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Term Rank Inequalities of Boolean Matrices and Linear Preservers

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부울 行列의 項 階數 不等式과 線型 保存者

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Term Rank Inequalities of Boolean Matrices and Linear Preservers

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CONTENTS

$<$Abstract $>$

TERM RANK INEQUALITIES OF BOOLEAN MATRICES AND LINEAR PRESERVERS

In this thesis, we research three topics on linear preserver problems, which have been researched in the international linear algebra society during last 100 years.

The first topic is to research the term ranks and their preservers of nonbinary Boolean matrices. We characterize the linear operators that preserve the sets of matrix pairs over nonbinary Boolean algebra which satisfy the extreme cases for certain term rank inequalities. We obtain these linear operators as $T(X)=PXQ$ or $T(X)=PXP^T$ with invertible Boolean matrices P and Q.

The second topic is to research the zero-term rank of nonbinary Boolean matrices. We characterize the linear operators that preserve the sets of matrix pairs over nonbinary Boolean algebra which satisfy the extreme cases for certain zero-term rank inequalities. We obtain those linear operators as $T(X)=PXQ$ or $T(X)=PXP^T$ with invertible Boolean matrices P and Q.

The third topic is to characterize the linear operators that preserve the regularity of nonbinary Boolean matrices. We obtain that a linear operator T strongly preserves regularity of nonbinary Boolean matrices if and only if T has the forms that $T(X)=UXV$ or $T(X)=U(\sum_{p=1}^{k}\sigma_p Y_p)V$ with invertible matrices U and V.

<국문초록>

부울 行列의 項 階數 不等式과 線型 保存者

本 論文에서는 國際線型代數學 分野에서 100년이 넘도록 研究되고 있는 線形保存者 問題의 一環으로 세 가지 主題를 중심으로 研究하였다.

첫째 主題는 일반화된 부울 代數 上의 行列의 項 係數와 관련된 行列 짝들의 集合을 보존하는 線形保存者를 糾明하는 研究이다.

一般的인 부울 代數 上의 集合에서 두 行列의 合과 곱에 대하여 項 階數의 값에 관한 不等式을 調査하여 그 不等式들이 等式이 되는 경우의 行列 짝들로 構成되는 여러 가지 極値 集合들을 構成하였다. 이 行列 짝들의 集合을 保存하는 線形保存者를 研究하여 그 形態를 糾明하였다. 곧, 이러한 行列 짝들의 集合을 保存하는 線形保存者의 形態는 $T(X)=PXQ$ 또는 $T(X)=PXP^T$ 와 같은 形態로 나타남을 보이고, 이들을 證明하였다.

둘째 主題는 일반화된 부울 代數 上의 行列의 零 項 係數와 관련된 行列짝들의 集合을 보존하는 線形保存者를 糾明하는 研究이다. 곧, 零 項 係數 不等式의 極値 집합들을 구성하여, 그 집합들을 保存하는 線型演算者의 形態를 $T(X)=PXQ$ 또는 $T(X)=PXP^T$ 와 같은 形態로 나타남을 보이고, 이들을 證明하였다.

셋째 主題는 일반화된 부울 代數 上에서 定規行列의 性質들을 調査 分析하고, 定規行列 X를 線形演算者로 보내어 다시 定規行列이 되게 할 경우에 그 線形演算者의 形態는 적당한 可逆行列 U와 V가 存在하여 $T(X)=UXV$ 또는 $T(X)=U(\sum_{p=1}^{k}\sigma_p Y_p)V$ 形態로 糾明됨을 밝혔고, 이를 부울 行列의 特性을 活用하여 證明하였다.

感謝의 글

수학에 대한 열정은 어느 누구 못지않게 대단하였지만 대학 졸업 후 여러 가지 이유로 더 이상 학업을 잇지 못한 것이 항상 마음 속에 아쉬움으로 자리잡고 있었습니다. 그 후 이십여 년간 가정일과 학교 업무로 바쁘지만 즐겁게 생활하면서도 시간이 지날수록 텅 비어가는 듯한 제 자신을 느끼게 되었습니다.

제 자신에게 변화와 재충전의 기회가 필요하다고 생각하면서 석사과정에 진학하겠다고 했을 때 가족들도 큰 반대없이 동의하여 주었습니다. 그렇게 시작한 지가 어느덧 7년을 넘기고 이렇게 큰 결실을 얻게 된 것은 저를 아끼고 배려해 준 분들의 도움이 없었다면 있을 수 없는 일이라 생각합니다.

학교에 근무하는 것을 배려하여 밤늦은 시간까지 강의해 주신 수학과 교수님들과 부족한 저를 세심한 지도로 이 자리까지 올 수 있게 해 주신 송석준 교수님께 더할 수 없는 존경과 감사의 마음을 전합니다.

석사논문 발표, 박사과정 강의 수강 등으로 가장 어려웠던 시기에 용기와 도움을 준 김녕구 식구들, 그리고 논문 마무리 과정에서 끝까지 최선을 다할 수 있도록 배려해 준 세화고 선생님들과 이 기쁨을 함께하고 싶습니다.

학교 업무, 대학원 수업 등으로 항상 시간에 쪼들리며 가사에 부족함이 많은 엄마 그리고 아내이지만 항상 신뢰하고 격려해 준 두 아들, 막내 딸 전영이 그리고 사랑하는 남편에게 감사하는 마음을 전합니다.

끝으로 힘든 투병 생활 속에서 삶의 끈을 놓지 않고 이겨내 준 사랑하는 친정어머니와 늙은 아내 병간호로 늘 고단하시면서도 딸의 학업과정을 꼭 챙기시며 흐뭇해하신 친정아버지께 모든 기쁨을 바칩니다.

2013년 1월

1 Introduction

One of the most active and fertile subjects in matrix theory during the past one hundred years is the linear preserver problem, which concerns the characterization of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant([24]). We call such a topic of research "Linear Preserver Problems". In 1887, Frobenius characterized the linear operators that preserve determinant of matrices over real field, which was the first results on linear preserver problems. After his result, many researchers have studied the linear operators that preserve some matrix functions, say, rank and permanent of matrices and so on([24]).

Recently, many researchers begin to research the matrices over semirings instead of fields([9] -[13]). There are many semirings such that (non)binary Boolean algebra, nonnegative integers, nonnegative reals, fuzzy semirings, max-algebra and so on([13]).

The results on linear preserver problems over semirigs are more applicable to linear preserver problems and combinatorics than those results over fields. The researches over a semiring are not easy to generalize those results over field since the system of semiring does not assume the additive inverse element for any element in the semiring. So we have to define many concepts for the properties of matrices over semirings to generalize the known definitions over fields.

Beasley and Guterman([2]) investigated rank inequalities of matrices over semirings. And they characterized the equality cases for some inequalities in [3]. These characterization problems are open even over fields(see [4]). The structure of matrix varieties which arise as extremal cases in these inequalities is far from being understood over fields, as well as over semirings. A usual way to generate elements of such a variety is to find a pair of matrices which belongs to it and to act on this pair by various linear operators that preserve this variety. The investigation of the corresponding problems over semirings for the column rank function was done in [4]. The complete classification of linear operators that preserve equality cases in matrix inequalities over fields was obtained in [7]. For details on linear operators preserving matrix invariants one can see [22] and [24]. Almost all researches on linear preserver problems over semirings have dealt with those semirings without zero-divisors to avoid the difficulties of multiplication arithmetic for the elements in those semirings([3]-[18]). But nonbinary Boolean algebra is not the case. That is, all elements except 0 and 1 in most nonbinary Boolean algebras are zero-divisors. So there are few results on the linear preserver problems for the matrices over nonbinary Boolean algebra([19], [20], [29]). Kirkland and Pullman characterized the linear operators that preserve rank of matrices over nonbinary Boolean algebra in [20].

Although there are many arithmetic difficulties of matrices over nonbinary Boolean algebra, we study the Boolean rank of matrices over nonbinary Boolean algebra and we characterize the linear operators that preserve pairs of matrices over nonbinary Boolean algebra which satisfy some term rank inequalities and zero-term rank inequalities.

In this thesis, we research three topics on the linear preserver problems.

The first topic is to characterize the linear operators that preserve the sets of matrix pairs over nonbinary Boolean algebra which satisfy the extreme cases for certain term rank inequalities. For this purpose, we study the inequalities of term rank for the sum or multiplication of matrices over nonbinary Boolean algebra. We also construct the sets of matrix pairs that satisfy the equalities for those term rank inequalities.

The second topic is to characterize the linear operators that preserve the sets of matrix pairs over nonbinary Boolean algebra which satisfy the extreme cases for certain zero-term rank inequalities. For this purpose, we also study the inequalities of zero-term rank for the sum or multiplication of matrices over nonbinary Boolean algebra. We also construct the sets of matrix pairs that satisfy the equalities for those zero-term rank inequalities.

The third topic is to characterize the linear operators that preserve regular matrices over nonbinary Boolean algebras.

The contents of this thesis are as follows:

In Chapter 2, we give some preliminaries and basic results for our purpose.

In Chapter 3, we study the extreme sets of matrix pairs for the term rank inequalities over nonbinary Boolean algebra and characterize the linear operators that preserve those extreme sets of matrix pairs.

In Chapter 4, we study the extreme sets of matrix pairs for the zero-term rank inequalities over nonbinary Boolean algebra and characterize the linear operators that preserve those extreme sets of matrix pairs.

In Chapter 5, we study the regular matrices over nonbinary Boolean algebra and characterize the linear operators that preserve those regular matrices.

2 Preliminaries and basic results

In this section, we give some definitions and construct sets of matrix pairs that arise as extremal cases in the term (and zero-term) rank inequalities of Boolean matrix sums and multiplications.

Definition 2.1. A *semiring* $\mathcal{S}$ consists of a set $\mathcal{S}$ and two binary operations, addition and multiplication, such that:

- $\mathcal{S}$ is an Abelian monoid under addition (identity denoted by 0);
- $\mathcal{S}$ is a semigroup under multiplication (identity, if any, denoted by 1);
- multiplication is distributive over addition on both sides;
- $s0 = 0s = 0$ for all $s \in \mathcal{S}$.

In this thesis we will always assume that there is a multiplicative identity 1 in $\mathcal{S}$ which is different from 0.

Definition 2.2. A semiring is called *antinegative* if the zero element is the only element with an additive inverse.

Definition 2.3. A semiring $\mathcal{S}$ is called a *Boolean algebra* if $\mathcal{S}$ is equivalent to a set of subsets of a given set M, the sum of two subsets is their union, and the product is their intersection. The zero element is the empty set and the identity element is the whole set M. That is, we denote $\phi = 0$ and $M = 1$.

Let $S_k = \{a_1, a_2, \cdots, a_k\}$ be a set of k-elements, $\mathcal{P}(S_k)$ be the set of all subsets of S_k and $\mathbb{B}_k$ be a Boolean algebra of subsets of $S_k = \{a_1, a_2, \cdots, a_k\}$, which is a subset of $\mathcal{P}(S_k)$.

Example 2.4. Let $S_3 = \{a_1, a_2, a_3\}$ be a set of 3-elements. Then,

$\mathbb{B}_3 = \{\phi, \{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}, \{a_1, a_2, a_3\}\}$

is a Boolean algebra of subsets of $S_3 = \{a_1, a_2, a_k\}$. Then $\{a_1, a_2\} \cdot \{a_3\} = 0$. That is, all elements, except ϕ and S_3, are zero-divisors. ■

It is straightforward to see that a Boolean algebra $\mathbb{B}_k$ is a commutative and antinegative semiring. If $\mathbb{B}_k$ consists of only the empty subset and M then it is called a *binary Boolean algebra*, which we denote $\mathbb{B}_1 = \{0, 1\}$. If $\mathbb{B}_k$ is not binary Boolean algebra then it is called a *nonbinary Boolean algebra*. Then all elements, except 0 and 1, are zero-divisors. Let $\mathbb{M}_{m,n}(\mathbb{B}_k)$ denote the set of $m \times n$ matrices with entries from the Boolean algebra $\mathbb{B}_k$. If $m = n$, we use the notation $\mathbb{M}_n(\mathbb{B}_k)$ instead of $\mathbb{M}_{n,n}(\mathbb{B}_k)$.

Throughout the thesis, we assume that $m \leq n$ and $\mathbb{B}_k$ denotes the nonbinary Boolean algebra, which contains at least 3 elements. The matrix I_n is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones and $O_{m,n}$ is the $m \times n$ zero matrix. We omit the subscripts when the order is obvious from the context and we write I, J and O, respectively. The matrix $E_{i,j}$, which is called a *cell*, denotes the matrix with exactly one nonzero entry, that being a one in the $(i, j)^{th}$ entry. A *weighted cell* is any nonzero scalar multiple of a cell, that is, $\alpha E_{i,j}$ is a weighted cell for any $0 \neq \alpha \in \mathbb{B}_k$. Let R_i denote the matrix whose i^{th} row is all ones and is zero elsewhere, and C_j denote the matrix whose j^{th} column is all ones and is zero elsewhere. We denote by $|A|$ the number of nonzero entries in the matrix A. We denote by A[i,j|r,s] the 2×2 submatrix of A which lies in the intersection of the i^{th} and j^{th} rows with the r^{th} and s^{th} columns.

Definition 2.5. ([4]) Let $\mathbb{B}_k$ be a nonbinary Boolean algebra. An operator $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ is called *linear* if it satisfies $T(X+Y) = T(X)+T(Y)$ and $T(\alpha X) = \alpha T(X)$ for all $X, Y \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ and $\alpha \in \mathbb{B}_k$.

Definition 2.6. A *line* of a matrix A is a row or a column of the matrix A.

Definition 2.7. The matrix $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ is said to be of *term rank* k ($t(A) = k$) if the least number of lines needed to include all nonzero elements of A is equal to k. Let us denote by $c(A)$ the least number of columns needed to include all nonzero elements of A and by $r(A)$ the least number of rows needed to include all nonzero elements of A.

Definition 2.8. The matrix $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ is said to be of *zero-term rank* k ($z(A) = k$) if the least number of lines needed to include all zero elements of A is equal to k.

The following rank functions are usual in the semiring context.

Definition 2.9. The matrix $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ is said to be of *factor rank* k ($rank(A) = k$) if there exist matrices $B \in \mathbb{M}_{m,k}(\mathbb{B}_k)$ and $C \in \mathbb{M}_{k,n}(\mathbb{B}_k)$ such that $A = BC$ and k is the smallest positive integer such that such a factorization exists. By definition the only matrix with factor rank equal to 0 is the zero matrix, O.

Example 2.10. Consider matrices $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ over $\mathbb{B}_k$. Then we can easily show that $t(A) = 2$, $z(A) = 2$, $rank(A) = 2$, $t(B) = 1$, $z(B) = 2$, and $rank(B) = 1$. ■

If $\mathcal{S}$ is a subsemiring of a certain field then there is a usual rank function $\rho(A)$ for any matrix $A \in \mathbb{M}_{m,n}(\mathcal{S})$. It is easy to see that these functions are not equal in general but the inequality $rank(A) \geq \rho(A)$ always holds.

The behaviour of the function ρ with respect to matrix multiplication and addition is given by the following inequalities:
The *rank-sum inequalities*:

$$| \rho(A) - \rho(B) | \leq \rho(A + B) \leq \rho(A) + \rho(B);$$

Sylvester's laws:

$$\rho(A) + \rho(B) - n \leq \rho(AB) \leq \min\{\rho(A), \rho(B)\}$$

and the *Frobenius inequality*:

$$\rho(AB) + \rho(BC) \leq \rho(ABC) + \rho(B),$$

where A, B, C are conformal matrices with coefficients from a field.

Arithmetic properties of term rank and zero-term rank of Boolean matrices are restricted by the following list of inequalities established in [2]:

1. $t(A + B) \leq t(A) + t(B)$;

2. $t(A + B) \geq \max\{t(A), t(B)\}$;

3. $t(AB) \leq \min(c(A), r(B))$

4. $t(AB) \geq t(A) + t(B) - n$;

5. If $\mathcal{S}$ is a subsemiring of positive reals then $\rho(AB) + \rho(BC) \leq t(ABC) + t(B)$;

6. $z(A + B) \geq 0$;

7. $z(A + B) \leq \min\{z(A), z(B)\}$;

8. $z(AB) \geq 0$;

9. $z(AB) \leq z(A) + z(B)$.

Below, we use the following notations in order to denote sets of Boolean matrices that arise as extremal cases in the inequalities listed above:

$$\mathbf{T}_{sa}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 | t(X + Y) = t(X) + t(Y)\};$$

$$\mathbf{T}_{sm}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 | t(X + Y) = \max(t(X), t(Y))\};$$

$$\mathbf{T}_{mn}(\mathbb{B}_k) = \Big\{(X, Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 | t(XY) = \min\{r(X), c(Y)\}\Big\};$$

$$\mathbf{T}_{ma}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 | t(XY) = t(X) + t(Y) - n\};$$

$$\mathbf{T}_{mt}(\mathbb{B}_k) = \{(X, Y, Z) \in \mathbb{M}_n(\mathbb{B}_k)^3 | t(XYZ) + t(Y) = rank(XY) + rank(YZ)\};$$

$$\mathbf{Z}_{sn}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 | z(X + Y) = \min\{z(X), z(Y)\}\};$$

$$\mathbf{Z}_{sz}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 | z(X + Y) = 0\};$$

$$\mathbf{Z}_{mz}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 | z(XY) = 0\};$$

$$\mathbf{Z}_{ms}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 | z(XY) = z(X) + z(Y)\}.$$

Definition 2.11. We say an operator, T, *preserves* a set $\mathcal{P}$ if $X \in \mathcal{P}$ implies that $T(X) \in \mathcal{P}$, or, if $\mathcal{P}$ is a set of ordered pairs, provided that $(X, Y) \in \mathcal{P}$ implies $(T(X), T(Y)) \in \mathcal{P}$, or, if $\mathcal{P}$ is a set of ordered triples, provided that $(X, Y, Z) \in \mathcal{P}$ implies $(T(X), T(Y), T(Z)) \in \mathcal{P}$.

Definition 2.12. The matrix $X \circ Y$ denotes the *Hadamard* or *Schur product*, i.e., the (i, j) entry of $X \circ Y$ is $x_{i,j}y_{i,j}$.

Definition 2.13. An operator T *strongly preserves* the set $\mathcal{P}$ if $X \in \mathcal{P}$ if and only if $T(X) \in \mathcal{P}$, or, if $\mathcal{P}$ is a set of ordered pairs, provided that $(X, Y) \in \mathcal{P}$ if and only if $(T(X), T(Y)) \in \mathcal{P}$, or, if $\mathcal{P}$ is a set of ordered triples, provided that $(X, Y, Z) \in \mathcal{P}$ if and only if $(T(X), T(Y), T(Z)) \in \mathcal{P}$.

Definition 2.14. An operator T is called a (P, Q, B)*-operator* if there exist permutation matrices P and Q, and a matrix B with no zero entries, such that $T(X) = P(X \circ B)Q$ for all $X \in \mathbb{M}_{m,n}(\mathcal{S})$, or, if $m = n$, $T(X) = P(X \circ B)^t Q$ for all $X \in \mathbb{M}_{m,n}(\mathcal{F})$. A (P, Q, B)-operator is called a (P, Q)-*operator* if $B = J$, the matrix of all ones.

It was shown in [1, 5, 7, 15] that linear preservers for extremal cases of classical matrix inequalities over fields are types of (U, V)-operators where U and V are arbitrary invertible matrices. On the other side, linear preservers for various rank functions over semirings have been the object of much study during the last years, see for example [10, 11, 12, 24], in particular term rank and zero term rank were investigated in the last years, see for example [8]. The aim of the present paper is to classify linear operators that preserve pairs of matrices that attain extreme cases in the inequalities 1 — 9.

Definition 2.15. We say that the matrix A *dominates* the matrix B if and only if $b_{i,j} \neq 0$ implies that $a_{i,j} \neq 0$, and we write $A \geq B$ or $B \leq A$.

Definition 2.16. If A and B are matrices and $A \geq B$ we let $A \backslash B$ denote the matrix C where

$$c_{i,j} = \begin{cases} 0 & \text{if } b_{i,j} \neq 0 \\ a_{i,j} & \text{otherwise} \end{cases} .$$

We begin with some basic results.

Theorem 2.17. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear operator. Then the following conditions are equivalent:*

(a) T is bijective;

(b) T is surjective;

(c) T is injective;

(d) there exists a permutation σ on $\{(i,j)|i = 1,2,\ldots,m; j = 1,2,\ldots,n\}$ such that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Proof. (a), (b) and (c) are equivalent since $\mathbb{M}_{m,n}(\mathbb{B}_k)$ is a finite set.

(d)$\Rightarrow$(b) For any $D \in \mathbb{M}_n(\mathbb{B}_k)$, we may write

$$D = \sum_{i=1}^{m}\sum_{j=1}^{n} d_{i,j}E_{i,j}.$$

Since σ is a permutation, there exist $\sigma^{-1}(i,j)$ and

$$D' = \sum_{i=1}^{m}\sum_{j=1}^{n} d_{\sigma^{-1}(i,j)}E_{\sigma^{-1}(i,j)}$$

such that

$$T(D') = T(\sum_{i=1}^{m}\sum_{j=1}^{n} d_{\sigma^{-1}(i,j)}E_{\sigma^{-1}(i,j)}) = \sum_{i=1}^{m}\sum_{j=1}^{n} d_{\sigma\sigma^{-1}(i,j)}E_{\sigma\sigma^{-1}(i,j)}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} d_{i,j}E_{i,j} = D.$$

(a)⇒(d) We assume that T is bijective. Suppose that $T(E_{i,j}) \neq E_{\sigma(i,j)}$ where σ be a permutation on $\{(i,j)|i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$. Then there exist some pairs (i,j) and (r,s) such that $T(E_{i,j}) = \alpha E_{r,s}(\alpha \neq 1)$ or some pairs $(i,j), (r,s)$ and (u,v) $((r,s) \neq (u,v))$ such that $T(E_{i,j}) = \alpha E_{r,s} + \beta E_{u,v} + Z(\alpha \neq 0, \beta \neq 0, Z \in \mathbb{M}_{m,n}(\mathbb{B}_k))$, where the $(r,s)^{th}$ and $(u,v)^{th}$ entries of Z are zeros.

Case 1) Suppose that there exist some pairs (i,j) and (r,s) such that $T(E_{i,j}) = \alpha E_{r,s}(\alpha \neq 1)$. Since T is bijective, there exist $X_{r,s} \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ such that $T(X_{r,s}) = E_{r,s}$. Then $\alpha T(X_{r,s}) = \alpha E_{r,s} = T(E_{i,j})$, and $T(\alpha X_{r,s}) = T(E_{i,j})$. Hence $\alpha X_{r,s} = E_{i,j}$, which contradicts the fact that $\alpha \neq 1$.

Case 2) Suppose that there exist some pairs $(i,j), (r,s)$ and (u,v) such that $T(E_{i,j}) = \alpha E_{r,s} + \beta E_{u,v} + Z(\alpha \neq 0, \beta \neq 0, Z \in \mathbb{M}_{m,n}(\mathbb{B}_k))$, where the $(r,s)^{th}$ and $(u,v)^{th}$ entries of Z are zeros. Since T is bijective, there exist $X_{r,s}$, $X_{u,v}$ and $Z' \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ such that $T(X_{r,s}) = \alpha E_{r,s}$, $T(X_{u,v}) = \beta E_{u,v}$, and $T(Z') = Z$. Thus $T(E_{i,j}) = \alpha E_{r,s} + \beta E_{u,v} + Z = T(X_{r,s}) + T(X_{u,v}) + T(Z') = T(X_{r,s} + X_{u,v} + Z')$. So $E_{i,j} = X_{r,s} + X_{u,v} + Z'$, a contradiction. ■

Remark 2.18. One can easily verify that if $m = 1$ or $n = 1$, then all operators under consideration are (P, Q, B)-operators and if $m = n = 1$, then all operators under consideration are (P, P^T, B)-operators.

Henceforth we will always assume that $m, n \geq 2$.

Lemma 2.19. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear operator which maps a line to a line and T be defined by the rule $T(E_{i,j}) = E_{\sigma(i,j)}$, where σ is a permutation on the set $\{(i,j)|i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$. Then T be a (P,Q)-operator.*

Proof. Since no combination of p rows and q columns can dominate J for any nonzero p and q with $p+q = m$, we have that either the image of each row is a row and the image of each column is a column, or $m = n$ and the image of each row is a column and image of each column is a row. Thus there are permutation matrices P and Q such that $T(R_i) \leq PR_iQ$, $T(C_j) \leq PC_jQ$ or, if $m = n$, $T(R_i) \leq P(R_i)^TQ$, $T(C_j) \leq P(C_j)^TQ$. Since each nonzero entry of a cell lies in the intersection of a row and a column and T maps cells to cells, it follows that $T(E_{i,j}) = PE_{i,j}Q$, or, if $m = n$, $T(E_{i,j}) = P(E_{i,j})^TQ$. ■

Lemma 2.20. *If $T(X) = X \circ B$ for all $X \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ and factor rank of B is 1, then there exist diagonal matrices D and E such that $T(X) = DXE$ for all $X \in \mathbb{M}_{m,n}(\mathbb{B}_k)$.*

Proof. Since factor rank of B is 1, there exist vectors $\mathbf{d} = [\mathbf{d_1}, \mathbf{d_2}, \ldots, \mathbf{d_m}]^{\mathbf{T}} \in \mathbb{M}_{\mathbf{m},\mathbf{1}}$ and $\mathbf{e} = [\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}] \in \mathbb{M}_{\mathbf{1},\mathbf{n}}$ such that $B = \mathbf{de}$ or $\mathbf{b_{i,j}} = \mathbf{d_i e_j}$. Let $D = diag\{d_1, d_2, \ldots, d_m\}$ and $E = diag\{e_1, e_2, \ldots, e_n\}$. Now the $(i,j)^{th}$

entry of $T(X)$ is $b_{i,j}x_{i,j}$ and the $(i,j)^{th}$ entry of DXE is $d_i x_{i,j} e_j = b_{i,j}x_{i,j}$. Hence $T(X) = DXE$. ■

Example 2.21. Consider the linear operator $T : \mathbb{M}_{3,3}(\mathbb{B}_3) \to \mathbb{M}_{3,3}(\mathbb{B}_3)$ defined by $T(X) = X \circ B$ for all $X \in \mathbb{M}_{3,3}(\mathbb{B}_3)$ with $\mathbb{B}_3 = \mathcal{P}(\{a,b,c\})$. Then $\mathrm{t}(B) = 3$ and $\mathrm{b}(B) = 1$ but we show that T does not preserves the term rank and the zero-term rank if $B \neq J$.

For, let $X = \begin{bmatrix} \{a,b\} & \{a,b,c\} & \{a,b\} \\ \{a,c\} & \{a,c\} & \{a,b\} \\ \{a\} & \{b,c\} & \{a,b\} \end{bmatrix}$ and $B = \begin{bmatrix} \{a\} & \{b\} & \{c\} \\ \{a\} & \{b\} & \{c\} \\ \{a\} & \{b\} & \{c\} \end{bmatrix}$.
Then $\mathrm{t}(X) = 3$, but

$$T(X) = X \circ B = \begin{bmatrix} \{a\} & \{b\} & 0 \\ \{a\} & 0 & 0 \\ \{a\} & \{b\} & 0 \end{bmatrix}.$$

That is, $\mathrm{t}(T(X)) = \mathrm{t}(X \circ B) = 2 \neq 3 = \mathrm{t}(X)$. Thus $\mathrm{t}(B) = 3$ but T does not preserves the term rank since every nonzero nonunit entry of B is a zero-divisor.

Moreover, $\mathrm{z}(T(X)) = \mathrm{z}(X \circ B) = 2 \neq 0 = \mathrm{z}(X)$. Thus $\mathrm{z}(B) = 0$ but T does not preserves the zero-term rank since every nonzero nonunit entry of B is a zero-divisor. ■

3 Extremes Preservers of Term Rank over Nonbinary Boolean Algebra

In this section, we characterize the linear operators that preserve the extreme set of matrix pairs, which are driven from the inequalities of the term ranks of matrices over nonbinary Boolean algebra.

We begin with a Lemma.

Lemma 3.1. *Let $\mathbb{B}_k$ be a nonbinary Boolean algebra, and $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a (P, Q)-operator. Then T preserves all term ranks.*

Proof. Assume that T is a (P, Q)-operator. For any $X \in \mathbb{M}_{m,n}(\mathbb{B}_k)$, we have

$$t(T(X)) = t(PXQ) = t(X)$$

or if $m = n$,

$$t(T(X)) = t(PX^tQ) = t(X^t) = t(X).$$

Hence any (P, Q)-operator preserves all term ranks. ∎

3.1 Characterization of linear operators that preserve $\mathbf{T}_{sa}(\mathbb{B}_k)$

Recall that

$$\mathbf{T}_{sa}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 | t(X + Y) = t(X) + t(Y)\}.$$

We show that $\mathbf{T}_{sa}(\mathbb{B}_2)$ is not an empty set.

Example 3.2. Let $\mathbb{B}_2 = \mathcal{P}(\{a,b\}) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$. Consider two matrices X and Y over $\mathbb{B}_2$:

$$X = \begin{bmatrix} \{a\} & \{a,b\} \\ 0 & 0 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 & 0 \\ \{a,b\} & \{b\} \end{bmatrix}.$$

Thus $\mathrm{t}(X) = \mathrm{t}(Y) = 1$ and

$$X + Y = \begin{bmatrix} \{a\} & \{a,b\} \\ \{a,b\} & \{b\} \end{bmatrix}$$

has term rank 2. Thus $(X, Y) \in \mathbf{T}_{sa}(\mathbb{B}_2)$. That is $\mathbf{T}_{sa}(\mathbb{B}_2) \neq \phi$. ■

Theorem 3.3. *Let $\mathbb{B}_k$ be a nonbinary Boolean algebra, $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a surjective linear map. Then T preserves the set $\mathbf{T}_{sa}(\mathbb{B}_k)$ if and only if T is a (P, Q)-operator, where P and Q are permutation matrices of appropriate sizes.*

Proof. ($\Leftarrow$) Assume that T is a (P, Q)-operator. Then T preserves all term ranks by Lemma 3.1. Therefore for any $(X, Y) \in \mathbf{T}_{sa}(\mathbb{B}_k)$, we have $\mathrm{t}(X+Y)$ $= \mathrm{t}(X) + \mathrm{t}(Y)$. Thus

$\mathrm{t}(T(X)+T(Y)) = \mathrm{t}(T(X+Y)) = \mathrm{t}(X+Y) = \mathrm{t}(X) + \mathrm{t}(Y) = \mathrm{t}(T(X)) + \mathrm{t}(T(Y))$.

Hence (P, Q)-operator preserves the set $\mathbf{T}_{sa}(\mathbb{B}_k)$.

($\Rightarrow$) If T is surjective, then by Theorem 2.17 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \leq i \leq m$, $1 \leq j \leq n$, where σ is a permutation on the set of pairs (i, j).

Let us show that T maps lines to lines. Suppose that the images of two cells are in the same line, but the cells are not, say $E_{i,j}, E_{k,l}$ are the cells such that $t(E_{i,j} + E_{k,l}) = 2$ and $t(T(E_{i,j} + E_{k,l})) = 1$. Then $(E_{i,j}, E_{k,l}) \in \mathbf{T}_{sa}(\mathbb{B}_k)$ but $(T(E_{i,j}), T(E_{k,l})) \notin \mathbf{T}_{sa}(\mathbb{B}_k)$, a contradiction. Thus T maps lines to lines. Thus by Lemma 2.19 T is a (P,Q)-operator where P and Q are permutation matrices of appropriate sizes. ■

Now we can improve Theorem 3.3 in the following way.

Theorem 3.4. *Let $\mathbb{B}_k$ be a nonbinary Boolean algebra, $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear map. Then T strongly preserves the set $\mathbf{T}_{sa}(\mathbb{B}_k)$ if and only if T is a (P,Q)-operator, where P and Q are permutation matrices of appropriate sizes.*

Proof. ($\Leftarrow$) By Lemma 3.1, (P,Q)-operator preserves the term rank. Hence it strongly preserves the set $\mathbf{T}_{sa}(\mathbb{B}_k)$ as we see in the proof of Theorem 3.3.

($\Rightarrow$) Suppose that T strongly preserves $\mathbf{T}_{sa}(\mathbb{B}_k)$ and $\mathbb{B}_k$ is finite and antinegative with identity 1. Then there exist positive integers $\alpha > \beta$ such that $\alpha \cdot 1 = \beta \cdot 1$. Also in this case there is some power of T which is idempotent, say $L = T^d$ and $L^2 = L$, see [11]. It is easy to see that L strongly preserves $\mathbf{T}_{sa}(\mathbb{B}_k)$.

Note that if $X \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ and $(X, X) \in \mathbf{T}_{sa}(\mathbb{B}_k)$ then necessarily $X = O$. Thus, if $A \neq O$, then $(A, A) \notin \mathbf{T}_{sa}(\mathbb{B}_k)$ and hence $(L(A), L(A)) \notin \mathbf{T}_{sa}(\mathbb{B}_k)$ since L strongly preserves $\mathbf{T}_{sa}(\mathbb{B}_k)$. Thus $L(A) \neq O$.

Suppose that there exists i, $1 \leq i \leq m$, such that $L(R_i)$ is not dominated by R_i. Then there is a pair of indexes (r, s) such that $E_{r,s}$ is not dominated by R_i and $L(R_i) \geq E_{r,s}$. Then $(R_i, E_{r,s}) \in \mathbf{T}_1$, and $L(R_i) = aE_{r,s} + X$ with $x_{r,s} = 0$.

Now,

$$\begin{aligned}
L(\beta R_i + (\alpha - \beta)aE_{r,s}) &= L(\beta R_i) + L((\alpha - \beta)aE_{r,s}) \\
&= L^2(\beta R_i) + L((\alpha - \beta)aE_{r,s}) \\
&= L(\beta L(R_i)) + L((\alpha - \beta)aE_{r,s}) \\
&= L(\beta(aE_{r,s} + X)) + L((\alpha - \beta)aE_{r,s}) \\
&= L(\beta aE_{r,s} + \beta X) + L((\alpha - \beta)aE_{r,s}) \\
&= L(\beta X) + L(\beta aE_{r,s}) + L((\alpha - \beta)aE_{r,s}) \\
&= L(\beta X) + L(\beta aE_{r,s} + (\alpha - \beta)aE_{r,s}) \\
&= L(\beta X) + L(\alpha aE_{r,s}) \\
&= L(\alpha X) + L(\alpha aE_{r,s}) \\
&= L(\alpha(X + aE_{r,s})) \\
&= L(\alpha L(R_i)) \\
&= L^2(\alpha R_i) \\
&= L(\alpha R_i) \\
&= L(\beta R_i).
\end{aligned}$$

Now, $(\beta R_i, (\alpha - \beta)aE_{r,s}) \in \mathbf{T}_{sa}(\mathbb{B}_k)$ but, $L(\beta R_i) + L((\alpha - \beta)aE_{r,s}) = L(\beta R_i + (\alpha - \beta)aE_{r,s}) = L(\beta R_i)$ and hence, $(L(\beta R_i), L((\alpha - \beta)aE_{r,s})) \notin \mathbf{T}_{sa}(\mathbb{B}_k)$, a contradiction.

We have established that $L(R_i) \leq R_i$ for all i. Similarly, $L(C_j) \leq C_j$ for all j. By considering that $E_{i,j}$ is dominated by both R_i and C_j we have that $L(E_{i,j}) \leq E_{i,j}$. Since $\mathcal{S}$ is antinegative, we have that T also maps a cell to a multiple of a cell, or $|T(E_{i,j})| = 1$ for all i, j, and $T(J)$ has all nonzero entries.

So T induces a permutation, σ, on the set of subscripts $\{1, 2, \cdots, m\} \times \{1, 2, \cdots, n\}$. That is, $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for some scalars $b_{i,j}$. But T does not preserve term rank if $b_{i,j} \neq 1$ from Example 2.21. So $T(E_{i,j}) = E_{\sigma(i,j)}$. Moreover we can show that T maps lines to lines by repeating the arguments used in the proof of Theorem 3.3. Therefore we obtain that T is a (P, Q)-operator. ■

3.2 Characterization of linear operators that preserve $\mathbf{T}_{sm}(\mathbb{B}_k)$

Recall that

$$\mathbf{T}_{sm}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 | t(X + Y) = \max(t(X), t(Y))\}.$$

We show that $\mathbf{T}_{sm}(\mathbb{B}_2)$ is not an empty set.

Example 3.5. Let $\mathbb{B}_2 = \mathcal{P}(\{a, b\}) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. Consider two matrices X and Y over $\mathbb{B}_2$:

$$X = \begin{bmatrix} \{a\} & \{a,b\} \\ 0 & \{b\} \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 & 0 \\ \{a,b\} & \{b\} \end{bmatrix}.$$

Thus $\mathrm{t}(X) = 2$, $\mathrm{t}(Y) = 1$ and

$$X + Y = \begin{bmatrix} \{a\} & \{a,b\} \\ \{a,b\} & \{b\} \end{bmatrix}$$

has term rank 2. Thus $(X, Y) \in \mathbf{T}_{sm}(\mathbb{B}_2)$. That is $\mathbf{T}_{sm}(\mathbb{B}_2) \neq \phi$. ■

Theorem 3.6. *Let $\mathbb{B}_k$ be a nonbinary Boolean algebra, $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a surjective linear map. Then T preserves the set $\mathbf{T}_{sm}(\mathbb{B}_k)$ if and only if T is a (P, Q)-operator, where P and Q are permutation matrices of appropriate sizes and elements.*

Proof. If T is surjective, then by Theorem 2.17 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \leq i \leq m$, $1 \leq j \leq n$, where σ is a permutation on the set of pairs (i, j).

Suppose that the images of two cells are not in the same line, but the cells are, say $E_{i,j}, E_{i,l}$ are the cells such that $T(E_{i,j}), T(E_{i,l})$ are not in the same line, i.e., $t(T(E_{i,j} + E_{i,l})) = 2$. Then $(E_{i,j}, E_{i,l}) \in \mathbf{T}_{sm}(\mathbb{B}_k)$ but $(T(E_{i,j}), T(E_{i,l})) \notin \mathbf{T}_{sm}(\mathbb{B}_k)$, a contradiction. Thus T^{-1} maps lines to lines. By Lemma 2.19 it follows that T^{-1} is a (P, Q)-operator where P and Q are permutation matrices of appropriate sizes. Hence, T is also of this type.

Conversely, by Lemma 3.1, any (P, Q)-operator preserves the term rank. Thus as we see in the proof of Theorem 3.3, any (P, Q)-operator preserves the set $\mathbf{T}_{sm}(\mathbb{B}_k)$. ∎

3.3 Characterization of linear operators that preserve $\mathbf{T}_{mn}(\mathbb{B}_k)$

Recall that

$$\mathbf{T}_{mn}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 | t(XY) = \min(r(X), c(Y))\}.$$

We show that $\mathbf{T}_{mn}(\mathbb{B}_2)$ is not an empty set.

Example 3.7. Let $\mathbb{B}_2 = \mathcal{P}(\{a, b\}) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. Consider two matrices X and Y over $\mathbb{B}_2$:

$$X = \begin{bmatrix} \{a\} & \{a, b\} \\ 0 & 0 \end{bmatrix} \text{ and } Y = \begin{bmatrix} \{a, b\} & 0 \\ \{b\} & 0 \end{bmatrix}.$$

Thus $\mathrm{r}(X) = 1$, $\mathrm{c}(Y) = 1$ and

$$XY = \begin{bmatrix} \{a\} + \{b\} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \{a, b\} & 0 \\ 0 & 0 \end{bmatrix}$$

has term rank 1. Thus $(X, Y) \in \mathbf{T}_{mn}(\mathbb{B}_2)$. That is $\mathbf{T}_{mn}(\mathbb{B}_2) \neq \phi$. ∎

Theorem 3.8. *Let $\mathbb{B}_k$ be a nonbinary Boolean algebra, $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a surjective linear map. Then T preserves the set $\mathbf{T}_{mn}(\mathbb{B}_k)$ if and only if T is a nontransposing (P, P^t)-operator, where P is a permutation matrix.*

Proof. ($\Leftarrow$) By similar proof of the Lemma 3.1, it is easy to see that any nontransposing (P, P^t)-operator preserves $t(A)$, $c(A)$ and $r(A)$. Therefore any nontransposing (P, P^t)-operator preserves the set $\mathbf{T}_{mn}(\mathbb{B}_k)$.

($\Rightarrow$) Assume that T preserves the set $\mathbf{T}_{mn}(\mathbb{B}_k)$. Since T is surjective, by Theorem 2.17 one has that $T(E_{i,j}) = E_{\sigma(i,j)}$.

Let us show that T transforms lines to lines. For all k one has that $(E_{i,j}, E_{j,k}) \in \mathbf{T}_{mn}(\mathbb{B}_k)$ since

$$t(E_{i,j}E_{j,k}) = t(E_{i,k}) = 1 = \min\{r(E_{i,j}), c(E_{j,k})\}.$$

Thus $(T(E_{i,j}), T(E_{j,k})) \in \mathbf{T}_{mn}(\mathbb{B}_k)$ by assumption, so $t(T(E_{i,j})T(E_{j,k})) = \min\{r(T(E_{i,j})), c(T(E_{j,k}))\} = 1$ since T transforms cells to cells. But $T(E_{i,j})T(E_{j,k}) = E_{\sigma(i,j)}E_{\sigma(j,k)}$ so that $E_{\sigma(j,k)}$ is in the same row as $E_{\sigma(j,1)}$ for every k. That is, T maps rows to rows. Similarly T maps columns to columns. That is, $T(X) = PXQ$ for some permutation matrices P and Q.

Therefore, $T(E_{i,j}) = E_{\sigma(i),\tau(j)}$ where σ is the permutation corresponding to P and τ is the permutation corresponding to Q^t. But, $(E_{1,i}, E_{i,1}) \in \mathbf{T}_{mn}(\mathbb{B}_k)$. Thus $(E_{\sigma(1),\tau(i)}, E_{\sigma(i),\tau(1)}) \in \mathbf{T}_{mn}(\mathbb{B}_k)$ by assumption, and hence $\tau \equiv \sigma$. This implies that $Q^t = P$ and hence T is a nontransposing (P, P^t)-operator. ■

3.4 Characterization of linear operators that preserve $\mathbf{T}_{ma}(\mathbb{B}_k)$

Recall that

$$\mathbf{T}_{ma}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 | t(XY) = t(X) + t(Y) - n\}.$$

We show that $\mathbf{T}_{ma}(\mathbb{B}_2)$ is not an empty set.

Example 3.9. Let $\mathbb{B}_2 = \mathcal{P}(\{a, b\}) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. Consider two matrices X and Y over $\mathbb{B}_2$:

$$X = \begin{bmatrix} \{a\} & \{a,b\} \\ \{a\} & \{a,b\} \end{bmatrix} \text{ and } Y = \begin{bmatrix} \{a,b\} & \{a\} \\ \{a,b\} & \{a\} \end{bmatrix}.$$

Thus $\mathrm{t}(X) = 2$, $\mathrm{t}(Y) = 2$ and

$$XY = \begin{bmatrix} \{a,b\} & \{a\} \\ \{a,b\} & \{a\} \end{bmatrix}$$

has term rank 2. Thus $(X, Y) \in \mathbf{T}_{ma}(\mathbb{B}_2)$. That is $\mathbf{T}_{ma}(\mathbb{B}_2) \neq \phi$. ■

To study linear preservers of the equality in the multiplicative low bound the following reduction is vital:

Lemma 3.10. *Let* $\mathbb{B}_k$ *be a nonbinary Boolean algebra, and* $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ *preserve the set* $\mathbf{T}_{ma}(\mathbb{B}_k)$. *Then* T *preserves the set of matrices with term rank* n.

Proof. Let $A = 0$ and let B be any matrix of term rank n over $\mathbb{B}_k$. Then, $t(A) = 0$, $t(AB) = 0$. Hence, $t(AB) = t(A) + t(B) - n$. It follows that $t(T(A)T(B)) = t(T(A)) + t(T(B)) - n$. That is $0 = 0 + t(T(B)) - n$. It follows that $t(T(B)) = n$. That is, T preserves term rank n. ■

Lemma 3.11. *Let* $\mathbb{B}_k$ *be a nonbinary Boolean algebra, and* $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ *be a surjective linear map. Then* T *preserves the set of matrices with term rank* n *if and only if* T *is a* (P, Q)*-operator where* P *and* Q *are permutation matrices of appropriate sizes.*

Proof. ($\Leftarrow$) By Lemma 3.1, any (P, Q)-operator preserves all the term ranks. Thus T preserves the set of matrices with term rank n.

($\Rightarrow$) By Theorem 2.17 one has that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \leq i, j \leq n$, where σ is a permutation on the set of pairs of indexes. Let us show that T^{-1} maps lines to lines. Assume that the pre-image of a row is not dominated by any line. Then there are indexes i, k, l such that $T^{-1}(E_{i,k}$ and $T^{-1}(E_{i,l})$ are not in one line. That is, there is indexes p, r, q, s, $p \neq r$, $q \neq s$, such that $T^{-1}(E_{i,k} + E_{i,l}) \leq E_{r,s} + E_{p,q}$, and $T^{-1}(E_{i,k} + E_{i,l})$ is not dominated by each of the cells $E_{r,s}$, $E_{p,q}$. By extending $E_{r,s} + E_{p,q}$ to a permutation matrix by adding $n - 2$ cells, we find a matrix A such that $t(A) = n$. Since T preserves term rank n by assumption, one has that $t(T(A)) = n$. On the other hand, $T(A)$ is dominated by $(n-1)$ lines since $T(E_{r,s}) = E_{i,k}$ and $T(E_{p,q}) = E_{i,l}$ lie in one row. This is a contradiction with $t(T(A)) = n$. Thus the pre-image of every row is a row or a column. Similarly, the pre-image of every column is a row or a column. It follows by Lemma 2.19 that T is a (P, Q)-operator. ∎

Theorem 3.12. *Let $\mathbb{B}_k$ be a nonbinary Boolean algebra, $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a surjective linear operator. Then T preserves the set $\mathbf{T}_{ma}(\mathbb{B}_k)$ if and only if T is a nontransposing (P, P^t)-operator, where P is a permutation matrix.*

Proof. ($\Leftarrow$) Let us prove that a nontransposing (P, P^t)-operator preserve the set $\mathbf{T}_{ma}(\mathbb{B}_k)$. By Lemma 3.1 any (P, Q)-operator preserves all the term ranks. Thus the right-hand side of the equality determining $\mathbf{T}_{ma}(\mathbb{B}_k)$ is not changed under the mapping by a nontransposing (P, P^t)-operator T and the left-hand side of the equality also is not changed since $t(T(X)T(Y)) = t(PXP^tPYP^t) = t(PXYP^t) = t(XY)$.

($\Rightarrow$) Assume that T preserves the set $\mathbf{T}_{ma}(\mathbb{B}_k)$. Then by Lemma 3.10 T preserves the set of matrices with term rank n. Since T is surjective, by applying Lemma 3.11 we obtain that T is a (P, Q)-operator.

Now, let us see that transposition transformation does not preserve the set $\mathbf{T}_{ma}(\mathbb{B}_k)$. Indeed, the pair $(X = E_{i,j}, Y = I - E_{j,j}) \in \mathbf{T}_{ma}(\mathbb{B}_k)$ since $t(XY) = t(0) = 0 = 1 + (n-1) - n = t(X) + t(Y) - n$. However, $(X^t = E_{j,i}, Y^t = I - E_{j,j}) \notin \mathbf{T}_{ma}(\mathbb{B}_k)$ since $t(X^tY^t) = t(E_{j,i}) = 1 \neq 0$.

It remains to prove that $PQ = I$ the identity matrix. Let us assume that a nontransposing (P, Q)-operator preserves the set $\mathbf{T}_{ma}(\mathbb{B}_k)$. Thus one has that $t(XY) = t(PXQPYQ) = t((XQPY)$ for all pairs $(X, Y) \in \mathbf{T}_{ma}(\mathbb{B}_k)$. The matrix QP is permutation matrix as a product of two permutation matrices. Assume that QP permutes i'th and j'th columns of X. Let $X = E_{i,i}$, $Y = \sum\limits_{j \neq i} E_{j,j}$. Thus $t(X) = 1$, $t(Y) = n - 1$, $t(XY) = t(0) = 0 = t(X) + t(Y) - n$, i.e., $(X, Y) \in \mathbf{T}_{ma}(\mathbb{B}_k)$. On the other side, $XQP = E_{i,j}$. Thus $XQPY = E_{i,j} \neq 0$. Hence, $(T(X), T(Y)) = (PXQ, PYQ) \notin \mathbf{T}_{ma}(\mathbb{B}_k)$. This contradiction concludes that $QP = I$ and hence T is a nontransposing (P, P^t)-operator. ∎

3.5 Characterization of linear operators that preserve $\mathbf{T}_{mt}(\mathbb{B}_k)$

Recall that

$$\mathbf{T}_{mt}(\mathbb{B}_k) = \{(X, Y, Z) \in \mathbb{M}_n(\mathbb{B}_k)^3 | t(XYZ) + t(Y) = t(XY) + t(YZ)\}.$$

We show that $\mathbf{T}_{mt}(\mathbb{B}_2)$ is not an empty set.

Example 3.13. Let $\mathbb{B}_2 = \mathcal{P}(\{a, b\}) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. Consider three matrices X, Y and Z over $\mathbb{B}_2$:

$$X = \begin{bmatrix} \{a,b\} & 0 \\ 0 & \{a,b\} \end{bmatrix}, Y = \begin{bmatrix} \{a\} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } Z = \begin{bmatrix} 0 & 0 \\ 0 & \{b\} \end{bmatrix}.$$

Thus $t(XYZ) = 0$, $t(Y) = 1$, $t(XY) = 1$ and $t(YZ) = 0$. Thus $(X, Y, Z) \in \mathbf{T}_{mt}(\mathbb{B}_2)$. That is $\mathbf{T}_{mt}(\mathbb{B}_2) \neq \phi$. ∎

Theorem 3.14. *Let $\mathbb{B}_k$ be a nonbinary Boolean algebra, and $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \rightarrow \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a surjective linear map. Then T preserves the set $\mathbf{T}_{mt}(\mathbb{B}_k)$ if and only if T is a nontransposing (P, P^t)-operator where P and Q are permutation matrices of appropriate sizes.*

Proof. ($\Leftarrow$) By Lemma 3.1, any (P, Q)-operator preserves all the term ranks. Thus as we see in the proof of Theorem 3.3, any nontransposing (P, P^t)-operator preserves the set $\mathbf{T}_{mt}(\mathbb{B}_k)$.

($\Rightarrow$) By Theorem 2.17 one has that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \leq i, j \leq n$, where σ is a permutation on the set of pairs of indices.

It can be directly checked that $(E_{i,j}, E_{j,k}, E_{k,l}) \in \mathbf{T}_{mt}(\mathbb{B}_k)$ for all l and for arbitrary fixed i, j, k. Thus

$$\begin{aligned} &t(T(E_{i,j})T(E_{j,k})) + t(T(E_{j,k})T(E_{k,l})) \\ &= t(T(E_{i,j})T(E_{j,k})T(E_{k,l})) + t(T(E_{j,k})). \end{aligned} \tag{1}$$

Let us denote $T(E_{i,j}) = E_{p,q}$, $T(E_{j,k}) = E_{r,s}$, and $T(E_{k,l}) = E_{u,v}$. Since $t(E_{r,s}) = 1 \neq 0$, it follows from the equality (1) that either $q = r$ or $s = u$ or both. If for all $l = 1, \ldots, n$ it holds that $q = r$ or for all $l = 1, \ldots, n$ it holds that $s = u$ then it is easy to see that T maps lines to lines. Assume that there exists an index l such that $r \neq q$. Thus by (1) $s = u$. Hence, for arbitrary m, $1 \leq m \leq n$ one has that $(E_{i,j}, E_{j,k}, E_{k,m}) \in \mathbf{T}_{mt}(\mathbb{B}_k)$. Denote, $T(E_{k,m}) = E_{w,z}$. Using the previous notations, one obtains that $(E_{p,q}, E_{r,s}, E_{w,z}) \in \mathbf{T}_{mt}(\mathbb{B}_k)$. Since $q \neq r$ it follows that $w = s$ and hence T maps kth row to sth row. Thus in this case we obtain that rows are transformed to rows. By the same arguments with the first matrix it is easy to see that columns are transformed to columns. In the other case $s \neq u$ and $q = r$ one obtains that rows are transformed to columns and columns to rows.

By Lemma 2.19 it follows that there exists a permutation matrices P and Q such that $T(X) = PXQ$ for all $X \in \mathcal{M}_n(\mathbb{B}_k)$ or $T(X) = PX^tQ$.

In order to show that the transposition transformation does not preserve $\mathbf{T}_{mt}(\mathbb{B}_k)$ it suffices to note that $(E_{i,j}, I, I - E_{j,j}) \in \mathbf{T}_{mt}(\mathbb{B}_k)$ and $(E_{j,i}, I, I - E_{j,j}) \notin \mathbf{T}_{mt}(\mathbb{B}_k)$.

In order to show that $Q = P^t$ it suffices to note that$(E_{i,j}, E_{j,j}, E_{j,i}) \in \mathbf{T}_{mt}(\mathbb{B}_k)$. Denote that $T(E_{i,j}) = E_{\sigma(i),\tau(j)}$ where σ is the permutation corresponding to P and τ is the permutation corresponding to Q^t. Therefore, $(E_{\sigma(i),\tau(j)}, E_{\sigma(j),\tau(j)}, E_{\sigma(j),\tau(i)}) \in \mathbf{T}_{mt}(\mathbb{B}_k)$ by assumption, and hence $\tau \equiv \sigma$. This implies that $Q^t = P$ and hence T is a nontransposing (P, P^t)-operator. ■

4 Extremes Preservers of Zero-Term Rank over Nonbinary Boolean Algebra

In this section, we characterize the linear operators that preserve the extreme set of matrix pairs, which are driven from the inequalities of the zero-term ranks of matrices over nonbinary Boolean algebra.

We begin with a Lemma.

Lemma 4.1. *Let $\mathbb{B}_k$ be a nonbinary Boolean algebra, and $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a (P, Q)-operator. Then T preserves all zero-term ranks.*

Proof. Assume that T is a (P, Q)-operator. For any $X \in \mathbb{M}_{m,n}(\mathbb{B}_k)$, we have

$$z(T(X)) = z(PXQ) = z(X)$$

or if m=n,

$$z(T(X)) = z(PX^tQ) = z(X^t) = z(X).$$

Hence any (P, Q)-operator preserves all zero-term ranks. ■

4.1 Characterization of linear operators that preserve $\mathbf{Z}_{sn}(\mathbb{B}_k)$

Recall that

$$\mathbf{Z}_{sn}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 | z(X + Y) = \min\{z(X), z(Y)\}\}.$$

We show that $\mathbf{Z}_{sn}(\mathbb{B}_2)$ is not an empty set.

Example 4.2. Let $\mathbb{B}_2 = \mathcal{P}(\{a,b\}) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$. Consider two matrices X and Y over $\mathbb{B}_2$:

$$X = \begin{bmatrix} \{a\} & \{a,b\} \\ 0 & 0 \end{bmatrix} \text{ and } Y = \begin{bmatrix} \{a,b\} & \{b\} \\ 0 & 0 \end{bmatrix}.$$

Thus $z(X) = z(Y) = 1$ and

$$X + Y = \begin{bmatrix} \{a,b\} & \{a,b\} \\ 0 & 0 \end{bmatrix}$$

has zero-term rank 1. Thus $(X,Y) \in \mathbf{Z}_{sn}(\mathbb{B}_2)$. That is $\mathbf{Z}_{sn}(\mathbb{B}_2) \neq \phi$. ■

Theorem 4.3. *Let $\mathbb{B}_k$ be a nonbinary Boolean algebra, and $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a surjective linear map. Then T preserves the set $\mathbf{Z}_{sn}(\mathbb{B}_k)$ if and only if T is a (P,Q)-operator where P and Q are permutation matrices of appropriate sizes.*

Proof. ($\Rightarrow$) By Theorem 2.17 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \leq i \leq m$, $1 \leq j \leq n$, where σ is a permutation on the set of pairs (i,j).

Let us show that T maps lines to lines. Suppose that the images of two cells are not in the same line, but the cells are, say $E_{i,j}, E_{i,k}$ are the cells such that $T(E_{i,j}), T(E_{i,k})$ are not in the same line. Then one has that $z((J - E_{i,j} - E_{i,k}) + E_{i,k}) = 1 = z(J - E_{i,j} - E_{i,k})$, i.e. $(J - E_{i,j} - E_{i,k}, E_{i,k}) \in \mathbf{Z}_{sn}(\mathbb{B}_k)$, as far as $z(T(J - E_{i,j} - E_{i,k}) + T(E_{i,k})) = 1 < 2 = min\{z(T(J - E_{i,j} - E_{i,k})), z(T(E_{i,k}))\}$, i.e. $(T(J - E_{i,j} - E_{i,k}), T(E_{i,k})) \notin \mathbf{Z}_{sn}(\mathbb{B}_k)$, a contradiction. Thus T maps lines to lines.

By Lemma 2.19 it follows that T is a (P,Q)-operator where P and Q are permutation matrices of appropriate sizes.

$(\Leftarrow)$ Assume that T is a (P,Q)-operator. Then T preserves all zero-term ranks by Lemma 4.1. Therefore for any $(X,Y) \in \mathbf{Z}_{sn}(\mathbb{B}_k)$, we have $z(X+Y) = \min\{z(X), z(Y)\}$. Thus $z(T(X)+T(Y)) = z(T(X+Y)) = z(X+Y) = \min\{z(X), z(Y)\} = \min\{z(T(X), z(T(Y)\}$. Hence (P,Q)-operator preserves the set $\mathbf{Z}_{sn}(\mathbb{B}_k)$. ■

4.2 Characterization of linear operators that preserve $\mathbf{Z}_{sz}(\mathbb{B}_k)$

Recall that

$$\mathbf{Z}_{sz}(\mathbb{B}_k) = \{(X,Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 | z(X+Y) = 0\}.$$

We show that $\mathbf{Z}_{sz}(\mathbb{B}_2)$ is not an empty set.

Example 4.4. Let $\mathbb{B}_2 = \mathcal{P}(\{a,b\}) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$. Consider two matrices X and Y over $\mathbb{B}_2$:

$$X = \begin{bmatrix} \{a\} & \{a,b\} \\ 0 & 0 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 & 0 \\ \{a,b\} & \{b\} \end{bmatrix}.$$

Thus $z(X) = z(Y) = 1$ but

$$X+Y = \begin{bmatrix} \{a\} & \{a,b\} \\ \{a,b\} & \{b\} \end{bmatrix}$$

has zero-term rank 0. Thus $(X,Y) \in \mathbf{Z}_{sz}(\mathbb{B}_2)$. That is $\mathbf{Z}_{sz}(\mathbb{B}_2) \neq \phi$. ■

Theorem 4.5. *Let $\mathbb{B}_k$ be a nonbinary Boolean algebra, and $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear map. Then T preserves the set $\mathbf{Z}_{sz}(\mathbb{B}_k)$ if and only if T is a permutation on the set of all cells.*

Proof. ($\Leftarrow$) Assume that T is a permutation on the set of all cells. That is, $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \leq i \leq m$, $1 \leq j \leq n$, where σ is a permutation on the set of pairs (i, j).

Consider $(A, B) \in \mathbf{Z}_{sz}(\mathbb{B}_k)$. Then $z(A + B) = 0$. From antinegativity it follows that sets of zero cells in A and B are disjoint. Thus the same holds for $T(A)$ and $T(B)$ since σ is a permutation. Hence in $(T(A) + T(B))$ there is no zero elements and hence $(T(A), T(B)) \in \mathbf{Z}_{sz}(\mathbb{B}_k)$. Thus such a linear operator T preserve the set $\mathbf{Z}_{sz}(\mathbb{B}_k)$.

($\Rightarrow$) Assume that T preserves the set $\mathbf{Z}_{sz}(\mathbb{B}_k)$. If T is not a permutation on the set of all cells, then there is two distinct cells $E_{i,j}, E_{h,k}$ such that $T(E_{i,j}) = T(E_{h,k}) = E_{p,q}$. Then $z(J) = 0$ but $z(T(J)) > 1$, and hence $(J, 0) \in \mathbf{Z}_{sz}(\mathbb{B}_k)$ but $(T(J), T(0)) \notin \mathbf{Z}_{sz}(\mathbb{B}_k)$, a contradiction. ∎

4.3 Characterization of linear operators that preserve $\mathbf{Z}_{mz}(\mathbb{B}_k)$

Recall that

$$\mathbf{Z}_{mz}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 | z(XY) = 0\}.$$

We show that $\mathbf{Z}_{mz}(\mathbb{B}_2)$ is not an empty set.

Example 4.6. Let $\mathbb{B}_2 = \mathcal{P}(\{a, b\}) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. Consider two matrices X and Y over $\mathbb{B}_2$:

$$X = \begin{bmatrix} \{a\} & \{a\} \\ \{b\} & \{b\} \end{bmatrix} \text{ and } Y = \begin{bmatrix} \{a, b\} & 0 \\ 0 & \{a, b\} \end{bmatrix}.$$

Then $XY = \begin{bmatrix} \{a\} & \{a\} \\ \{b\} & \{b\} \end{bmatrix}$, and hence $z(XY) = 0$. ∎

Theorem 4.7. *Let $\mathbb{B}_k$ be a nonbinary Boolean algebra, and $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$ be a linear surjective map. Then T preserves the set $\mathbf{Z}_{mz}(\mathbb{B}_k)$ if and only if T is a nontransposing (P, P^t)-operator, where P is a permutation matrix.*

Proof. (⇐) By Lemma 4.1, nontransposing (P, P^t)-operators preserve all the zero-term ranks. Let $(X, Y) \in \mathbf{Z}_{mz}(\mathbb{B}_k)$. Then $z(XY) = 0$ and hence XY has no zero entries. Since T is a nontransposing (P, P^t)-operator, one has $T(X)T(Y) = PXP^tPYP^t = PXYP^t$, which has no zero entries. Thus $(T(X), T(Y)) \in \mathbf{Z}_{mz}(\mathbb{B}_k)$. Hence T preserves the set $\mathbf{Z}_{mz}(\mathbb{B}_k)$.

(⇒) By Theorem 2.17 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \le i \le m$, $1 \le j \le n$, where σ is a permutation on the set of pairs (i, j).

Let us show that T maps lines to lines. Suppose that the images of two cells are in the same line, but the cells are not, say $E_{i,j}, E_{i,k}$ are the cells such that $T^{-1}(E_{i,j}), T^{-1}(E_{i,k})$ are not in the same line. Let us consider $A = T^{-1}(J \backslash R_i)$. Thus there are no zero rows of A since T is a permutation on the set of cells and not all elements of i'th row lie in one row by the choice of i. Hence AJ does not have zero elements by the antinegativity and $z(AJ) = 0$.

Thus $(A, J) \in \mathbf{Z}_{mz}(\mathbb{B}_k)$ as far as $(T(A), T(J)) = (J \setminus R_i, T(J)) \notin \mathbf{Z}_{mz}(\mathbb{B}_k)$, a contradiction. Thus T^{-1} maps lines to lines. Hence T maps lines to lines.

By Lemma 2.19 it follows that T is a (P, Q)-operator where P and Q are permutation matrices of appropriate sizes.

In order to prove that transposition operator does not preserve $\mathbf{Z}_{mz}(\mathbb{B}_k)$ it suffices to take the pair (C_1, R_1). That is, $(C_1, R_1) \in \mathbf{Z}_{mz}(\mathbb{B}_k)$ but $(C_1^t, R_1^t) = (R_1, C_1) \notin \mathbf{Z}_{mz}(\mathbb{B}_k)$.

Now, let us show that $Q = P^t$. Assume in the contrary that $QP \neq I$. Thus there exists indexes i, j such that QP transforms i'th column into j'th column. In this case we take matrices $A = J \setminus (E_{1,1} + \ldots + E_{1,n}) + E_{1,i}$, $B = J \setminus E_{j,n}$. Thus AB has no zero elements, i.e., $z(AB) = 0$. However, the $(1,1)$'th element of $QT(A)T(B)P$ is zero, i.e., $z(T(A)T(B)) \neq 0$. This contradiction concludes that $Q = P^t$. Thus T is a nontransposing (P, P^t)-operator. ■

4.4 Characterization of linear operators that preserve $\mathbf{Z}_{ms}(\mathbb{B}_k)$

Recall that

$$\mathbf{Z}_{ms}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 | z(XY) = z(X) + z(Y)\}.$$

We show that $\mathbf{Z}_{ms}(\mathbb{B}_2)$ is not an empty set.

Example 4.8. Let $\mathbb{B}_2 = \mathcal{P}(\{a, b\}) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. Consider two matrices X and Y over $\mathbb{B}_2$:

$$X = \begin{bmatrix} \{a\} & \{b\} \\ 0 & \{b\} \end{bmatrix} \text{ and } Y = \begin{bmatrix} \{a\} & \{b\} \\ \{a\} & \{b\} \end{bmatrix}.$$

Then $XY = \begin{bmatrix} \{a\} & \{b\} \\ 0 & \{b\} \end{bmatrix}$ and hence $(X, Y) \in \mathbf{Z}_{ms}(\mathbb{B}_k)$. ■

Theorem 4.9. *Let $\mathbb{B}_k$ be a nonbinary Boolean algebra, and $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$ be a linear surjective map. Then T preserves the set $\mathbf{Z}_{ms}(\mathbb{B}_k)$ if and only if T is a nontransposing (P, P^t)-operator, where P and Q are permutation matrices of order n.*

Proof. ($\Leftarrow$) By Lemma 4.1, nontransposing (P, P^t)-operators preserve all the zero-term ranks. Let $(X, Y) \in \mathbf{Z}_{ms}(\mathbb{B}_k)$. Then $z(XY) = z(X) + z(Y)$. Since T is a nontransposing (P, P^t)-operator, one has $T(X)T(Y) = PXP^tPYP^t = PXYP^t$, which has the same zero-term rank as $z(XY)$. And $z(T(X)) + z(T(Y)) = z(X) + z(Y)$. Thus $(T(X), T(Y)) \in \mathbf{Z}_{ms}(\mathbb{B}_k)$. Hence T preserves the set $\mathbf{Z}_{ms}(\mathbb{B}_k)$.

($\Rightarrow$) By Theorem 2.17 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \le i \le m$, $1 \le j \le n$, where σ is a permutation on the set of pairs (i, j).

Let us show that T maps lines to lines. Suppose that the images of two cells are not in the same line, but the cells are, say $E_{i,j}, E_{i,k}$ are the cells such that $T(E_{i,j}), T(E_{i,k})$ are not in the same line. Note that

$$z((J \setminus R_i)J) = z(J \setminus R_i) = 1 = 1 + 0 = z(J \setminus R_i) + z(J).$$

Thus $(J \setminus R_i, J) \in \mathbf{Z}_{ms}(\mathbb{B}_k)$. On the other hand, $T(J) = J$ and $T(J \setminus R_i)$ has at least two lines containing zero entries, so one has $z(T(J \setminus R_i)) + z(T(J)) \ge 2$. But $T(J \setminus R_i)$ has no rows containing only zero entries and $T(J) = J$, so one has $z(T(J \setminus R_i)T(J)) = z(J) = 0$. Hence $(T(J \setminus R_i), T(J)) \notin \mathbf{Z}_{ms}(\mathbb{B}_k)$. This contradiction shows that T maps lines to lines.

By Lemma 2.19 it follows that T is a (P, Q)-operator where P and Q are permutation matrices of appropriate sizes.

In order to prove that transposition operator does not preserve $\mathbf{Z}_{ms}(\mathbb{B}_k)$ it suffices to take the pair of matrices $X = J \setminus R_1$, $Y = J \setminus C_1$ since $(X, Y) \in \mathbf{Z}_{ms}(\mathbb{B}_k)$ but $(X^t, Y^t) \notin \mathbf{Z}_{ms}(\mathbb{B}_k)$.

Now, let us show that $Q = P^t$. Assume in the contrary that $QP \neq I$. Thus there exist indexes i, j such that QP transforms i'th column into j'th column. In this case we take matrices $A = J \setminus C_i$, $B = R_i$. Thus $AB = 0$ and hence $z(AB) = n$. And $z(A) + Z(B) = n$. Therefore $(A, B) \in \mathbf{Z}_{ms}(\mathbb{B}_k)$. However, $T(A)T(B) = PAQPBQ = P(J \setminus C_j)R_iQ = PJQ = J$ has zero-term rank 0 while $z(T(A)) + z(T(B)) = z(PAQ) + z(PBQ) = z(A) + z(B) = n$. Therefore $(T(A), T(B)) \notin \mathbf{Z}_{ms}(\mathbb{B}_k)$. This contradiction concludes that $Q = P^t$. Thus T is a nontransposing (P, P^t)-operator. ∎

5 Regular matrices preservers over (non)binary Boolean Algebra

In this section, we study some properties of regular matrices over nonbinary Boolean algebras $\mathbb{B}_k$. We also determine the linear operators on $\mathbb{M}_n(\mathbb{B}_k)$ that strongly preserve regular matrices.

5.1 Some basic properties of regular matrices

A matrix X in $\mathbb{M}_n(\mathbb{B}_k)$ is said to be *invertible* if there is a matrix Y in $\mathbb{M}_n(\mathbb{B}_k)$ such that $XY = YX = I_n$.

In 1952, Luce [21] showed a matrix A in $\mathcal{M}_n(\mathbb{B}_1)$ possesses a two-sided inverse if and only if A is an orthogonal matrix in the sense that $AA^t = I_n$, and that, in this case, A^t is a two-sided inverse of A. In 1963, Rutherford [27] showed if a matrix A in $\mathcal{M}_n(\mathbb{B}_1)$ possesses a one-sided inverse, then the inverse is also a two-sided inverse. Furthermore such an inverse, if it exists, is unique and is A^T. Also, it is well known that the $n \times n$ permutation matrices are the only $n \times n$ invertible matrices over the binary Boolean algebra.

Let $\sigma_1 = \{a_1\}$, $\sigma_p = \{a_p\}$ for $p = 1, 2, ..., k$. For any matrix $A = [a_{i,j}]$ in $\mathbb{M}_n(\mathbb{B}_k)$, the p^{th} *constituent*, A_p, of A is the matrix in $\mathcal{M}_n(\mathbb{B}_1)$ whose $(i,j)^{th}$ entry is 1 if and only if $a_{i,j} \supseteq \sigma_p$. Via the constituents, A can be written uniquely as $A = \sum_{p=1}^{k} \sigma_p A_p$ which is called the *canonical form* of A. It follows from the uniqueness of the decomposition and the fact that the singletons are mutually orthogonal idempotents that for all matrices $A, B, C \in \mathbb{M}_n(\mathbb{B}_k)$ and for all $\alpha \in \mathbb{B}_k$,

$$(A + B)_p = A_p + B_p, \quad (BC)_p = B_p C_p \quad \text{and} \quad (\alpha A)_p = \alpha_p A_p \tag{5.1.1}$$

for all $p = 1, \ldots, k$.

Lemma 5.1. *([20]) For any matrix A in $\mathbb{M}_n(\mathbb{B}_k)$ with $k \geq 1$, A is invertible if and only if its all constituents are permutation matrices. In particular, if A is invertible, then $A^{-1} = A^T$.*

The notion of generalized inverse of an arbitrary matrix apparently originated in the work of Moore [23], and the generalized inverses have applications in network and switching theory and information theory ([14]).

Let A be a matrix in $\mathbb{M}_n(\mathbb{B}_k)$. Consider a matrix $X \in \mathbb{M}_n(\mathbb{B}_k)$ in the equation

$$AXA = A. \tag{5.1.2}$$

If (5.1.2) has a solution $X \in \mathbb{M}_n(\mathbb{B}_k)$, then X is called a *generalized inverse* of A. Furthermore A is called *regular* if there is a solution of (5.1.2).

The equation (5.1.2) has been studied by several authors ([17], [23], [25], [26]). Rao and Rao [26] characterized all regular matrices in $\mathcal{M}_n(\mathbb{B}_1)$. Also Plemmons [25] published algorithms for computing generalized inverses of regular matrices in $\mathcal{M}_n(\mathbb{B}_1)$ under certain conditions.

Matrices J and O in $\mathbb{M}_n(\mathbb{B}_k)$ are regular because $JGJ = J$ and $OGO = O$ for all cells G in $\mathbb{M}_n(\mathbb{B}_k)$. Therefore in general, a solution of (5.1.2), although it exists, is not necessarily unique. Furthermore each cell E in $\mathbb{M}_n(\mathbb{B}_k)$ is regular because $EE^tE = E$.

Proposition 5.2. *Let A be a matrix in $\mathbb{M}_n(\mathbb{B}_k)$. If U and V are invertible matrices in $\mathbb{M}_n(\mathbb{B}_k)$, then the following are equivalent:*

(i) A *is regular*;

(ii) UAV *is regular*;

(iii) A^T *is regular.*

Proof. The proof is an easy exercise. ∎

Also we can easily show that

$$A \text{ is regular if and only if } \begin{bmatrix} A & O \\ O & B \end{bmatrix} \text{ is regular} \tag{5.1.3}$$

for all matrices $A \in \mathbb{M}_n(\mathbb{B}_k)$ and for all regular matrices $B \in \mathcal{M}_m(\mathbb{B}_k)$. In particular, all idempotent matrices in $\mathbb{M}_n(\mathbb{B}_k)$ are regular.

For any zero-one matrices $A = [a_{i,j}]$ and $B = [b_{i,j}]$ in $\mathbb{M}_n(\mathbb{B}_k)$, we define $A \setminus B$ to be the zero-one matrix $C = [c_{i,j}]$ such that $c_{i,j} = 1$ if and only if $a_{i,j} = 1$ and $b_{i,j} = 0$ for all i and j.

Define an upper triangular matrix Λ_n in $\mathbb{M}_n(\mathbb{B}_k)$ by

$$\Lambda_n = [\lambda_{i,j}] \equiv \left(\sum_{i \le j}^{n} E_{i,j} \right) \setminus E_{1,n} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 \\ & 1 & \cdots & 1 & 1 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}.$$

Then the following Lemma shows that Λ_n is not regular for $n \ge 3$.

Lemma 5.3. Λ_n *is regular in* $\mathbb{M}_n(\mathbb{B}_k)$ *if and only if* $n \le 2$.

Proof. For $n \le 2$, clearly Λ_n is regular because $\Lambda_n I_n \Lambda_n = \Lambda_n$.

Conversely, assume that Λ_n is regular for some $n \ge 3$. Then there is a nonzero matrix $B = [b_{i,j}]$ in $\mathbb{M}_n(\mathbb{B}_k)$ such that $\Lambda_n = \Lambda_n B \Lambda_n$. From $0 = \lambda_{1,n} =$

$\sum_{i=1}^{n-1}\sum_{j=2}^{n} b_{i,j}$, we obtain all entries of the second column of B are zero except for the entry $b_{n,2}$. From $0 = \lambda_{2,1} = \sum_{i=2}^{n} b_{i,1}$, we have all entries of the first column of B are zero except for $b_{1,1}$. Also, from $0 = \lambda_{3,2} = \sum_{i=3}^{n}\sum_{j=1}^{2} b_{i,j}$, we obtain $b_{n,2} = 0$. If we combine these three results, we conclude all entries of the first two columns are zero except for $b_{1,1}$. But we have $1 = \lambda_{2,2} = \sum_{i=2}^{n}\sum_{j=1}^{2} b_{i,j} = 0$, a contradiction. Hence Λ_n is not regular for all $n \geq 3$. ■

In particular, $\Lambda_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is not regular in $\mathcal{M}_3(\mathbb{B}_k)$. Let

$$\Phi_n = \begin{bmatrix} \Lambda_3 & O \\ O & O \end{bmatrix} \tag{5.1.4}$$

for all $n \geq 3$. Then Φ_n is not regular in $\mathbb{M}_n(\mathbb{B}_k)$ by (5.1.3).

Note that for a matrix $A = [a_{i,j}]$ in $\mathbb{M}_n(\mathbb{B}_k)$, the p^{th} constituent, A_p, of A is the matrix in $\mathcal{M}_n(\mathbb{B}_1)$ whose $(i,j)^{th}$ entry is 1 if and only if $a_{i,j} \supseteq \sigma_p$.

Example 5.4. Let $k \geq 2$. Consider the matrix

$$A = \begin{bmatrix} 1 & \sigma_1 & 0 \\ 0 & \sigma_1 & \sigma_1 \\ 0 & 0 & \sigma_1 \end{bmatrix} \in \mathcal{M}_3(\mathbb{B}_k).$$

Then $A_1 = \Lambda_3$ is not regular in $\mathcal{M}_3(\mathbb{B}_1)$, while $A_p = E_{1,1}$ is regular in $\mathcal{M}_3(\mathbb{B}_1)$ for all $p = 2, 3, \ldots, k$. The below Theorem shows that A is not regular in $\mathcal{M}_3(\mathbb{B}_k)$. ■

Theorem 5.5. *Let A be a matrix in $\mathbb{M}_n(\mathbb{B}_k)$. Then A is regular in $\mathbb{M}_n(\mathbb{B}_k)$ if and only if its all constituents are regular in $\mathcal{M}_n(\mathbb{B}_1)$.*

Proof. If A is regular in $\mathbb{M}_n(\mathbb{B}_k)$, then all constituents of A are regular in $\mathcal{M}_n(\mathbb{B}_1)$ by (5.1.1).

Conversely, assume that each constituent A_p of A is regular in $\mathcal{M}_n(\mathbb{B}_1)$ for all $p = 1, \ldots, k$. Then there are matrices $G_1, \ldots, G_k$ in $\mathcal{M}_n(\mathbb{B}_1)$ such that $A_p G_p A_p = A_p$ for all $p = 1, \ldots, k$. If $G = \sum_{p=1}^{k} \sigma_p G_p$, then we can easily show that $AGA = A$ and hence A is regular in $\mathbb{M}_n(\mathbb{B}_k)$. ∎

Theorem 5.5 shows that the regularity of a matrix A in $\mathbb{M}_n(\mathbb{B}_k)$ depends only on the regularities of its all constituents in $\mathcal{M}_n(\mathbb{B}_1)$. Henceforth we suffice to consider properties of regular matrices in $\mathcal{M}_n(\mathbb{B}_1)$.

The *Boolean factor rank*([9]) of a nonzero matrix $A \in \mathbb{M}_n(\mathbb{B}_k)$ is defined as the least integer r for which there are Boolean matrices B and C of orders $n \times r$ and $r \times n$, respectively such that $A = BC$. We denote $rank(A)$ as $b(A)$ for any $A \in \mathbb{M}_n(\mathbb{B}_k)$. The rank of a zero matrix is zero. Also we can easily obtain that

$$0 \leq b(A) \leq n \quad \text{and} \quad b(AB) \leq \min\{b(A), b(B)\} \tag{5.1.5}$$

for all $A, B \in \mathbb{M}_n(\mathbb{B}_k)$.

Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ be a matrix in $\mathbb{M}_n(\mathbb{B}_k)$, where $\mathbf{a}_j$ denotes the j^{th} column of A for all $j = 1, \ldots, n$. Then the *column space* of A is the set $\left\{ \sum_{j=1}^{n} \alpha_j \mathbf{a}_j \,\middle|\, \alpha_j \in \mathbb{B}_k \right\}$, and denoted by $< A >$; the *row space* of A is $< A^T >$.

For a matrix $A \in \mathbb{M}_n(\mathbb{B}_k)$ with $b(A) = r$, A is said to be *space decomposable* if there are matrices B and C of orders $n \times r$ and $r \times n$, respectively such that $A = BC$, $< A > = < B >$ and $< A^T > = < C^T >$.

Theorem 5.6. *([26]) A is regular in $\mathcal{M}_n(\mathbb{B}_1)$ if and only if A is space decomposable.*

Let A be a matrix in $\mathbb{M}_n(\mathbb{B}_k)$. By Theorem 5.5 and 5.6, A is regular in $\mathbb{M}_n(\mathbb{B}_k)$ if and only if its all constituents are space decomposable in $\mathcal{M}_n(\mathbb{B}_1)$.

Lemma 5.7. *If A is a matrix in $\mathcal{M}_n(\mathbb{B}_1)$ with $b(A) \leq 2$, then A is regular.*

Proof. If $b(A) = 0$, then $A = O$ is clearly regular. If $b(A) = 1$, then there exist permutation matrices P and Q such that $PAQ = \begin{bmatrix} J & O \\ O & O \end{bmatrix}$, and hence PAQ is regular by (5.1.3). It follows from Proposition 5.2 that A is regular.

Suppose $b(A) = 2$. Then there are matrices $B = [\mathbf{b}_1 \quad \mathbf{b}_2]$ and $C = [\mathbf{c}_1 \quad \mathbf{c}_2]^T$ of orders $n \times 2$ and $2 \times n$, respectively such that $A = BC$, where $\mathbf{b}_1$ and $\mathbf{b}_2$ are distinct nonzero columns of B, and $\mathbf{c}_1$ and $\mathbf{c}_2$ are distinct nonzero columns of C^T. Then we can easily show that all columns of A are of the forms $\mathbf{0}, \mathbf{b}_1, \mathbf{b}_2$ and $\mathbf{b}_1 + \mathbf{b}_2$ so that $< A >=< B >$. Similarly, all columns of A^T are of the forms $\mathbf{0}, \mathbf{c}_1, \mathbf{c}_2$ and $\mathbf{c}_1 + \mathbf{c}_2$ so that $< A^T >=< C^T >$. Therefore A is space decomposable and hence A is regular by Theorem 5.6. ■

For matrices $A = [a_{i,j}]$ and $B = [b_{i,j}]$ in $\mathbb{M}_n(\mathbb{B}_k)$, we say B *dominates* A (written $B \geq A$ or $A \leq B$) if $b_{i,j} = 0$ implies $a_{i,j} = 0$ for all i and j. This provides a reflexive and transitive relation on $\mathbb{M}_n(\mathbb{B}_k)$.

The number of nonzero entries of a matrix A in $\mathbb{M}_n(\mathbb{B}_k)$ is denoted by $|A|$. The number of elements in a set $\mathbb{S}$ is also denoted by $|\mathbb{S}|$.

Corollary 5.8. *Let A be a nonzero matrix in $\mathcal{M}_n(\mathbb{B}_1)$, where $n \geq 3$.*

(i) *If* $|A| \leq 4$, *then* A *is regular*;

(ii) *If* $|A| \leq 2$, *there is a matrix* B *such that* $|A+B| = 5$ *and* $A+B$ *is not regular*;

(iii) *If* $|A| = 3$ *and* $b(A) = 2$ *or* 3, *there is a matrix* C *with* $|C| = 2$ *such that* $A+C$ *is not regular*;

(iv) *If* $|A| = 5$ *and* A *has a row or a column that has at least* 3 *nonzero entries, then* A *is regular.*

Proof. (i) By Lemma 5.7, we lose no generality in assuming that $b(A) \geq 3$ so that $b(A) = 3$ or 4. Consider the matrix $X = \begin{bmatrix} A & O \\ O & 0 \end{bmatrix}$ in $\mathcal{M}_{n+1}(\mathbb{B}_1)$. Since $|A| \leq 4$ and $b(A) = 3$ or 4, we can easily show that there are permutation matrices P and Q of orders $n+1$ such that $PXQ = \begin{bmatrix} Y & O \\ O & O \end{bmatrix}$ for some idempotent matrix Y in $\mathcal{M}_4(\mathbb{B}_1)$ with $|Y| = 3$ or 4. By (5.1.3) and Proposition 5.2, X is regular and hence A is regular by (5.1.3).

(ii) If $|A| \leq 2$, we can easily show that there are permutation matrices P and Q such that $PAQ \leq \Phi_n$. Let $B' = \Phi_n \backslash PAQ$. Then we have $PAQ+B' = \Phi_n$ so that $A + P^T B' Q^T = P^T \Phi_n Q^T$ is not regular by Proposition 5.2. If we let $B = P^T B' Q^T$, then we have $|A+B| = 5$ and $A+B$ is not regular.

(iii) Similar to (ii).

(iv) If $|A| = 5$ and A has a row or a column that has at least 3 nonzero entries, then we can easily show that $b(A) \leq 3$. By Lemma 5.7, it suffices to consider $b(A) = 3$. Then A has either a row or a column that has just 3 nonzero entries. Suppose that a row of A has just 3 nonzero entries. Since $b(A) = 3$, there are permutation matrices P and Q such that

$$PAQ = E_{1,1} + E_{1,2} + E_{1,3} + E_{2,i} + E_{3,j}$$

for some $i, j \in \{1, \ldots, n\}$ with $i < j$. If $j \geq 4$, then PAQ is regular by the above result (i) and (5.1.4), and hence A is regular by Proposition 5.2. If $1 \leq i < j \leq 3$, then there are permutation matrices P' and Q' such that $P'PAQQ' = \begin{bmatrix} D & O \\ O & O \end{bmatrix}$, where $D = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We can easily show that D is idempotent in $\mathcal{M}_3(\mathbb{B}_1)$, and hence D is regular. It follows from (5.1.3) and Proposition 5.2 that A is regular.

If a column of A has just 3 nonzero entries, a parallel argument shows that A is regular. ■

Linearity of operators on $\mathbb{M}_n(\mathbb{B}_k)$ is defined as for vector spaces over fields. A linear operator on $\mathbb{M}_n(\mathbb{B}_k)$ is completely determined by its behavior on the set of cells in $\mathbb{M}_n(\mathbb{B}_k)$.

An operator T on $\mathbb{M}_n(\mathbb{B}_k)$ is said to be *singular* if $T(X) = O$ for some nonzero matrix $X \in \mathbb{M}_n(\mathbb{B}_k)$; Otherwise T is *nonsingular*.

An operator T on $\mathbb{M}_n(\mathbb{B}_k)$

(1) *preserve regularity* if $T(A)$ is regular whenever A is regular in $\mathbb{M}_n(\mathbb{B}_k)$;

(2) *strongly preserve regularity* provided that $T(A)$ is regular if and only if A is regular in $\mathbb{M}_n(\mathbb{B}_k)$.

Example 5.9. Let A be any regular matrix in $\mathbb{M}_n(\mathbb{B}_k)$, where at least one entry of A is 1. Define an operator T on $\mathbb{M}_n(\mathbb{B}_k)$ by

$$T(X) = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} x_{i,j}\right)A$$

for all $X = [x_{i,j}] \in \mathbb{M}_n(\mathbb{B}_k)$. Then we can easily show that T is nonsingular and T is a linear operator that preserves regularity. But T does not preserve any matrix that is not regular in $\mathbb{M}_n(\mathbb{B}_k)$. ■

Thus, we are interested in linear operators on $\mathbb{M}_n(\mathbb{B}_k)$ that strongly preserve regularity.

Lemma 5.10. *Let $n \geq 3$. If T is a linear operator on $\mathcal{M}_n(\mathbb{B}_1)$ that strongly preserves regularity, then T is nonsingular.*

Proof. If $T(X) = O$ for some nonzero matrix X in $\mathcal{M}_n(\mathbb{B}_1)$, then we have $T(E) = O$ for all cells $E \leq X$. By Corollary 5.8(ii), there is a matrix B such that $|B| = 4$ and $E + B$ is not regular, while B is regular by Corollary 5.8(i). Nevertheless, $T(E + B) = T(B)$, a contradiction to the fact that T strongly preserves regularity. Hence $T(X) \neq O$ for all nonzero matrix X in $\mathcal{M}_n(\mathbb{B}_1)$. Therefore T is nonsingular. ■

If $n \leq 2$, then all matrices in $\mathcal{M}_n(\mathbb{B}_1)$ are regular by (5.1.5) and Lemma 5.7. Therefore all matrices in $\mathbb{M}_n(\mathbb{B}_k)$ are also regular by Theorem 5.5. This proves:

Theorem 5.11. *If $n \leq 2$, then all operators on $\mathbb{M}_n(\mathbb{B}_k)$ strongly preserve regularity.*

5.2 Characterization of linear operators that strongly preserve regular matrices over the binary Boolean algebra

In this section we have characterizations of the linear operators that strongly preserve regular matrices over the binary Boolean algebra $\mathbb{B}_1$.

As shown in Theorem 5.11, each operator T on $\mathcal{M}_n(\mathbb{B}_1)$ strongly preserves regularity if $n \leq 2$. Thus in the followings, unless otherwise stated, we assume that T is a linear operator on $\mathcal{M}_n(\mathbb{B}_1)$ that strongly preserves regularity for $n \geq 3$.

The following lemma and proposition are necessary to prove the main Theorem.

Lemma 5.12. *Let A be a matrix in $\mathcal{M}_n(\mathbb{B}_1)$ with $|A| = k$ and $b(A) = k$. Then $J \setminus A$ is regular if and only if $k \leq 2$.*

Proof. If $k \leq 2$, then there are permutation matrices P and Q such that $P(J \setminus A)Q = J \setminus (aE_{1,1} + bE_{2,2})$, where $a, b \in \{0, 1\}$, and hence

$$P(J \setminus A)Q = \begin{bmatrix} a' & 1 \\ 1 & b' \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

so that $b(J \setminus A) = b(P(J \setminus A)Q) \leq 2$, where $a + a' = b + b' = 1$ with $a \neq a'$ and $b \neq b'$. Thus we have $J \setminus A$ is regular by Lemma 5.7.

Conversely, assume that $J \setminus A$ is regular for some $k \geq 3$. It follows from $|A| = k$ and $b(A) = k$ that there are permutation matrices U and V such that

$$U(J \setminus A)V = J \setminus \sum_{t=1}^{k} E_{t,t}.$$

Let $J \setminus \left(\sum_{t=1}^{k} E_{t,t} \right) = X = [x_{i,j}]$. By Proposition 5.2, X is regular, and hence there is a nonzero matrix $B = [b_{i,j}] \in \mathcal{M}_n(\mathbb{B}_1)$ such that $X = XBX$. Then the $(t,t)^{th}$ entry of XBX becomes

$$\sum_{i \in I} \sum_{j \in J} b_{i,j} \tag{5.2.1}$$

for all $t = 1, \ldots, k$, where $I = J = \{1, \ldots, n\} \setminus \{t\}$. From $x_{1,1} = 0$ and (5.2.1), we have

$$b_{i,j} = 0 \quad \text{for all } i, j \in \{2, \ldots, n\}. \tag{5.2.2}$$

Consider the first row and the first column of B. It follows from $x_{2,2} = 0$ and (5.2.1) that

$$b_{i,1} = 0 = b_{1,j} \quad \text{for all } i, j \in \{1, 3, 4, \ldots, n\}. \tag{5.2.3}$$

Also, from $x_{3,3} = 0$, we obtain $b_{1,2} = b_{2,1} = 0$, and hence $B = O$ by (5.2.2) and (5.2.3). This contradiction shows that $k \leq 2$. ∎

Proposition 5.13. *Let A and B be matrices in $\mathcal{M}_n(\mathbb{B}_1)$ such that $A \leq B$ and $|A| < |B|$. If $|B| \leq (n-2)n$, then we have $|T(A)| < |T(B)|$.*

Proof. Suppose that $|T(A)| = |T(B)|$ for some $A, B \in \mathcal{M}_n(\mathbb{B}_1)$ with $A \leq B$, $|A| < |B|$ and $|B| \leq (n-2)n$. Then $T(A) = T(B)$ and there is a cell E such that $E \leq B$ and $E \not\leq A$. Since $|A| < (n-2)n$, there must be two distinct cells

F and G different from E such that $F \not\leq A$, $G \not\leq A$ and $b(E+F+G) = 3$. Let $C = J \setminus (E+F+G)$. Then

$$A + C = J \setminus (E+F+G) \quad \text{and} \quad B + C = J \setminus (F+G).$$

It follows from $T(A) = T(B)$ that $T(J \setminus (E+F+G)) = T(J \setminus (F+G))$, a contradiction to the fact that T strongly preserves regularity because $J \setminus (F+G)$ is regular, while $J \setminus (E+F+G)$ is not regular by Lemma 5.12. Hence the result follows. ■

Let A be a matrix in $\mathcal{M}_3(\mathbb{B}_1)$. If $|A| \leq 4$, then A is regular by Corollary 5.8(i). And if $|A| \geq 7$, then $b(A) \leq 2$ and so A is regular by Lemma 5.7. Hence, if $A \in \mathcal{M}_3(\mathbb{B}_1)$ is not regular, then $|A| = 5$ or 6 and there are permutation matrices P and Q such that PAQ is of the form of following :

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Furthermore, if E is a cell with $E \leq C$, then there are permutation matrices P' and Q' such that $P'(C \setminus E)Q' = B$ and hence $C \setminus E$ is not regular.

Lemma 5.14. *For every cell E in $\mathcal{M}_3(\mathbb{B}_1)$, $T(E)$ is a cell.*

Proof. Suppose that $|T(E_1)| \geq 2$ for some cell $E_1 \in \mathcal{M}_3(\mathbb{B}_1)$. Let $A \in \mathcal{M}_3(\mathbb{B}_1)$ be a matrix that is not regular with $E_1 \leq A$ and $|A| = 5$. Then $T(A)$ is not regular and so $|T(A)| \in \{5, 6\}$. Let $B \in \mathcal{M}_3(\mathbb{B}_1)$ be a matrix with $B \leq A$ and $|B| = 4$. If $|T(B)| \geq 5$, then $T(B)$ is not regular, while B is regular by Corollary 5.8(i), a contradiction. Hence there is not a matrix B with $B \leq A$ and $|B| = 4$ such that $|T(B)| \geq 5$.

Write $A = \sum_{i=1}^{5} E_i$ for distinct cells $E_1, \ldots, E_5$. It follows from Proposition 5.13 that

$$|T(E_1)| < |T(E_1 + E_2)| < |T(E_1 + E_2 + E_3)|$$

and hence $4 \leq |T(E_1+E_2+E_3)| \leq |T(A)|$ because $|T(E_1)| \geq 2$. Thus we have $|T(E_1 + E_2 + E_3)| = 4$. Since $T\Big(\sum_{i=1}^{3} E_i\Big) \leq T\Big(\sum_{i=1}^{4} E_i\Big)$ and $\Big|T\Big(\sum_{i=1}^{4} E_i\Big)\Big| \geq 5$ are impossible, we have

$$T\Big(\sum_{i=1}^{3} E_i\Big) = T\Big(\sum_{i=1}^{4} E_i\Big),$$

and hence $T(E_1 + E_2 + E_3 + E_5) = T(A)$, a contradiction because A is not regular, while $E_1 + E_2 + E_3 + E_5$ is regular by Corollary 5.8(i). Thus we have $|T(E)| \leq 1$ and hence $|T(E)| = 1$ for every cell E by Lemma 5.10. Consequently, $T(E)$ is a cell for every cell E. ■

For any $k \in \{1, 2, \ldots, n^2\}$, let S_k denote a sum of arbitrary distinct cells in $\mathcal{M}_n(\mathbb{B}_1)$ with $|S_k| = k$.

Proposition 5.15. (i) *If $n = 2t$ and $t \geq 2$, then $|T(S_{tn-1})| \leq n^2 - 3$ for all $S_{tn-1} \in \mathcal{M}_n(\mathbb{B}_1)$,*

(ii) *If $n = 2t + 1$ and $t \geq 2$, then*

$$|T(S_{(t+1)n-(t+1)})| \leq n^2 - 2$$

for all $S_{(t+1)n-(t+1)} \in \mathcal{M}_n(\mathbb{B}_1)$.

Proof. (i) Let $n = 2t$ with $t \geq 2$. Suppose that $|T(S_{tn-1})| \geq n^2 - 2$ for some $S_{tn-1} \in \mathcal{M}_n(\mathbb{B}_1)$. Since $|S_{tn-1}| = tn - 1$, there must be three distinct cells E_1, E_2 and E_3 such that they are not dominated by S_{tn-1} and $b(E_1 + E_2 + E_3) = 3$. Hence there is a matrix $A \in \mathcal{M}_n(\mathbb{B}_1)$ such that $S_{tn-1} + A = J \setminus (E_1 + E_2 + E_3)$. It follows from $|T(S_{tn-1})| \geq n^2 - 2$ that $|T(J \setminus (E_1 + E_2 + E_3))| \geq n^2 - 2$ and hence $B = T(J \setminus (E_1 + E_2 + E_3))$ is regular by Lemma 5.7 because $b(B) \leq 2$. But $J \setminus (E_1 + E_2 + E_3)$ is not regular by Lemma 5.12, a contradiction. Hence the result follows.

(ii) Similar to (i). ■

The next Lemma will be important in order to show that if E is any cell in $\mathcal{M}_n(\mathbb{B}_1)$ with $n \geq 4$, then $T(E)$ is also a cell for any linear operator on $\mathcal{M}_n(\mathbb{B}_1)$ that strongly preserves regularity.

Lemma 5.16. (i) *Let $n = 2t$, $t \geq 2$ and $h \in \{0, 1, 2, \ldots, tn - 2\}$. Then*

$$|T(S_{tn-1-h})| \leq n^2 - 3 - 2h$$

for all $S_{tn-1-h} \in \mathcal{M}_n(\mathbb{B}_1)$,

(ii) *Let $n = 2t + 1$, $t \geq 2$ and $h \in \{0, 1, 2, \ldots, (t+1)n - (t+2)\}$. Then*

$$|T(S_{(t+1)n-(t+1)-h})| \leq n^2 - 2 - 2h$$

for all $S_{(t+1)n-(t+1)-h} \in \mathcal{M}_n(\mathbb{B}_1)$.

Proof. (i) The proof proceeds by induction on h. If $h = 0$, the result is obvious by Proposition 5.15(i). Next, we assume that for some $h \in \{0, 1, 2, \ldots, tn - 3\}$, the argument holds. That is,

$$|T(S_{tn-1-h})| \leq n^2 - 3 - 2h \tag{5.2.4}$$

for all $S_{tn-1-h} \in \mathcal{M}_n(\mathbb{B}_1)$. Now we will show that $|T(S_{tn-2-h})| \leq n^2 - 5 - 2h$ for all $S_{tn-2-h} \in \mathcal{M}_n(\mathbb{B}_1)$. Suppose that $|T(S_{tn-2-h})| \geq n^2 - 4 - 2h$ for some $S_{tn-2-h} \in \mathcal{M}_n(\mathbb{B}_1)$. By (5.2.4) and Proposition 5.13, we have $|T(S_{tn-2-h})| = n^2 - 4 - 2h$ and

$$|T(S_{tn-2-h} + F)| = n^2 - 3 - 2h$$

for all cells F with $F \not\leq S_{tn-2-h}$. This means that for all cell F with $F \not\leq S_{tn-2-h}$, there is only cell C_F such that

$$C_F \not\leq T(S_{tn-2-h}), \quad C_F \leq T(F) \quad \text{and} \quad T(S_{tn-2-h} + F) = T(S_{tn-2-h}) + C_F \tag{5.2.5}$$

because $|T(S_{tn-2-h})| = n^2 - 4 - 2h$. Let $\mathcal{E}_n$ be the set of all cells in $\mathcal{M}_n(\mathbb{B}_1)$ and let

$$\Omega = \{C_F \,|\, F \in \mathcal{E}_n \quad \text{and} \quad F \not\leq S_{tn-2-h}\}.$$

Suppose that $C_H \neq C_F$ for all distinct cells F and H that are not dominated by S_{tn-2-h}. Then we have $|\Omega| = n^2 - (tn - 2 - h)$. Since $C_F \not\leq T(S_{tn-2-h})$ for any cell F with $F \not\leq S_{tn-2-h}$, we have $|\Omega| \leq n^2 - (n^2 - 4 - 2h)$ because $|T(S_{tn-2-h})| = n^2 - 4 - 2h$. This is impossible. Hence $C_H = C_F$ for some two distinct cells F and H that are not dominated by S_{tn-2-h}. It follows from (5.2.5) that

$$\begin{aligned} T(S_{tn-2-h} + F + H) &= T(S_{tn-2-h} + F) + T(S_{tn-2-h} + H) \\ &= T(S_{tn-2-h}) + C_F = T(S_{tn-2-h} + F). \end{aligned}$$

But Proposition 5.13 implies that $|T(S_{tn-2-h} + F)| < |T(S_{tn-2-h} + F + H)|$ because $|S_{tn-2-h} + F + H| \leq tn \leq (n-1)n$, a contradiction. Hence the result follows.

(ii) Similar to (i). ∎

Corollary 5.17. *$T(E)$ is a cell for all cells E.*

Proof. For $n = 3$, the result was proved in Lemma 5.14. If $n = 2t$ with $t \geq 2$, let $h = tn - 2$ in Lemma 5.16(i). Then $|T(S_1)| \leq 1$ for all $S_1 \in \mathcal{M}_n(\mathbb{B}_1)$. If $n = 2t + 1$ with $t \geq 2$, let $h = (t + 1)n - (t + 2)$ in Lemma 5.16(ii). Then $|T(S_1)| \leq 1$ for all $S_1 \in \mathcal{M}_n(\mathbb{B}_1)$. It follows from Lemma 5.10 that $|T(S_1)| = 1$ for all $S_1 \in \mathcal{M}_n(\mathbb{B}_1)$, equivalently $|T(E)| = 1$ for any cell E in $\mathcal{M}_n(\mathbb{B}_1)$. Therefore $T(E)$ is a cell for any cell E in $\mathcal{M}_n(\mathbb{B}_1)$. ∎

Lemma 5.18. *T is bijective on the set of cells.*

Proof. By Corollary 5.17, it suffices to show that $T(E) \neq T(F)$ for all distinct cells E and F in $\mathcal{M}_n(\mathbb{B}_1)$. Suppose $T(E) = T(F)$ for some distinct cells E and F. Then we have $T(E + F) = T(E)$. But this is impossible because $|T(E) < |T(E + F)|$ by Proposition 5.13. Thus the result follows. ∎

A matrix $L \in \mathcal{M}_n(\mathbb{B}_1)$ is called a *line matrix* if $L = \sum_{k=1}^{n} E_{i,k}$ or $L = \sum_{l=1}^{n} E_{l,j}$ for some $i, j \in \{1, \ldots, n\}$; $R_i = \sum_{k=1}^{n} E_{i,k}$ is an i^{th} *row matrix* and $C_j = \sum_{l=1}^{n} E_{l,j}$ is a j^{th} *column matrix*. Cells $E_1, E_2, \ldots, E_k$ are called *collinear* if $\sum_{i=1}^{k} E_i \leq L$ for some line matrix L.

A matrix $A \in \mathcal{M}_n(\mathbb{B}_1)$ is an *s-star matrix* if $|A| = s$ and there are cells $E_1, \ldots, E_s$ such that $A = \sum_{i=1}^{s} E_i$ and $A \leq L$ for some line matrix L. By Lemma 5.7, all line matrices and all s-star matrices are regular in $\mathcal{M}_n(\mathbb{B}_1)$.

Lemma 5.19. *T preserves all line matrices.*

Proof. By Lemma 5.18, T is bijective on the set of cells. First, we show that T preserves all 3-star matrices. If T does not preserve a 3-star matrix $A \in \mathcal{M}_n(\mathbb{B}_1)$, then we have $b(T(A)) = 2$ or 3 with $|T(A)| = 3$. By Corollary 5.8 (iii), there is a matrix $C \in \mathcal{M}_n(\mathbb{B}_1)$ with $|C| = 2$ such that $T(A) + C$ is not regular. Furthermore we can write $C = T(E_1 + E_2)$ for some distinct cells E_1 and E_2. Thus we have

$$T(A) + C = T(A + E_1 + E_2).$$

But $A + E_1 + E_2$ is regular by Corollary 5.8(i) or (iv). This contradicts to the fact that T strongly preserves regularity. Hence T preserves all 3-star matrices.

Suppose that T does not preserve a line matrix L in $\mathcal{M}_n(\mathbb{B}_1)$. Then there are two distinct cells F_1 and F_2 dominated by L such that two cells $T(F_1)$ and $T(F_2)$ are not collinear. Let F_3 be a cell such that $F_1 + F_2 + F_3$ is a 3-star matrix. By the above result, $T(F_1 + F_2 + F_3)$ is a 3-star matrix, and hence $b(T(F_1 + F_2 + F_3)) = 1$. Thus, the three cells $T(F_1)$, $T(F_2)$ and $T(F_3)$ are collinear. This contradicts to the fact that the two cells $T(F_1)$ and $T(F_2)$ are not collinear. Therefore T preserves all line matrices. ■

A linear operator T on $\mathbb{M}_n(\mathbb{B}_k)$ is called a (U, V)-*operator* if there are invertible matrices U and V such that $T(X) = UXV$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$ or $T(X) = UX^TV$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$.

We remind the $n \times n$ permutation matrices are the only invertible matrices in $\mathcal{M}_n(\mathbb{B}_1)$.

Now, we are ready to prove the main Theorem.

Theorem 5.20. *Let T be a linear operator on $\mathcal{M}_n(\mathbb{B}_1)$ with $n \geq 3$. Then T strongly preserves regularity if and only if T is a (U, V)-operator.*

Proof. If T is a (U, V)-operator on $\mathcal{M}_n(\mathbb{B}_1)$, clearly T strongly preserves regularity by Proposition 5.2.

Conversely, assume that T strongly preserves regularity. Then T is bijective on the set of cells by Lemma 5.18 and T preserves all line matrices by Lemma 5.19. Since no combination of s row matrices and t column matrices can dominate J_n where $s + t = n$ unless $s = 0$ or $t = 0$, we have that either

(1) the image of each row matrix is a row matrix and the image of each column matrix is a column matrix, or

(2) the image of each row matrix is a column matrix and the image of each column matrix is a row matrix.

If (1) holds, then there are permutations σ and τ of $\{1, \ldots, n\}$ such that $T(R_i) = R_{\sigma(i)}$ and $T(C_j) = C_{\tau(j)}$ for all $i, j \in \{1, 2, \ldots, n\}$. Let U and V be permutation (i.e., invertible) matrices corresponding to σ and τ, respectively. Then we have

$$T(E_{i,j}) = E_{\sigma(i),\tau(j)} = UE_{i,j}V$$

for all cells $E_{i,j}$ in $\mathcal{M}_n(\mathbb{B}_1)$. Let $X = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{i,j} E_{i,j}$ be any matrix in $\mathcal{M}_n(\mathbb{B}_1)$. By the action of T on the cells, we have $T(X) = UXV$. If (2) holds, then a parallel argument shows that there are invertible matrices U and V such that $T(X) = UX^T V$ for all $X \in \mathcal{M}_n(\mathbb{B}_1)$. ∎

Thus, as shown in Theorems 5.11 and 5.12, we have characterizations of the linear operators that strongly preserve regular matrices over the binary Boolean algebra.

5.3 Characterization of linear operators that strongly preserve regular matrices over the nonbinary Boolean algebra

If T is a linear operator on $\mathbb{M}_n(\mathbb{B}_k)$ with $k \geq 1$, for each $p \in \{1, 2, \ldots, k\}$, define its p^{th} *constituent operator*, T_p, by $T_p(B) = (T(B))_p$ for all $B \in \mathcal{M}_n(\mathbb{B}_1)$. By the linearity of T, we have

$$T(A) = \sum_{p=1}^{k} \sigma_p T_p(A_p)$$

for all $A \in \mathbb{M}_n(\mathbb{B}_k)$.

Lemma 5.21. *If T is a linear operator on $\mathbb{M}_n(\mathbb{B}_k)$ that strongly preserves regularity, then its all constituent operators on $\mathcal{M}_n(\mathbb{B}_1)$ strongly preserve regularity.*

Proof. Let A be any matrix in $\mathcal{M}_n(\mathbb{B}_1)$. Obviously, A is the matrix in $\mathbb{M}_n(\mathbb{B}_k)$ such that $A_p = A$ for all $p = 1, \ldots, k$. If A is regular in $\mathcal{M}_n(\mathbb{B}_1)$, then A is regular in $\mathbb{M}_n(\mathbb{B}_k)$ by Theorem 5.5. Since T preserves regularity, we have $T(A) = \sum\limits_{p=1}^{k} \sigma_p T_p(A_p)$ is also regular in $\mathbb{M}_n(\mathbb{B}_k)$. Again by Theorem 5.5, each $T_p(A_p)$ is regular in $\mathcal{M}_n(\mathbb{B}_1)$ so that $T_p(A)$ is regular in $\mathcal{M}_n(\mathbb{B}_1)$ for all $p = 1, \ldots, k$.

Conversely, if $T_p(A)$ is regular in $\mathcal{M}_n(\mathbb{B}_1)$ for all $p = 1, \ldots, k$, then $T(A) = \sum_{p=1}^{k} \sigma_p T_p(A_p)$ is regular in $\mathbb{M}_n(\mathbb{B}_k)$ by Theorem 5.5. Since T strongly preserves regularity, A is regular in $\mathbb{M}_n(\mathbb{B}_k)$. Hence by Theorem 5.5, $A(= A_p)$ is regular in $\mathcal{M}_n(\mathbb{B}_1)$. ■

Example 5.22. Let $n \geq 3$. Define an operator T on $\mathcal{M}_n(\mathbb{B}_3)$ by

$$T(X) = \sigma_1 X_1 + \sigma_2 X_2^T + \sigma_3 X_3$$

for all $X = \sum_{p=1}^{3} \sigma_p X_p$ in $\mathcal{M}_n(\mathbb{B}_3)$. Then we can easily show that T is not a (U, V)-operator on $\mathcal{M}_n(\mathbb{B}_3)$ while its all constituent operators are (U, V)-operators on $\mathcal{M}_n(\mathbb{B}_1)$. Furthermore the below theorem shows that T strongly preserves regularity.

Theorem 5.23. *Let T be a linear operator on $\mathbb{M}_n(\mathbb{B}_k)$ with $n \geq 3$. Then the following statements are equivalent:*

(i) *T strongly preserves regularity on $\mathbb{M}_n(\mathbb{B}_k)$;*

(ii) *All constituent operators of T strongly preserve regularity on $\mathcal{M}_n(\mathbb{B}_1)$;*

(iii) *There are invertible matrices U and V such that*

$$T(X) = UXV \quad \text{for all } X \in \mathbb{M}_n(\mathbb{B}_k), \quad \text{or} \tag{5.3.1}$$

$$T(X) = U\Big(\sum_{p=1}^{k} \sigma_p Y_p\Big)V \quad \text{for all } X \in \mathbb{M}_n(\mathbb{B}_k), \tag{5.3.2}$$

where $Y_p = X_p$ or X_p^T for all $p = 1, \ldots, k$.

Proof. It follows from Lemma 5.21 that (i) implies (ii).

Assume (ii) holds. That is, each constituent operator T_p of T strongly preserves regularity on $\mathcal{M}_n(\mathbb{B}_1)$ for all $p = 1, \ldots, k$. Let $X = \sum_{p=1}^{k} \sigma_p X_p$ be any matrix in $\mathbb{M}_n(\mathbb{B}_k)$. Then we have $T(X) = \sum_{p=1}^{k} \sigma_p T_p(X_p)$. By Theorem 5.20, each T_p has the form

$$T_p(X_p) = U_p X_p V_p, \tag{5.3.3}$$

or

$$T_p(X_p) = U_p X_p^T V_p, \tag{5.3.4}$$

where U_p and V_p are permutation matrices for all $p = 1, \ldots, k$.

Assume that only (5.3.3) are possible for all $p = 1, \ldots, k$. Then we have

$$T(X) = \sum_{p=1}^{k} \sigma_p U_p X_p V_p = \Big(\sum_{p=1}^{k} \sigma_p U_p\Big)\Big(\sum_{p=1}^{k} \sigma_p X_p\Big)\Big(\sum_{p=1}^{k} \sigma_p V_p\Big).$$

If we let $U = \Big(\sum_{p=1}^{k} \sigma_p U_p\Big)$ and $V = \Big(\sum_{p=1}^{k} \sigma_p V_p\Big)$, then U and V are invertible matrices in $\mathbb{M}_n(\mathbb{B}_k)$ by Lemma 5.1, and hence (5.3.1) is satisfied.

If both (5.3.3) and (5.3.4) are possible, then $T(X) = \sum_{p=1}^{k} \sigma_p U_p Y_p V_p$, where $Y_p = X_p$ or X_p^T for each $p \in \{1, \ldots, k\}$, equivalently

$$T(X) = \Big(\sum_{p=1}^{k} \sigma_p U_p\Big)\Big(\sum_{p=1}^{k} \sigma_p Y_p\Big)\Big(\sum_{p=1}^{k} \sigma_p V_p\Big).$$

If we let $U = \Big(\sum_{p=1}^{k} \sigma_p U_p\Big)$ and $V = \Big(\sum_{p=1}^{k} \sigma_p V_p\Big)$, then (5.3.2) is satisfied. Therefore (ii) implies (iii).

Assume (iii) holds. If T has a form (5.3.1), then we are done by Proposition 5.2. Thus we assume (5.3.2). If $X = \sum_{p=1}^{k} \sigma_p X_p$ is regular in $\mathbb{M}_n(\mathbb{B}_k)$, then so is X_p in $\mathcal{M}_n(\mathbb{B}_1)$ for all $p = 1, \ldots, k$ by Theorem 5.5. Thus there are matrices $G_p \in \mathcal{M}_n(\mathbb{B}_1)$ such that $X_p G_p X_p = X_p$ for all $p = 1, \ldots, k$. Let $G = V^T \Big(\sum_{p=1}^{k} \sigma_p H_p \Big) U^T$, where $H_p = G_p$ or G_p^T according as $Y_p = X_p$ or X_p^T. Then we can easily show that $T(X)GT(X) = T(X)$ so that $T(X)$ is regular in $\mathbb{M}_n(\mathbb{B}_k)$. Conversely, if $T(X)$ is regular in $\mathbb{M}_n(\mathbb{B}_k)$, then each constituent $T_p(X_p) = U_p Y_p V_p$ is regular in $\mathcal{M}_n(\mathbb{B}_1)$ for all $p = 1, \ldots, k$. By Proposition 5.2, each X_p is regular in $\mathcal{M}_n(\mathbb{B}_1)$ because $Y_p = X_p$ or X_p^T for all $p = 1, \ldots, k$. Hence X is regular in $\mathbb{M}_n(\mathbb{B}_k)$ by Theorem 5.5. Therefore (i) is satisfied. ■

Thus, as shown in Theorems 5.11 and 5.23, we have characterizations of the linear operators that strongly preserve regular matrices over general Boolean algebras.

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