CONVERGENCE ANALYSIS OF AN ITERATIVE ALGORITHM FOR THE IDENTIFICATION OF CONTINUOUS-TIME SYSTEMS

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ABSTRACT. In this paper the convergence of an iterative algorithm for the identification of continuous-time systems is discussed. It's convergent order is given. An initial guess is improved by means of the extrapolative scheme as well. Thus the improved algorithm could be used for both time-invariant and time-variant systems. It also makes the on-line identification possible. The numerical examples show that the computed results and computer cost are acceptable for time-variant systems.

I. Introduction

Consider a linear system

(1,1) $$\dot{X} = AX + BU$$

where $X \in R^n$, $U \in R^r$, $U \in R^{n \times n}$ and $B \in R^{n \times r}$ are the objects of the identification.

The discrete-time model of (1.1) is

(1.2) $$X(k+1) = FX(k) + GU(k)$$

where

(1.3) $$F = \exp\{AT\} = \sum_{k=0}^{\infty} \frac{1}{k!}(AT)^k$$

(1.4) $$G = \int_0^T \exp\{At\}Bdt = A^{-1}\sum_{k=1}^{\infty}\frac{1}{k!}(AT)^k B$$

and T denotes the uniform sampling interval which satisfies $\| AT \| < 0.5$ where $\| \quad \|$ is the spectral norm.

Let k denotes the sampling order number and

(1.5) $$X_k = (X(1), \quad X(2), \quad \cdots \quad , X(k))$$

(1.6) $$P_k = \begin{pmatrix} X(0) & X(1) & \cdots & X(k) \\ U(0) & U(1) & \cdots & U(k) \end{pmatrix}$$

(1.7) $$P_k{}^+ = P_k{}^T(P_k P_k{}^T)^{-1}$$

then the Least-Squares estimates of F and G can be determined using the following equation

(1.8) $$(\widehat{F_k}\widehat{G_k}) = X_k P_k{}^+$$

In order to estimate A and B a fixed-point iteration was proposed in [1]. The interative scheme given in [1] is

(1.9) $$(AT)_k^{(l+1)} = (AT)_k^{(l)} + \widehat{F_k^{-1}}(\widehat{F_k} - F_k^{(l)})$$

(1.10) $$F_k^{(l)} = I + (AT)_k^{(l)} + \frac{1}{2!}[(AT)_k^{(l)}]^2 + \frac{1}{3!}[(AT)_k^{(l)}]^3 + \cdots$$

and

(1.11) $$(AT)_k^{(0)} = \frac{1}{2}(\widehat{F_k} - \widehat{F_{k-1}})$$

where $(AT)_k^{(l)}$ is l-th iterative value of (AT) based upon the first k sampling points.

Unfortunately the convergence of (1.8)-(1.9) was not proved in [1], [2] and the initial guess (1.11) could not be given recursively such that the algorithm is inconvenient for on-line identification.

In Ⅱ the convergence of (1.9)-(1.11) is proved and its convergent order is given.

In Ⅲ an improved initial guess is suggested and an on-line identification scheme is considered.

Ⅱ. The Convergence of the Algorithm

Theorem 2.1 Let $A_k^* T$ be the true solution of AT at kth sampling point. We can associate error matrices ε_l defined by

(2.1)
$$\varepsilon_l = (AT)_k^{(l)} - A_k^* T \qquad l \geq 0$$

then the algorithm (1.9)-(1.11) is convergent if

(2.2)
$$||\varepsilon_0|| < \frac{6}{5} - \delta_0$$

where δ_0 is a positive number and less than 1.

Proof.

Based upon (1.9), (1,10) we have

(2.3)
$$\begin{aligned}(AT)_k^{(l+1)} &= (AT)_k^{(l)} + I - \widehat{F}_k^{-1} F_k^{(l)} \\ &= (AT)_k^{(l)} + I - \exp\{(AT)_k^{(l)} - A_k^* T\}\end{aligned}$$

Subtracting $A_k^* T$ from both sides of (2.3) and considering the definition of ε_l in (2.1) we get

(2.4)
$$\begin{aligned}\varepsilon_{l+1} &= \varepsilon_l + I - \exp\{\varepsilon_l\} \\ &= -\sum_{k=2}^{\infty} \frac{1}{k!} \varepsilon_l^k\end{aligned}$$

taking the norm of the matrices in (2.4), we obtain

(2.5) $$||\varepsilon_{l+1}|| \leq \frac{||\varepsilon_l||^2}{2}\sum_{k=0}^{\infty}\left(\frac{||\varepsilon_l||}{3}\right)^k \qquad l \geq 0$$

Specially we can conclude

(2.6) $$||\varepsilon_1|| < \frac{||\varepsilon_0||^2}{2}\frac{1}{1-||\varepsilon_0||/3} < (1-\delta)||\varepsilon_0||$$

where δ is a positive number and $\delta < 1$. The inequality (2.6) is resulted from the assumption (2.2).

Again considering (2.5) and by induction it follows that

(2.7) $$||\varepsilon_l|| < (1-\delta)^l||\varepsilon_0|| \qquad l > 0$$

Thus

$$\lim_{l\to\infty}||\varepsilon_l|| = 0.$$

The proof is completed.

Theorem 2.2 If the assumption (2.2) is satisfied, then the convergent order of the algorithm (1.9)-(1.11) is 2.

Furthermore , there exists a positive number α such that

(2.8) $$\lim_{l\to\infty}\frac{||\varepsilon_{l+1}||}{||\varepsilon_l||^2} = \alpha$$

and $\frac{1}{6} \leq \alpha \leq \frac{1}{2}$.

Proof.

From (2.2), (2.5) and (2.7) it follows that

(2.9) $$||\varepsilon_l|| < \frac{6}{5}$$

and

(2.10) $$\frac{||\varepsilon_{l+1}||}{||\varepsilon_l||^2} < \frac{1}{2}\frac{1}{1-||\varepsilon_l||/3} < \frac{1}{2}$$

respectively

On the other hand, from (2.9) and (2.4) we can get

$$||\varepsilon_{l+1}|| \geq \frac{||\varepsilon_l||^2}{2}\left|1-\left(\frac{||\varepsilon_l||}{3}+\frac{||\varepsilon_l||^2}{4\cdot 3}+\cdots\right)\right|,$$

i.e.,

(2.11) $$\frac{||\varepsilon_{l+1}||}{||\varepsilon_l||^2} > \frac{1}{2}(1-\frac{2}{3}) = \frac{1}{6} \qquad l > 0.$$

Combining (2.11) and (2.10) we can conclude

(2.12) $$\frac{1}{6} < \frac{||\varepsilon_{l+1}||}{||\varepsilon_l||^2} < \frac{1}{2}$$

Again from (2.7), it follows that $\{\|\varepsilon_l\|\}$ is a monotonic decreasing sequence.

Considering the convergence of the algorithm and the inequalities (2.10) and (2.9) we can say that

(2.13) $$\frac{||\varepsilon_{l+1}||/||\varepsilon_l||^2}{||\varepsilon_l||/||\varepsilon_{l-1}||^2} = \frac{||\varepsilon_{l+1}||\cdot||\varepsilon_{l-1}||^2}{||\varepsilon_l||^3} < 1$$

when $l > 1$

The inequality (2.13) means that the sequence $\{\frac{||\varepsilon_{l+1}||}{||\varepsilon_l||^2}\}$ is also monotonically decrescent after some iterations.

Hence there exists a limit of $\frac{||\varepsilon_{l+1}||}{||\varepsilon_l||^2}$ such that

(2.14) $$\lim_{l\to\infty}\frac{||\varepsilon_{l+1}||}{||\varepsilon_l||^2} = \alpha$$

and

$$\frac{1}{6} \leq \alpha \leq \frac{1}{2} \tag{2.15}$$

This complete our proof.

Ⅲ. Improved Initial Guess for an on-line Identification

For practical purposes it is important to identify a time-variant system on-line. In these cases, an improved initial guess is necessary for reducing the consuming computer time and identifying recursively. Thus the following initial guess, which can be recursively obtained, could be used instead of (1.11)

$$(AT)^{(0)}_{k+l} = 2(\widehat{A}T)_{k+l-1} - (\widehat{A}T)_{k+l-2} \tag{3.1}$$

where k denotes the size of the initial sampling data and $l=2,3,\cdots$. Then $(\widehat{A}T)_n$ is the final computed value of (AT) at nth sampling point. To start the iteration the first two initial guesses $(AT)^{(0)}_{k+1}$ and $(AT)^{(0)}_{k}$ are still determined by (1.11).

Our experience show that initial guess in (3.1) does work in many cases.

Essentially speaking the formula (3.1) is just the extrapolated mean of the $(\widehat{A}T)_{k+l-1}$ and $(\widehat{A}T)_{k+l-2}$. Thus the recursive estimating sequences of A and B could be computed as following

$$\widehat{A}_{k+l} = \frac{1}{T}(\widehat{A}T)_{k+l} \tag{3.2}$$

$$\widehat{B}_{k+l} = \widehat{R}^{-1}_{k+l}\widehat{G}_{k+l} \tag{3.3}$$

$$\widehat{R}_{k+l} = IT + \frac{T}{2!}[(\widehat{A}T)_{k+l}] + \frac{T}{3!}[(\widehat{A}T)_{k+l}]^2 + \cdots + \frac{T}{(M+1)!}[(\widehat{A}T)_{k+l}]^M \tag{3.4}$$

where M is some needed order number.

Ⅳ. Numerical examples

Several simplified examples are computed for both time-invariant and time-variant system. The examples show that initial guess(3.1) could provide not only satisfied accuracy as well as (1.11) but also convenience for on-line identification.

Example 4.1. A two dimensional time-invariant system is given as below.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

with $u(t) \equiv 1$.The observed data are

Table 4.1

t	$x_1(t)$	$x_2(t)$	$u(t)$
0.0	.0000000	.0000000	1
0.1	.0045280	.0861067	1
0.2	.0164293	.1484107	1
0.3	.0335876	.1920066	1
0.4	.0543444	.2209911	1
0.5	.0774090	.2386512	1
0.6	.1017856	.2476174	1
0.7	.1267132	.2499884	1
0.8	.1516194	.2474324	1
0.9	.1760797	.2412708	1
1.0	.1997883	.2325441	1
1.1	.2225305	.2220680	1
1.2	.2441649	.2104763	1
1.3	.2646052	.1982581	1
1.4	.2838078	.1857872	1
1.5	.3017632	.1733432	1
1.6	.3184854	.1611340	1
1.7	.3340034	.1493103	1
1.8	.3483633	.1379749	1
1.9	.3616166	.1271980	1
2.0	.3738227	.1170190	1

Based on all of 20 sampling points and the initial guess (1.11) the computed identification matrices are

$$\widehat{A}_{21} = \begin{pmatrix} 0.0000047 & 1.0000190 \\ -2.0000000 & -3.0000250 \end{pmatrix}$$

and

$$\widehat{B}_{21} = \begin{pmatrix} -0.0000047 \\ 1.0000020 \end{pmatrix}$$

respectively

Now taking the first 11 points in table 4.1 as the initial sampling data and using the initial guess given by the extrapolated scheme (3.1) we have the identified matrices $\widehat{A}_l$ and $\widehat{B}_l$ $(11 \le l \le 20)$ as follows.

table 4.2

t	matrix $\widehat{A}_{11+l}$	matrix $\widehat{B}_{11+l}$
1.1	$\begin{pmatrix} -0.0000087 & 1.0000130 \\ -1.9999880 & -3.0000120 \end{pmatrix}$	$\begin{pmatrix} -0.0000037 \\ 1.0000000 \end{pmatrix}$
1.2	$\begin{pmatrix} -0.0000007 & 1.0000190 \\ -1.9999970 & -3.0000130 \end{pmatrix}$	$\begin{pmatrix} -0.0000022 \\ 1.0000030 \end{pmatrix}$
⋯	⋯ ⋯	⋯
2.0	$\begin{pmatrix} 0.0000049 & 1.0000190 \\ -2.0000000 & -3.0000250 \end{pmatrix}$	$\begin{pmatrix} -0.0000048 \\ 1.0000020 \end{pmatrix}$

Example 4.2. A two dimensional time-variant system given below is considered

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} t & 1 \\ 0 & t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

with $u(t) \equiv 1$.

In the time-variant case, our algorithm gives the identified matrices $\int_0^t A(r)\,dr$ only.

Some numerical differentiation technique will be needed to determine the state transform matrices $A(t)$ and $B(t)$

Table 4.3

t	identified results of $\int_0^t A(r)dr$	real value of $\int_0^t A(r)dr$
0.25	Iteration number = 4 $\begin{pmatrix} .0312499 & .2500000 \\ .0000000 & .0312499 \end{pmatrix}$	$\begin{pmatrix} 0.03150 & 0.25 \\ 0 & 0.03150 \end{pmatrix}$
0.30	Iteration number = 4 $\begin{pmatrix} .0450001 & .3000000 \\ .0000000 & .0450001 \end{pmatrix}$	$\begin{pmatrix} 0.04500 & 0.3 \\ 0 & 0.04500 \end{pmatrix}$
0.35	Iteration number = 4 $\begin{pmatrix} .0612501 & .3500000 \\ .0000000 & .0612501 \end{pmatrix}$	$\begin{pmatrix} 0.06125 & 0.35 \\ 0 & 0.01625 \end{pmatrix}$
0.40	Iteration number = 4 $\begin{pmatrix} .0800000 & .4000000 \\ .0000000 & .0800000 \end{pmatrix}$	$\begin{pmatrix} 0.08000 & 0.4 \\ 0 & 0.08000 \end{pmatrix}$
0.45	Iteration number = 3 $\begin{pmatrix} .1012499 & .4500000 \\ .0000000 & .1012499 \end{pmatrix}$	$\begin{pmatrix} 0.10125 & 0.45 \\ 0 & 0.10125 \end{pmatrix}$
0.50	Iteration number = 3 $\begin{pmatrix} .1250000 & .5000000 \\ .0000000 & .1250000 \end{pmatrix}$	$\begin{pmatrix} 0.12500 & 0.5 \\ 0 & 0.12500 \end{pmatrix}$
...	... ...	... ...
2.00	$\begin{pmatrix} 1.9999970 & 2.0000000 \\ .0000000 & 1.9999970 \end{pmatrix}$	$\begin{pmatrix} 2.00000 & 2.00000 \\ 0.00000 & 0.00000 \end{pmatrix}$

REFERENCE

〔1〕. LASTMAN, G,J., Transformation algorithm for identification of continuous-time multivariable system from discrete data, Electron. Lett.,1981, 17, pp 779-780

〔2〕. LASTMAN, G.J., Identification of continuous-time multivariable systems from sampled data, Int.J.Control,1982,35,pp 117-126

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