\}q(gx_0, gx_1) \\
&\preceq (\frac{k_1 t_1^{n-1}}{1-t_1} + \frac{k_2 t_2^{n-1}}{1-t_2})q(gx_1, gx_2) + \beta(\frac{k_1 t_1^{n-2}}{1-t_1} + \frac{k_2 t_2^{n-2}}{1-t_2})q(gx_0, gx_1) \\
&\to \theta
\end{aligned}$$

as $n \to \infty$. Therefore $\{gx_n\}$ is a Cauchy sequence in $g(X)$ by Lemma 1.14 (3). Since $g(X)$ is complete, there exists $x' \in g(X)$ such that $gx_m \to x'$ as $m \to \infty$. By definition 1.8(q3)

$$q(gx_n, x') \preceq (\frac{k_1 t_1^{n-1}}{1-t_1} + \frac{k_2 t_2^{n-1}}{1-t_2})q(gx_1, gx_2) + \beta(\frac{k_1 t_1^{n-2}}{1-t_1} + \frac{k_2 t_2^{n-2}}{1-t_2})q(gx_0, gx_1)$$

Since P is a normal cone with normal constant K, we have

$$\begin{aligned}
\|q(gx_n, gx_m)\| &\le K\|(\frac{k_1 t_1^{n-1}}{1-t_1} + \frac{k_2 t_2^{n-1}}{1-t_2})q(gx_1, gx_2) + \beta(\frac{k_1 t_1^{n-2}}{1-t_1} + \frac{k_2 t_2^{n-2}}{1-t_2})q(gx_0, gx_1)\| \\
&\le K(\frac{k_1 t_1^{n-1}}{1-t_1} + \frac{k_2 t_2^{n-1}}{1-t_2})\|q(gx_1, gx_2)\| + K\beta(\frac{k_1 t_1^{n-2}}{1-t_1} + \frac{k_2 t_2^{n-2}}{1-t_2})\|q(gx_0, gx_1)\| \\
&\to 0
\end{aligned}$$

as $n \to \infty$. Also

$$\begin{aligned}
\|q(gx_n, x')\| &\le K\|(\frac{k_1 t_1^{n-1}}{1-t_1} + \frac{k_2 t_2^{n-1}}{1-t_2})q(gx_1, gx_2) + \beta(\frac{k_1 t_1^{n-2}}{1-t_1} + \frac{k_2 t_2^{n-2}}{1-t_2})q(gx_0, gx_1)\| \\
&\le K(\frac{k_1 t_1^{n-1}}{1-t_1} + \frac{k_2 t_2^{n-1}}{1-t_2})\|q(gx_1, gx_2)\| + K\beta(\frac{k_1 t_1^{n-2}}{1-t_1} + \frac{k_2 t_2^{n-2}}{1-t_2})\|q(gx_0, gx_1)\| \\
&\to 0
\end{aligned}$$

as $n \to \infty$.

Suppose that x' is not a point of coincidence of f and g. Then by assumption,

$$\begin{aligned} 0 &< \inf\{\|q(gx, x')\| + \|q(fx, x')\| + \|q(gx, fx)\| \,:\, x \in X\} \\ &\leq \inf\{\|q(gx_n, x')\| + \|q(fx_n, x')\| + \|q(gx_n, fx_n)\| \,:\, n \in \mathbb{N}\} \\ &= \inf\{\|q(gx_n, x')\| + \|q(gx_{n+2}, x')\| + \|q(gx_n, gx_{n+2})\| \,:\, x \in \mathbb{N}\} \\ &= 0 \end{aligned}$$

which is a contradiction. Therefore x' is a point of coincidence of f and g. So there exists $x \in X$ such that $fx = gx = x'$. If there exists $w \in X$ such that $fy = gy = w$ for some $y \in X$,

$$\begin{aligned} q(x', x') &= q(fx, fx) \\ &\preceq a_1 q(gx, gx) + a_2 q(gx, fx) + a_3 q(gx, fx) + a_4 q(gx, fx) + a_5 q(gx, fx) \\ &= (a_1 + a_2 + a_3 + a_4 + a_5) q(x', x'). \end{aligned}$$

Hence

$$q(x', x') = \theta. \tag{2.5}$$

Similarly

$$q(w, w) = \theta. \tag{2.6}$$

Now by (2.5) and (2.6)

$$\begin{aligned} q(x', w) &= q(fx, fy) \\ &\preceq a_1 q(gx, gy) + a_2 q(gx, fx) + a_3 q(gy, fy) + a_4 q(gx, fy) + a_5 q(gy, fx) \\ &= a_1 q(x', w) + a_2 q(x', x') + a_3 q(w, w) + a_4 q(x', w) + a_5 q(w, x') \\ &= (a_1 + a_4) q(x', w) + a_5 q(w, x'). \end{aligned}$$

Similarly $q(w, x') \preceq (a_1 + a_4) q(w, x') + a_5 q(x', w)$. Thus

$$q(x', w) + q(w, x') \preceq (a_1 + a_4 + a_5)\{q(x', w) + q(w, x')\}.$$

Therefore $q(x', w) + q(w, x') = \theta$ which implies

$$q(x', w) = q(w, x') = \theta. \tag{2.7}$$

By (2.6),(2.7) and Lemma 1.14(1), $x' = w$. Consequently x' is a unique point of coincidence of f and g.

Moreover if f and g are weakly compatible,

$$gx' = ggx = gfx = fgx = fx'$$

which implies gx' is a point of coincidence of f and g. By uniqueness of the point of coincidence , $fx' = gx' = x'$. In other words, x' is the unique common fixed point of f and g. □

Corollary 2.3 *Let $(X,\ d)$ be a complete cone metric space and let P be a normal cone with normal constant K and q be a c-distance on X. Let $f : X \to X$ be a self mapping . Suppose that there exist nonnegative constants $a_i \in [0,1)$, $i = 1,2,3,4,5$ with $a_1 + 2a_2 + 2a_3 + 3a_4 + a_5 < 1$ such that the following contractive condition holds for all $x, y \in X$:*

$$q(fx, fy) \preceq a_1 q(x,y) + a_2 q(x, fx) + a_3 q(y, fy) + a_4 q(x, fy) + a_5 q(y, fx)$$

and that

$$\inf\{\|q(x,y)\| + \|q(fx,y)\| + \|q(x,fx)\| \,:\, x \in X\} > 0$$

if $fy \neq y$. Then f has a unique fixed point in X.

Proof. Take $g = I$ in the above theorem. □

Corollary 2.4 *Let $(X,\ d)$ be a complete cone metric space and let P be a normal cone with normal constant K and q be a c-distance on X. Let $f : X \to X$ be a continuous self mapping . Suppose that there exist nonnegative constants $a_i \in [0,1)$, $i = 1,2,3,4$ with $a_1 + 2a_2 + 2a_3 + 3a_4 < 1$ such that*

$$q(fx, fy) \preceq a_1 q(x,y) + a_2 q(x, fx) + a_3 q(y, fy) + a_4 q(x, fy).$$

Then f has a unique fixed point in X.

Proof. Assume there exists $y \in X$ such that $fy \neq y$ and

$$\inf\{\|q(x,y)\| + \|q(fx,y)\| + \|q(x,fx)\| \,:\, x \in X\} = 0.$$

Then we can construct $\{x_n\}$ in X such that

$$\inf\{\|q(x_n,y)\| + \|q(fx_n,y)\| + \|q(x_n,fx_n)\| \,:\, n \in \mathbb{N}\} = 0.$$

Hence

$$q(x_n, y) \to \theta, \quad q(fx_n, y) \to \theta, \quad q(x_n, fx_n) \to \theta.$$

By Lemma 1.14(2), $fx_n \to y$. By the contractive condition, we have

$$\begin{aligned} q(fx_n, f^2x_n) &\preceq a_1 q(x_n, fx_n) + a_2 q(x_n, fx_n) + a_3 q(fx_n, f^2x_n) + a_4 q(x_n, f^2x_n) \\ &\preceq a_1 q(x_n, fx_n) + a_2 q(x_n, fx_n) + a_3 q(fx_n, f^2x_n) \\ &+ a_4 q(x_n, fx_n) + a_4 q(fx_n, f^2x_n). \end{aligned}$$

Therefore $q(fx_n, f^2x_n) \preceq \frac{a_1+a_2+a_4}{1-a_3-a_4} q(x_n, fx_n)$. Hence

$$\begin{aligned} q(x_n, f^2x_n) &\preceq q(x_n, fx_n) + q(fx_n, f^2x_n) \\ &\preceq q(x_n, fx_n) + \frac{a_1+a_2+a_4}{1-a_3-a_4} q(x_n, fx_n) \to \theta \end{aligned}$$

as $n \to \infty$. This implies $q(x_n, f^2x_n) \to \theta$. Consequently, $f^2x_n \to y$ by Lemma 1.14(2) . Since f is continuous, we have

$$fy = f(\lim_{n\to\infty} fx_n) = \lim_{n\to\infty} f^2x_n = y$$

which is a contradiction. Therefore if $fy \neq y$, then

$$\inf\{\|q(x,y)\| + \|q(fx,y)\| + \|q(x,fx)\| \,:\, x \in X\} > 0.$$

By Corollary 2.3, the proof is done. □

Example 2.5 *Let* $X = \{0,1,2,3\}$ *,*$E = \mathbb{R}$ *and* $P = \{x \in \mathbb{R} : x \geq 0\}$*. Define* $d : X \times X \to E$ *by* $d(x,y) = |x-y|$*. Then* (X,d) *is a complete cone metric space. Define* $q : X \times X \to E$ *by the following :*

$$\begin{aligned} q(0,0) &= 0, \quad q(0,1) = 1, \quad q(0,2) = 1.1, \quad q(0,3) = 0.5, \\ q(1,0) &= 1, \quad q(1,1) = 0, \quad q(1,2) = 0.1, \quad q(1,3) = 0.5, \\ q(2,0) &= 1, \quad q(2,1) = 1, \quad q(2,2) = 0, \quad q(2,3) = 0.5, \\ q(3,0) &= 1, \quad q(3,1) = 0.5, \quad q(3,2) = 0.6, \quad q(3,3) = 0. \end{aligned}$$

Then q *is a* $c-$*distance. In fact Definition 1.8 (q1),(q3) are obvious. If we put* $e = 0.01$*, (q4) is also clear. For (q2),*

$$1 = q(0,1) \leq q(0,2) + q(2,1) = 2.1,$$

$$1 = q(0,1) \leq q(0,3) + q(3,1) = 1,$$

$$1.1 = q(0,2) \leq q(0,1) + q(1,2) = 1.1,$$

$$1.1 = q(0,2) \leq q(0,3) + q(3,2) = 1.1,$$

$$0.5 = q(0,3) \leq q(0,1) + q(1,3) = 1.5,$$

$$0.5 = q(0,3) \leq q(0,2) + q(2,3) = 1.6,$$

$$1 = q(1,0) \leq q(1,2) + q(2,0) = 1.1,$$

$$1 = q(1,0) \leq q(1,3) + q(3,0) = 1.5,$$

$$0.1 = q(1,2) \leq q(1,0) + q(0,2) = 2.1,$$

$$0.1 = q(1,2) \leq q(1,3) + q(3,2) = 1.1,$$

$$0.5 = q(1,3) \leq q(1,0) + q(0,3) = 1.5,$$

$$0.5 = q(1,3) \leq q(1,2) + q(2,3) = 0.6,$$

$$1 = q(2,0) \leq q(2,1) + q(1,0) = 2,$$

$$1 = q(2,0) \leq q(2,3) + q(3,0) = 1.5,$$

$$1 = q(2,1) \leq q(2,0) + q(0,1) = 2,$$

$$1 = q(2,1) \leq q(2,3) + q(3,1) = 1,$$

$$0.5 = q(2,3) \leq q(2,0) + q(0,3) = 1.5,$$

$$0.5 = q(2,3) \leq q(2,1) + q(1,3) = 1.5,$$

$$1 = q(3,0) \leq q(3,1) + q(1,0) = 1.5,$$

$$1 = q(3,0) \leq q(3,2) + q(2,0) = 1.6,$$

$$0.5 = q(3,1) \leq q(3,0) + q(0,1) = 2,$$

$$0.5 = q(3,1) \leq q(3,2) + q(2,1) = 1.6,$$

$$0.6 = q(3,2) \leq q(3,0) + q(0,2) = 2.1,$$

$$0.6 = q(3,2) \leq q(3,1) + q(1,2) = 0.6.$$

Thus (q2) is checked and so q *is a* $c-$*distance.*

Define $f : X \to X$ *by* $f0 = 1, f1 = 2, f2 = 2, f3 = 2$ *and define* $g : X \to X$ *by* $gx = x$. *Then* $f(X) \subseteq g(X)$.

Consider $x = 2, y = 0$. *Then* $q(f2, f0) = q(2,1) = 1$ *and*

$$\begin{aligned} a_1q(g2,g0) &+ a_2q(g2,f2) + a_3q(g0,f0) + a_4q(g2,f0) \\ &= a_1q(2,0) + a_2q(2,2) + a_3q(0,1) + a_4q(2,1) \\ &= a_1 + a_3 + a_4 \leq a_1 + a_3 + 2a_4 < 1 \end{aligned}$$

for any nonnegative real numbers $a_i\,(i=1,2,3,4)$ *with* $a_1+a_2+a_3+2a_4<1$. *Hence the contractive condition of Theorem 2.1 is not satisfied and so Theorem 2.1 can not be applied to this example.*

But Theorem 2.2 can be applied to this example. In fact we take $a_1=0.14, a_2=a_3=a_4=0$ *and* $a_5=0.85$. *Then*

$$\begin{aligned} 0.1 &= q(f0,f1) < a_1q(g0,g1)+a_5q(g1,f0)=0.14,\\ 0.1 &= q(f0,f2) < a_1q(g0,g2)+a_5q(g2,f0)=1.004,\\ 0.1 &= q(f0,f3) < a_1q(g0,g3)+a_5q(g3,f0)=0.495,\\ 1 &= q(f1,f0) < a_1q(g1,g0)+a_5q(g0,f1)=1.075,\\ 1 &= q(f2,f0) < a_1q(g2,g0)+a_5q(g0,f2)=1.075,\\ 1 &= q(f3,f0) < a_1q(g3,g0)+a_5q(g0,f3)=1.075. \end{aligned}$$

Also

$$\begin{aligned} \inf\{\|q(gx,0)\|+\|q(fx,0)\|+\|q(gx,fx)\| : x\in X\} &= 2>0\\ \inf\{\|q(gx,1)\|+\|q(fx,1)\|+\|q(gx,fx)\| : x\in X\} &= 1.1>0\\ \inf\{\|q(gx,3)\|+\|q(fx,3)\|+\|q(gx,fx)\| : x\in X\} &= 1>0. \end{aligned}$$

Hence the hypotheses are satisfied and so by Theorem 2.2 f *and* g *have a unique point of coincidence. Since* $f2=2$ *and* $g2=2$, *2 is a unique point of coincidence. Since* $2=gf2=fg2$, f *and* g *are weakly compatible. 2 is the unique common fixed point of* f *and* g.

3 Common fixed point results on cone metric spaces

Theorem 3.1 *([9]) Let (X,d) be a cone metric space, P be a cone and q be a $c-$distance on X. Let $f,g : X \to X$ be two self mappings such that $f(X) \subseteq g(X)$ and $g(X)$ be a complete subset of X. Suppose that there exists nonnegative constants $a_i \in [0,1)$, $i = 1,2,3$ with $a_1 + a_2 + a_3 < 1$ such that the following contractive condition holds for all $x, y \in X$:*

$$q(fx, fy) \preceq a_1 q(gx, gy) + a_2 q(gx, fx) + a_3 q(gy, fy)$$

and that

$$\inf\{q(gx,y) + q(fx,y) + q(gx,fx) : x \in X\} \succ \theta$$

for all $y \in X$ with y is not a point of coincidence of f and g. Then f and g have a unique point of coincidence in X.

Moreover if f and g are weakly compatible then f and g have a unique common fixed point in X.

In ([4]), Z.M. Fadail, A.G.B Ahmad and S. Radenovic proved the following theorem 3.2 without the condition

$$\inf\{q(gx,y) + q(fx,y) + q(gx,fx) : x \in X\} \succ \theta \tag{3.1}$$

for all $y \in X$ which is not a point of coincidence of f and g. In fact in Theorem 3.1 of ([4]) it is necessary that

$$\inf\{q(gx,y) + q(fx,y) + q(gx,fx) : x \in X\} \succ \theta$$

for all $y \in X$ which is not a point of coincidence of f and g. Hence we obtain the following theorem.

Theorem 3.2 *Let (X,d) be a cone metric space, P be a cone and q be a $c-$distance on X. Let $f,g : X \to X$ be two self mappings such that $f(X) \subseteq g(X)$ and $g(X)$ be a complete subset of X. Suppose that there exist nonnegative constants $a_i \in [0,1)$, $i = 1,2,3,4$ with $a_1 + a_2 + a_3 + 2a_4 < 1$ such that the following contractive condition holds for all $x, y \in X$:*

$$q(fx, fy) \preceq a_1 q(gx, gy) + a_2 q(gx, fx) + a_3 q(gy, fy) + a_4 q(gx, fy)$$

and that

$$\inf\{q(gx,y) + q(fx,y) + q(gx,fx) : x \in X\} \succ \theta$$

for all $y \in X$ with y is not a point of coincidence of f and g. Then f and g have a unique point of coincidence in X.

Moreover if f and g are weakly compatible then f and g have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Choose a point $x_1 \in X$ such that $gx_1 = fx_0$. This can be done because $f(X) \subseteq g(X)$. Continuing this process we obtain a sequence $\{x_n\}$ in X such that $gx_{n+1} = fx_n$. Then we have

$$\begin{aligned} q(gx_n, gx_{n+1}) &= q(fx_{n-1}, fx_n) \\ &\preceq a_1 q(gx_{n-1}, gx_n) + a_2 q(gx_{n-1}, fx_{n-1}) + a_3 q(gx_n, fx_n) + a_4 q(gx_{n-1}, fx_n) \\ &= a_1 q(gx_{n-1}, gx_n) + a_2 q(gx_{n-1}, gx_n) + a_3 q(gx_n, gx_{n+1}) + a_4 q(gx_{n-1}, gx_{n+1}) \\ &\preceq a_1 q(gx_{n-1}, gx_n) + a_2 q(gx_{n-1}, gx_n) + a_3 q(gx_n, gx_{n+1}) + a_4 \{q(gx_{n-1}, gx_n) \\ &+ q(gx_n, gx_{n+1})\}. \end{aligned}$$

and so

$$\begin{aligned} q(gx_n, gx_{n+1}) &\preceq \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} q(gx_{n-1}, gx_n) \\ &= hq(gx_{n-1}, gx_n) \\ &\preceq h^2 q(gx_{n-2}, gx_{n-1}) \\ &\vdots \\ &\preceq h^n q(gx_0, gx_1). \end{aligned}$$

where $0 \leq h = \frac{a_1+a_2+a_4}{1-a_3-a_4} < 1$.

Let $m > n \geq 1$. It follows that

$$\begin{aligned} q(gx_n, gx_m) &\preceq q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_{n+2}) + \cdots + q(gx_{m-1}, gx_m) \\ &\preceq (h^n + h^{n+1} + \cdots + h^{m-1}) q(gx_0, gx_1) \\ &\preceq \frac{h^n}{1-h} q(gx_0, gx_1). \end{aligned}$$

Hence $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, there exists $x' \in g(X)$ such that $gx_m \to x'$ as $m \to \infty$. By definition 1.8 (q3),

$$q(gx_n, gx_m) \preceq \frac{h^n}{1-h} q(gx_0, gx_1).$$

Suppose that $x' \in X$ is not a point of coincidence of f and g.Then by assumption,

$$\begin{aligned} \theta &\prec \inf\{q(gx, x') + q(fx, x') + q(gx, fx) : x \in X\} \\ &\preceq \inf\{q(gx_n, x') + q(fx_n, x') + q(gx_n, fx_n) : n \in \mathbb{N}\} \\ &= \inf\{q(gx_n, x') + q(gx_{n+1}, x') + q(gx_n, gx_{n+1}) : x \in \mathbb{N}\} = \theta \end{aligned}$$

which is a contradiction. Therefore x' is a point of coincidence of f and g. So there exists $x \in X$ such that $fx = gx = x'$. If there exists $w \in X$ such that $fy = gy = w$ for some $y \in X$,

$$\begin{aligned} q(x', x') &= q(fx, fx) \\ &\preceq a_1 q(gx, gx) + a_2 q(gx, fx) + a_3 q(gx, fx) + a_4 q(gx, fx) \\ &= (a_1 + a_2 + a_3 + a_4) q(x', x'). \end{aligned}$$

Hence $q(x', x') = \theta$. Similarly $q(w, w) = \theta$. Now

$$\begin{aligned} q(x', w) &= q(fx, fy) \\ &\preceq a_1 q(gx, gy) + a_2 q(gx, fx) + a_3 q(gy, fy) + a_4 q(gx, fy) \\ &= a_1 q(x', w) + a_2 q(x', x') + a_3 q(w, w) + a_4 q(x', w) \\ &= (a_1 + a_4) q(x', w). \end{aligned}$$

Therefore $q(x', w) = \theta$ which means $x' = w$. Consequently x' is a unique point of coincidence of f and g. Moreover if f and g are weakly compatible,

$$gx' = ggx = gfx = fgx = fx'$$

which implies gx' is a point of coincidence of f and g. By uniqueness of the point of coincidence , $fx' = gx' = x'$. In other words, x' is the unique common fixed point of f and g. □

Example 3.3 *(the case of a nonnormal cone) Consider Example 1.12. Define the mappings* $f : X \to X$ *by* $fx = \frac{x^2}{4}$ *and* $g : X \to X$ *by* $gx = \frac{x}{2}$ *for all* $x \in X$*. It is clear that* $f(X) \subseteq g(X)$ *and* $g(X)$ *is a complete subset of* X*. From the direct calculation, we obtain that*

$$\begin{aligned} q(fx, fy)(t) &= fy \cdot e^t = \frac{y^2}{4} e^t \\ &\preceq \frac{1}{2}\frac{y}{2} e^t = \frac{1}{2}(gy \cdot e^t) = a_1 q(gx, gy)(t) \\ &\preceq a_1 q(gx, gy)(t) + a_2 q(gx, fx)(t) + a_3 q(gy, fy)(t) + a_4 q(gx, fy)(t), \end{aligned}$$

where $a_1 = \frac{1}{2}, a_2 = a_3 = \frac{1}{8}, a_4 = \frac{1}{16}$ *and* $a_1 + a_2 + a_3 + 2a_4 = \frac{7}{8} < 1$*. Also*

$$\inf\{q(gx, y) + q(fx, y) + q(gx, fx) : x \in X\} = \inf\{ye^t + ye^t + \frac{x^2}{4} e^t : x \in X\} \succ \theta,$$

if y *is not a zero element. Hence*

$$\inf\{q(gx, y) + q(fx, y) + q(gx, fx) : x \in X\} \succ \theta$$

for all $y \in X$ which y is not a point of coincidence of f and g.

Also, f and g are weakly compatible at $x = 0$. Therefore all conditions of Theorem 3.2 are satisfied. Hence f and g have a unique common fixed point $x = 0$ and $f(0) = g(0) = 0$ with $q(0,0) = 0$.

Theorem 3.4 *Let (X, d) be a cone metric space, P be a cone and q be a $c-$distance on X. Let $f, g : X \to X$ be two self mappings such that $f(X) \subseteq g(X)$ and $g(X)$ be a complete subset of X. Suppose that there exist nonnegative constants $a_i \in [0,1), i = 1,2,3,4,5$ with $a_1 + 2a_2 + 2a_3 + 3a_4 + a_5 < 1$ such that the following contractive condition holds for all $x, y \in X$:*

$$q(fx, fy) \preceq a_1 q(gx, gy) + a_2 q(gx, fx) + a_3 q(gy, fy) + a_4 q(gx, fy) + a_5 q(gy, fx)$$

and that

$$\inf\{q(gx, y) + q(fx, y) + q(gx, fx) : x \in X\} \succ \theta$$

for all $y \in X$ with y is not a point of coincidence of f and g. Then f and g have a unique point of coincidence in X.

Moreover if f and g are weakly compatible then f and g have a unique common fixed point in X.

Proof. Let $x_0, x_1 \in X$. Using the fact that $f(X) \subseteq g(X)$, construct $\{x_{2n}\}, \{x_{2n+1}\}$ such that $gx_{2n} = fx_{2n-2}$ and $gx_{2n+1} = fx_{2n-1} \quad (n \in \mathbb{N})$. Then we have

$$\begin{aligned}
q(gx_{2n}, gx_{2n+1}) &= q(fx_{2n-2}, fx_{2n-1}) \\
&\preceq a_1 q(gx_{2n-2}, gx_{2n-1}) + a_2 q(gx_{2n-2}, fx_{2n-2}) + a_3 q(gx_{2n-1}, fx_{2n-1}) \\
&+ a_4 q(gx_{2n-2}, fx_{2n-1}) + a_5 q(gx_{2n-1}, fx_{2n-2}) \\
&= a_1 q(gx_{2n-2}, gx_{2n-1}) + a_2 q(gx_{2n-2}, gx_{2n}) \\
&+ a_3 q(gx_{2n-1}, gx_{2n+1}) + a_4 q(gx_{2n-2}, gx_{2n+1}) + a_5 q(gx_{2n-1}, gx_{2n}) \\
&\preceq a_1 q(gx_{2n-2}, gx_{2n-1}) + a_2\{q(gx_{2n-2}, gx_{2n-1}) + q(gx_{2n-1}, gx_{2n})\} \\
&+ a_3\{q(gx_{2n-1}, gx_{2n}) + q(gx_{2n}, gx_{2n+1})\} \\
&+ a_4\{q(gx_{2n-2}, gx_{2n-1}) + q(gx_{2n-1}, gx_{2n}) + q(gx_{2n}, gx_{2n+1})\} \\
&+ a_5 q(gx_{2n-1}, gx_{2n}).
\end{aligned}$$

Hence

$$\begin{aligned}
q(gx_{2n}, gx_{2n+1}) &\preceq \frac{a_2 + a_3 + a_4 + a_5}{1 - a_3 - a_4} q(gx_{2n-1}, gx_{2n}) \\
&+ \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} q(gx_{2n-2}, gx_{2n-1}). \qquad (3.2)
\end{aligned}$$

Similarly,

$$\begin{aligned} q(gx_{2n-1}, gx_{2n}) &\preceq \frac{a_2+a_3+a_4+a_5}{1-a_3-a_4} q(gx_{2n-2}, gx_{2n-1}) \\ &+ \frac{a_1+a_2+a_4}{1-a_3-a_4} q(gx_{2n-3}, gx_{2n-2}). \end{aligned} \tag{3.3}$$

Clearly $0 \le \frac{a_2+a_3+a_4+a_5}{1-a_3-a_4} < 1$ and $0 \le \frac{a_1+a_2+a_4}{1-a_3-a_4} < 1$. Set

$$b_1 = \alpha = \frac{a_2+a_3+a_4+a_5}{1-a_3-a_4} \quad \text{and} \quad c_1 = \beta = \frac{a_1+a_2+a_4}{1-a_3-a_4}.$$

Applying (3.2) and (3.3) and putting $b_2 = c_1 + \alpha b_1 = \beta + \alpha b_1, c_2 = \beta b_1$,

$$\begin{aligned} q(gx_{2n}, gx_{2n+1}) &\preceq b_1 q(gx_{2n-1}, gx_{2n}) + c_1 q(gx_{2n-2}, gx_{2n-1}) \\ &\preceq b_2 q(gx_{2n-2}, gx_{2n-1}) + c_2 q(gx_{2n-3}, gx_{2n-2}) \\ &\vdots \\ &\preceq b_{2n-1} q(gx_1, gx_2) + c_{2n-1} q(gx_0, gx_1), \end{aligned} \tag{3.4}$$

where $b_{2n-1} = \beta b_{2n-3} + \alpha b_{2n-2}$ and $c_{2n-1} = \beta b_{2n-2}$.

Similarly

$$q(gx_{2n-1}, gx_{2n}) \preceq b_{2n-2} q(gx_1, gx_2) + c_{2n-2} q(gx_0, gx_1) \tag{3.5}$$

where $b_{2n-2} = \beta b_{2n-4} + \alpha b_{2n-3}$ and $c_{2n-2} = \beta b_{2n-3}$. From (3.4) and (3.5),

$$q(gx_{n+1}, gx_{n+2}) \preceq b_n q(gx_1, gx_2) + c_n q(gx_0, gx_1)$$

where $b_n = \beta b_{n-2} + \alpha b_{n-1}$ and $c_n = \beta b_{n-1}$.

Consider

$$b_{n+2} = \alpha b_{n+1} + \beta b_n \quad (0 \le \alpha, \beta < 1, b_1, b_2 \ge 0).$$

Then $b_n \ge 0$ for all $n \in \mathbb{N}$. Its characteristic equation is that $t^2 - \alpha t - \beta = 0$. If $1-\alpha-\beta > 0$ and $1+\alpha-\beta > 0$ then it has two roots t_1, t_2 such that $-1 < t_1 \le 0 \le t_2 < 1$. Also the hypothesis $a_1+2a_2+2a_3+3a_4+a_5 < 1$ implies $1-\alpha-\beta > 0$ and $1+\alpha-\beta > 0$. For such t_1 and t_2, $b_n = k_1(t_1)^n + k_2(t_2)^n$ for some $k_1, k_2 \in \mathbb{R}$.

Let $m > n \ge 1$. It follows that

$$\begin{aligned} q(gx_n, gx_m) &\preceq q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_{n+2}) + \cdots + q(gx_{m-1}, gx_m) \\ &\preceq (b_{n-1} + b_n + \cdots + b_{m-2}) q(gx_1, gx_2) + (c_{n-1} + c_n + \cdots + c_{m-2}) q(gx_0, gx_1) \\ &\preceq \{k_1(t_1^{n-1} + t_1^n + \cdots + t_1^{m-2}) + k_2(t_2^{n-1} + \cdots + t_2^{m-2})\} q(gx_1, gx_2) \\ &+ \beta\{k_1(t_1^{n-2} + \cdots + t_1^{m-3}) + k_2(t_2^{n-2} + \cdots + t_2^{m-3})\} q(gx_0, gx_1) \\ &\preceq (\frac{k_1 t_1^{n-1}}{1-t_1} + \frac{k_2 t_2^{n-1}}{1-t_2}) q(gx_1, gx_2) + \beta(\frac{k_1 t_1^{n-2}}{1-t_1} + \frac{k_2 t_2^{n-2}}{1-t_2}) q(gx_0, gx_1) \\ &\to \theta \end{aligned}$$

as $n \to \infty$. Therefore $\{gx_n\}$ is a Cauchy sequence in $g(X)$ by lemma 1.14(3). Since $g(X)$ is complete, there exists $x' \in g(X)$ such that $gx_m \to x'$ as $m \to \infty$. By definition 1.8(q3)

$$q(gx_n, x') \preceq (\frac{k_1 t_1^{n-1}}{1-t_1} + \frac{k_2 t_2^{n-1}}{1-t_2})q(gx_1, gx_2) + \beta(\frac{k_1 t_1^{n-2}}{1-t_1} + \frac{k_2 t_2^{n-2}}{1-t_2})q(gx_0, gx_1) \to \theta$$

as $n \to \infty$. Suppose that x' is not a point of coincidence of f and g. Then by assumtion,

$$\begin{aligned} \theta &\prec \inf\{q(gx, x') + q(fx, x') + q(gx, fx) : x \in X\} \\ &\preceq \inf\{q(gx_n, x') + q(fx_n, x') + q(gx_n, fx_n) : n \in \mathbb{N}\} \\ &= \inf\{q(gx_n, x') + q(gx_{n+2}, x') + q(gx_n, gx_{n+2}) : x \in \mathbb{N}\} = \theta \end{aligned}$$

which is a contradiction. Therefore x' is a point of coincidence of f and g. So there exists $x \in X$ such that $fx = gx = x'$. If there exists $w \in X$ such that $fy = gy = w$ for some $y \in X$,

$$\begin{aligned} q(x', x') &= q(fx, fx) \\ &\preceq a_1 q(gx, gx) + a_2 q(gx, fx) + a_3 q(gx, fx) + a_4 q(gx, fx) + a_5 q(gx, fx) \\ &= (a_1 + a_2 + a_3 + a_4 + a_5) q(x', x'). \end{aligned}$$

Hence

$$q(x', x') = \theta. \tag{3.6}$$

Similarly

$$q(w, w) = \theta. \tag{3.7}$$

Now by (3.6) and (3.7)

$$\begin{aligned} q(x', w) &= q(fx, fy) \\ &\preceq a_1 q(gx, gy) + a_2 q(gx, fx) + a_3 q(gy, fy) + a_4 q(gx, fy) + a_5 q(gy, fx) \\ &= a_1 q(x', w) + a_2 q(x', x') + a_3 q(w, w) + a_4 q(x', w) + a_5 q(w, x') \\ &= (a_1 + a_4) q(x', w) + a_5 q(w, x'). \end{aligned}$$

Similarly $q(w, x') \preceq (a_1 + a_4) q(w, x') + a_5 q(x', w)$. Thus

$$q(x', w) + q(w, x') \preceq (a_1 + a_4 + a_5)\{q(x', w) + q(w, x')\}.$$

Therefore $q(x', w) + q(w, x') = \theta$ which implies

$$q(x', w) = q(w, x') = \theta. \tag{3.8}$$

By (3.7),(3.8) and Lemma 1.14(1), $x' = w$. Consequently x' is a unique point of coincidence of f and g. Moreover if f and g are weakly compatible,

$$gx' = ggx = gfx = fgx = fx'$$

which implies gx' is a point of coincidence of f and g. By uniqueness of the point of coincidence , $fx' = gx' = x'$. In other words, x' is the unique common fixed point of f and g. □

Corollary 3.5 *Let $(X,\ d)$ be a complete cone metric space and let P be a cone and q be a c-distance on X. Let $f : X \to X$ be a self mapping. Suppose that there exist nonnegative constants $a_i \in [0, 1)$, $i = 1, 2, 3, 4, 5$ with $a_1 + 2a_2 + 2a_3 + 3a_4 + a_5 < 1$ such that the following contractive condition holds for all $x, y \in X$:*

$$q(fx, fy) \preceq a_1q(x, y) + a_2q(x, fx) + a_3q(y, fy) + a_4q(x, fy) + a_5q(y, fx)$$

and that

$$\inf\{q(x, y) + q(fx, y) + q(x, fx) : x \in X\} \succ \theta$$

if $fy \neq y$. Then f has a unique fixed point in X.

Proof. Take $g = I$ in Theorem 3.4. □

Corollary 3.6 *Let $(X,\ d)$ be a complete cone metric space and let P be a cone and q be a c-distance on X. Let $f : X \to X$ be a continuous self mapping . Suppose that there exist nonnegative constants $a_i \in [0, 1)$, $i = 1, 2, 3, 4$ with $a_1 + 2a_2 + 2a_3 + 3a_4 < 1$ such that*

$$q(fx, fy) \preceq a_1q(x, y) + a_2q(x, fx) + a_3q(y, fy) + a_4q(x, fy).$$

Then f has a unique fixed point in X.

Proof. Assume there exists $y \in X$ such that $fy \neq y$ and

$$\inf\{q(x, y) + q(fx, y) + q(x, fx) : x \in X\} = \theta.$$

Then we can construct $\{x_n\}$ in X such that

$$\inf\{q(x_n, y) + q(fx_n, y) + q(x_n, fx_n) : n \in \mathbb{N}\} = \theta.$$

Hence

$$q(x_n, y) \to \theta, \quad q(fx_n, y) \to \theta, \quad q(x_n, fx_n) \to \theta.$$

By Lemma 1.14(2), $fx_n \to y$. By the contractive condition, we have

$$\begin{aligned} q(fx_n, f^2x_n) &\preceq a_1q(x_n, fx_n) + a_2q(x_n, fx_n) + a_3q(fx_n, f^2x_n) + a_4q(x_n, f^2x_n) \\ &\preceq a_1q(x_n, fx_n) + a_2q(x_n, fx_n) + a_3q(fx_n, f^2x_n) \\ &+ a_4q(x_n, fx_n) + a_4q(fx_n, f^2x_n). \end{aligned}$$

Therefore

$$q(fx_n, f^2x_n) \preceq \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} q(x_n, fx_n).$$

Hence

$$\begin{aligned} q(x_n, f^2x_n) &\preceq q(x_n, fx_n) + q(fx_n, f^2x_n) \\ &\preceq q(x_n, fx_n) + \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} q(x_n, fx_n) \to \theta \end{aligned}$$

as $n \to \infty$. This implies $q(x_n, f^2x_n) \to \theta$. Consequently, $f^2x_n \to y$. Since f is continuous, we have

$$fy = f(\lim_{n\to\infty} fx_n) = \lim_{n\to\infty} f^2x_n = y$$

which is a contradiction. Therefore if $fy \neq y$, then

$$\inf\{q(x, y) + q(fx, y) + q(x, fx) : x \in X\} \succ \theta.$$

By Theorem 3.4, the proof is done. □

4 Fixed point on partially ordered cone metric spaces

Theorem 4.1 *([2]) Let $(X,\sqsubseteq)$ be a partially ordered set and suppose that (X,d) is a complete cone metric space. Let q be a $c-$distance on X and $f: X \to X$ be a nondecreasing mapping with respect to $\sqsubseteq$.(without the assumption of continuity of f) Suppose that the following three assertions hold:*

(i) there exist nonnegative numbers $a_i, i = 1, 2$ with $a_1 + a_2 < 1$ such that

$$q(fx, fy) \preceq a_1 q(x, y) + a_2 q(x, fx)$$

for all $x, y \in X$ with $x \sqsubseteq y$;

(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.

(iii) if $\{x_n\}$ is nondecreasing mapping with respect to $\sqsubseteq$ and converges to x then $x_n \sqsubseteq x$ as $n \to \infty$.

Then f has a fixed point $x \in X$. If $v = fv$ then $q(v, v) = \theta$.

Theorem 4.2 *([3])Let $(X,\sqsubseteq)$ be a partially ordered set and suppose that (X,d) is a complete cone metric space. Let q be a $c-$distance on X and $f: X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following two assertions hold:*

(i) there exist $a_i \geq 0, i = 1, 2, 3$ with $a_1 + a_2 + a_3 < 1$ such that

$$q(fx, fy) \preceq a_1 q(x, y) + a_2 q(x, fx) + a_3 q(y, fy)$$

for all $x, y \in X$ with $x \sqsubseteq y$;

(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.

Then f has a fixed point $x \in X$. If $v = fv$, then $q(v, v) = \theta$.

Theorem 4.3 *([2]) Let $(X,\sqsubseteq)$ be a partially ordered set and suppose that (X,d) is a complete cone metric space. Let q be a $c-$distance on X and $f: X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following two assertions hold:*

(i) there exist $a_i \geq 0, i = 1, 2, 3, 4$ with $a_1 + a_2 + a_3 + 2a_4 < 1$ such that

$$q(fx, fy) \preceq a_1 q(x, y) + a_2 q(x, fx) + a_3 q(y, fy) + a_4 q(x, fy)$$

for all $x, y \in X$ with $x \sqsubseteq y$;

(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.

Then f has a fixed point $x \in X$. If $v = fv$ then $q(v, v) = \theta$.

Theorem 4.4 *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that $(X,\ d)$ is a complete cone metric space. Let q be a $c-$distance on X. Let $f : X \to X$ be a continuous self mapping which is nondecreasing with respect to $\sqsubseteq$. Suppose that the following two assertions hold:*

(i) there exist nonnegative constants $a_i \in [0,1)$ $i = 1,2,3,4,5$ with $a_1 + 2a_2 + 2a_3 + 3a_4 + a_5 < 1$ such that

$$q(fx, fy) \preceq a_1 q(x,y) + a_2 q(x, fx) + a_3 q(y, fy) + a_4 q(x, fy) + a_5 q(y, fx)$$

for all $x, y \in X$ with $x \sqsubseteq y$.

(ii) there exist $x_0, x_1 \in X$ such that $x_0 \sqsubseteq x_1 \sqsubseteq fx_0$.
Then f has a fixed point in X. If $v = fv$,then $q(v,v) = \theta$

Proof. Since f is nondecreasing with respect to $\sqsubseteq$, we have

$$x_0 \sqsubseteq x_1 \sqsubseteq fx_0 = x_2 \sqsubseteq fx_1 = x_3 \sqsubseteq \cdots .$$

Then we have

$$\begin{aligned}
q(x_{2n}, x_{2n+1}) &= q(fx_{2n-2}, fx_{2n-1}) \\
&\preceq a_1 q(x_{2n-2}, x_{2n-1}) + a_2 q(x_{2n-2}, fx_{2n-2}) + a_3 q(x_{2n-1}, fx_{2n-1}) \\
&+ a_4 q(x_{2n-2}, fx_{2n-1}) + a_5 q(x_{2n-1}, fx_{2n-2}) \\
&= a_1 q(x_{2n-2}, x_{2n-1}) + a_2 q(x_{2n-2}, x_{2n}) + a_3 q(x_{2n-1}, x_{2n+1}) \\
&+ a_4 q(x_{2n-2}, x_{2n+1}) + a_5 q(x_{2n-1}, x_{2n}) \\
&\preceq a_1 q(x_{2n-2}, x_{2n-1}) + a_2 \{q(x_{2n-2}, x_{2n-1}) + q(x_{2n-1}, x_{2n})\} \\
&+ a_3 \{q(x_{2n-1}, x_{2n}) + q(x_{2n}, x_{2n+1})\} \\
&+ a_4 \{q(x_{2n-2}, x_{2n-1}) + q(x_{2n-1}, x_{2n}) + q(x_{2n}, x_{2n+1})\} + a_5 q(x_{2n-1}, x_{2n}).
\end{aligned}$$

Hence

$$q(x_{2n}, x_{2n+1}) \preceq \frac{a_2 + a_3 + a_4 + a_5}{1 - a_3 - a_4} q(x_{2n-1}, x_{2n}) + \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} q(x_{2n-2}, x_{2n-1}).$$

Similarly,

$$q(x_{2n-1}, x_{2n}) \preceq \frac{a_2 + a_3 + a_4 + a_5}{1 - a_3 - a_4} q(x_{2n-2}, x_{2n-1}) + \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} q(x_{2n-3}, x_{2n-2}).$$

Clearly $0 \leq \frac{a_2+a_3+a_4+a_5}{1-a_3-a_4} < 1$ and $0 \leq \frac{a_1+a_2+a_4}{1-a_3-a_4} < 1$. Set

$$b_1 = \alpha = \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} \quad \text{and} \quad c_1 = \beta = \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4}.$$

Applying the above inequalities and putting $b_2 = c_1 + \alpha b_1 = \beta + \alpha b_1,\ c_2 = \beta b_1$,

$$\begin{aligned} q(x_{2n}, x_{2n+1}) &\preceq b_1 q(x_{2n-1}, x_{2n}) + c_1 q(x_{2n-2}, x_{2n-1}) \\ &\preceq b_2 q(x_{2n-2}, x_{2n-1}) + c_2 q(x_{2n-3}, x_{2n-2}) \\ &\vdots \\ &\preceq b_{2n-1} q(x_1, x_2) + c_{2n-1} q(x_0, x_1), \end{aligned} \tag{4.1}$$

where $b_{2n-1} = \beta b_{2n-3} + \alpha b_{2n-2}$ and $c_{2n-1} = \beta b_{2n-2}$.

Similarly

$$q(x_{2n-1}, x_{2n}) \preceq b_{2n-2} q(x_1, x_2) + c_{2n-2} q(x_0, x_1) \tag{4.2}$$

where $b_{2n-2} = \beta b_{2n-4} + \alpha b_{2n-3}$ and $c_{2n-2} = \beta b_{2n-3}$. From (4.1) and (4.2),

$$q(x_{n+1}, x_{n+2}) \preceq b_n q(x_1, x_2) + c_n q(x_0, x_1)$$

where $b_n = \beta b_{n-2} + \alpha b_{n-1}$ and $c_n = \beta b_{n-1}$.

Consider

$$b_{n+2} = \alpha b_{n+1} + \beta b_n \quad (0 \le \alpha, \beta \le 1, b_1, b_2 \ge 0).$$

Then $b_n \ge 0$ for all $n \in \mathbb{N}$. Its characteristic equation is that $t^2 - \alpha t - \beta = 0$. If $1-\alpha-\beta > 0$ and $1+\alpha-\beta > 0$ then it has two roots t_1, t_2 such that $-1 < t_1 \le 0 \le t_2 < 1$. Also the hypothesis $a_1 + 2a_2 + 2a_3 + 3a_4 + a_5 < 1$ implies $1-\alpha-\beta > 0$ and $1+\alpha-\beta > 0$. For such t_1, t_2, $b_n = k_1(t_1)^n + k_2(t_2)^n$ for some $k_1, k_2 \in \mathbb{R}$.

Let $m > n \ge 1$. It follows that

$$\begin{aligned} q(x_n, x_m) &\preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\ &\preceq (b_{n-1} + b_n + \cdots + b_{m-2}) q(x_1, x_2) + (c_{n-1} + c_n + \cdots + c_{m-2}) q(x_0, x_1) \\ &\preceq \{k_1(t_1^{n-1} + t_1^n + \cdots + t_1^{m-2}) + k_2(t_2^{n-1} + \cdots + t_2^{m-2})\} q(x_1, x_2) \\ &+ \beta\{k_1(t_1^{n-2} + \cdots + t_1^{m-3}) + k_2(t_2^{n-2} + \cdots + t_2^{m-3})\} q(x_0, x_1) \\ &\preceq \left(\frac{k_1 t_1^{n-1}}{1-t_1} + \frac{k_2 t_2^{n-1}}{1-t_2}\right) q(x_1, x_2) + \beta\left(\frac{k_1 t_1^{n-2}}{1-t_1} + \frac{k_2 t_2^{n-2}}{1-t_2}\right) q(x_0, x_1) \\ &\to \theta \end{aligned}$$

as $n \to \infty$. Therefore $\{x_n\}$ is a Cauchy sequence in X by Lemma 1.14(3). Since X is complete, there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$. Using the continuity of f,

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f x_{n-2} = fx.$$

Therefore x is a fixed point of f. Moreover suppose that $v = fv$. Then we have

$$\begin{aligned} q(v,v) = q(fv,fv) &\preceq a_1q(v,v) + a_2q(v,fv) + a_3q(v,fv) + a_4q(v,fv) + a_5q(v,fv) \\ &= (a_1+a_2+a_3+a_4+a_5)q(v,v). \end{aligned}$$

Since $0 \le a_1+a_2+a_3+a_4+a_5 < 1$, we have $q(v,v) = \theta$. □

Example 4.5 *Let* $X = \{0,1,2,3\}$, $E = \mathbb{R}$ *and* $P = \{x \in \mathbb{R} : x \ge 0\}$ *in Example 2.5. Define* $d : X \times X \to E$ *by* $d(x,y) = |x-y|$ *and define* $\sqsubseteq$ *by*

$$x \sqsubseteq y \quad \Leftrightarrow \quad x \ge y.$$

Then (X,d) *is a complete cone metric space and* X *is a partially ordered set. Define* $q : X \times X \to E$ *by the following :*

$$\begin{aligned} q(0,0) &= 0, \quad q(0,1) = 1, \quad q(0,2) = 1.1, \quad q(0,3) = 0.5, \\ q(1,0) &= 1, \quad q(1,1) = 0, \quad q(1,2) = 0.1, \quad q(1,3) = 0.5, \\ q(2,0) &= 1, \quad q(2,1) = 1, \quad q(2,2) = 0, \quad q(2,3) = 0.5, \\ q(3,0) &= 1, \quad q(3,1) = 0.5, \quad q(3,2) = 0.6, \quad q(3,3) = 0. \end{aligned}$$

Then q *is a* $c-$*distance as in Example 2.5.*

Define $f : X \to X$ *by* $f0 = 1, f1 = 2, f2 = 2, f3 = 2$. *Then* f *is nondecreasing. Consider* $x = 2, y = 0$. *Then* $q(f2, f0) = q(2,1) = 1$ *and*

$$\begin{aligned} a_1q(2,0) &+ a_2q(2,f2) + a_3q(0,f0) + a_4q(2,f0) \\ &= a_1q(2,0) + a_2q(2,2) + a_3q(0,1) + a_4q(2,1) \\ &= a_1 + a_3 + a_4 \le a_1 + a_3 + 2a_4 < 1 \end{aligned}$$

for any nonnegative real numbers $a_i\,(i = 1,2,3,4)$ *with* $a_1 + a_2 + a_3 + 2a_4 < 1$. *Hence the contractive condition of Theorem 4.3 is not satisfied and so Theorem 4.3 can not be applied to this example.*

But Theorem 4.4 can be applied to this example. In fact we take $a_1 = 0.14, a_2 = a_3 = a_4 = 0$ *and* $a_5 = 0.85$. *Then*

$$\begin{aligned} 1 &= q(f1,f0) < a_1q(1,0) + a_5q(0,f1) = 1.075, \\ 1 &= q(f2,f0) < a_1q(2,0) + a_5q(0,f2) = 1.075, \\ 1 &= q(f3,f0) < a_1q(3,0) + a_5q(0,f3) = 1.075. \end{aligned}$$

Set $x_0 = 3$ *and* $x_1 = 2$. *Then* $x_0 \sqsubseteq x_1 \sqsubseteq fx_0$. *Clearly* f *is continuous. Hence the hypotheses are satisfied and so by Theorem 4.4* f *has a fixed point* 2.

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<국문초록>

c−거리를 사용한 원뿔 거리공간(cone metric space) 에서의 부동점 정리

X를 임의의 집합, f를 X에서 X로 가는 함수라 하자. fx=x를 만족하는 X의 원소 x를 f의 부동점이라 한다. 부동점 이론은 현대수학의 가장 강력하고 풍부한 결과를 낳는 소재중 하나이며, 1922년에 바나흐(Banach)가 '완비거리공간에서 축소사상은 부동점을 갖는다'는 정리를 증명한 후에 많은 수학자들이 이 정리의 일반화된 결과를 얻었으며 이 정리는 비선형 해석학 이론의 주요 핵심 도구로 사용되고 있다.

Huang 그리고 Zhang([5]) 은 거리공간의 일반화된 공간인 원뿔 거리공간 (cone metric space) 개념을 도입하고 그 공간 내에서의 여러 부동점 정리를 얻었다. 그 후 특히 c−거리를 사용한 원뿔 거리공간에서의 부동점 정리 등 여러 부동점 정리의 개선을 위한 일련의 논문들이 나왔다.

공통부동점의 개념은 Jungck([6])에 의해 시작되었고 Wang 그리고 Guo([12])는 c−거리를 사용하여 원뿔 거리공간에서의 두 개의 함수에 관한 공통부동점에 관한 결과를 얻었다.

본 논문에서는 거리공간의 일반화된 cone 거리공간(cone metric space)에서 c−거리를 이용하여 축소조건을 만족하는 함수쌍에 대하여 유일한 공통일치점이 존재하기 위한 충분조건과 약한 양립 (weakly compatible)을 이용한 유일한 공통부동점을 갖기 위한 충분조건에 대한 결과를 얻었으며, 또한 부분순서를 갖는 완비 cone 거리공간에서 축소조건을 만족하는 감소하지 않는 연속함수에 대하여 부동점이 존재하기 위한 충분조건에 대한 결과를 얻었다.