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Linear Preservers of Regularity and Extreme Sets of Matrix Inequalities over Boolean Algebras

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Linear Preservers of Regularity and Extreme Sets of Matrix Inequalities over Boolean Algebras

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CONTENTS

<Abstract>

Linear Preservers of Regularity and Extreme Sets of Matrix Inequalities over Boolean Algebras

In this thesis, we research two topics on linear preserver problems.

One topic is to characterize the linear operators that preserve the regularity of binary Boolean matrices. A matrix M is called regular if there exists a matrix X such that $MXM=M$. We obtain that a linear operator T strongly preserves regularity of binary Boolean matrices if and only if T has the forms that $T(X)=UXV$ or $T(X)=UX^TV$ with invertible matrices U and V.

Another topic is to characterize the linear operators that preserve the sets of matrix pairs over general Boolean algebras which satisfy the extreme cases for certain Boolean rank inequalities. For this purpose we construct the following 8 sets of matrix pairs;

$\mathcal{R}_{SA}(\mathbb{B}_k)=\{(X,Y)\in\mathbb{M}_{m,n}(\mathbb{B}_k)^2|\ \mathrm{b}(X+Y)=\mathrm{b}(X)+\mathrm{b}(Y)\}$,

$\mathcal{R}_{S1}(\mathbb{B}_k)=\{(X,Y)\in\mathbb{M}_{m,n}(\mathbb{B}_k)^2|\ \mathrm{b}(X+Y)=1\}$,

$\mathcal{R}_{SD}(\mathbb{B}_k)=\{(X,Y)\in\mathbb{M}_{m,n}(\mathbb{B}_k)^2|\ \mathrm{b}(X+Y)=|\mathrm{b}(X)-\mathrm{b}(Y)|\}$,

$\mathcal{R}_{MM}(\mathbb{B}_k)=\{(X,Y)\in\mathbb{M}_{n}(\mathbb{B}_k)^2|\ \mathrm{b}(XY)=\min\{\mathrm{b}(X),\mathrm{b}(Y)\}\}$,

$\mathcal{R}_{M0}(\mathbb{B}_k)=\{(X,Y)\in\mathbb{M}_{n}(\mathbb{B}_k)^2|\ \mathrm{b}(XY)=0\}$,

$\mathcal{R}_{M1}(\mathbb{B}_k)=\{(X,Y)\in\mathbb{M}_{n}(\mathbb{B}_k)^2|\ \mathrm{b}(XY)=1\}$,

$\mathcal{R}_{MA}(\mathbb{B}_k)=\{(X,Y)\in\mathbb{M}_{n}(\mathbb{B}_k)^2|\ \mathrm{b}(XY)=\mathrm{b}(X)+\mathrm{b}(Y)-n\}$,

$\mathcal{R}_{M3}(\mathbb{B}_k)=\{(X,Y,Z)\in\mathbb{M}_{n}(\mathbb{B}_k)^3|\ \mathrm{b}(XYZ)+\mathrm{b}(Y)=\mathrm{b}(XY)+\mathrm{b}(YZ)\}$.

We characterize those linear operators that preserve these 8 sets as $T(X)=PXQ$, $T(X)=PXP^T$ or $T(X)=PX^TQ$ with invertible Boolean matrices P and Q.

1 Introduction

There are many papers on linear operators that preserve certain properties of matrices ([1] -[16], [24] -[27]). We call such topic of research as "Linear Preserver Problems". This linear preserver problems have been studied for the various characterizations of matrices during a century. In 1887, Frobenius characterized the linear operators that preserve determinant of matrices over real field, which was the first results on linear preserver problems. After his result, many researchers have studied the linear operators that preserve some matrix functions, say, rank and permanent of matrices and so on([20]).

Recently, Beasley & Pullman began to research the matrices over semirings or Boolean algebras([9] -[11]). There are many semirings such that nonnegative integers, nonnegative reals, fuzzy semirings and (non)binary Boolean algebra and so on([11]).

The results on linear preserver problems over semirigs are more applicable to linear preserver problems and combinatorics than those results over fields. The researches over a semiring are not easy to generalize those results over field since the system of semiring does not assume the additive inverse element for any element in the semiring. So we have to re-define many concepts for the properties of matrices over semiring to generalize the known definitions over field.

Now, almost all researches on linear preserver problems have dealt with those semirings without zero-divisors to avoid the difficulties of multiplication arithmetic for the elements in those semirings([3], [4], [9], [14]). But general Boolean algebra is not the case. That is, all elements except 0 and 1 in the general Boolean algebra are zero-divisors. So there are few results on the linear preserver problems for the matrices over general Boolean algebra([16], [25] , [27]). Although there are many arithmetic difficulties of matrices over general Boolean algebra, we study the Boolean rank of matrices over general Boolean algebra and that we characterize the linear operators that preserve pairs of matrices over general Boolean algebra which satisfy some rank inequalities.

In this thesis, we research two topics on the linear preserver problems. One topic is to characterize the linear operators that preserve the regularity of binary Boolean matrices. Another topic is to characterize the linear operators that preserve the sets of matrix pairs over general Boolean algebra which satisfy the extreme cases for certain Boolean rank inequalities. For this purpose, we study the inequalities of Boolean rank for the sum or multiplication of matrices over general Boolean algebra. We also construct the sets of matrix pairs that satisfy the equalities for those Boolean rank inequalities.

The contents of this thesis are as follows:

In Chapter 2, we study the regularity of matrices over binary Boolean algebra and characterize the linear operators that preserve the regularity.

In Chapter 3, we study the extreme sets of matrix pairs for the Boolean rank inequalities over general Boolean algebra and characterize the linear operators that preserve the extreme sets of matrix pairs.

2 Regularity preservers of matrices over binary Boolean algebra

2.1 Properties of regularity and singularity of Boolean matrices

The *binary Boolean algebra*([15]) is the set $\mathbb{B}_1 = \{0, 1\}$ equipped with two operations, addition($+$) and multiplication($\cdot$), defined as follows:

$$\begin{array}{cc} 0+0=0 & 0\cdot 0=0 \\ 0+1=1+0=1 & 0\cdot 1=1\cdot 0=0 \\ 1+1=1 & 1\cdot 1=1. \end{array}$$

For all $a, b \in \mathbb{B}_1$, we suppress the dot of $a \cdot b$ and simply write ab. Let $\mathbb{M}_{m,n}(\mathbb{B}_1)$ denote the set of all $m \times n$ Boolean matrices with entries in the binary Boolean algebra $\mathbb{B}_1$. The usual definitions for addition and multiplication of matrices over fields are applied to Boolean matrices as well. If $m = n$, we use the notation $\mathbb{M}_n(\mathbb{B}_1)$ instead of $\mathbb{M}_{m,n}(\mathbb{B}_1)$.

Boolean matrices play an important role in linear algebra, combinatorics, graph theory and network theory. And many problems in the theory of nonnegative matrices depend only on the distribution of nonzero entries. In such cases the relevant property of each entry is whether it is zero or nonzero, and the problem can be often simplified by substituting for the given matrix the Boolean $(0, 1)$-matrix.

Several authors characterized those linear operators on $\mathbb{M}_{m,n}(\mathbb{B}_1)$ that (strongly) preserve various properties and functions defined on $\mathbb{M}_{m,n}(\mathbb{B}_1)$([9], [24], [25]).

In this chapter, we study some properties of Boolean regular matrices. We also determine the linear operators on $\mathbb{M}_{m,n}(\mathbb{B}_1)$ that strongly preserve Boolean regular matrices.

The matrix I_n is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones, $O_{m,n}$ is the $m \times n$ zero matrix. We will suppress the subscripts on these matrices when the orders are evident from the context. For any matrix $A \in \mathbb{M}_{m,n}(\mathbb{B}_1)$, A^T is denoted by the transpose of A. A matrix in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ with only one nonzero entry is called a *cell*. If the nonzero entry occurs in the i^{th} row and the j^{th} column, we denote this cell by $E_{i,j}$.

Definition 2.1.1. A matrix in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ is called an i^{th} *row matrix*, denoted by R_i, if $R_i = \sum_{j=1}^{n} E_{i,j}$ for some $i \in \{1, \ldots, m\}$. Similarly, a matrix in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ is called a j^{th}

column matrix, denoted by C_j, if $C_j = \sum_{i=1}^{m} E_{i,j}$ for some $j \in \{1, \ldots, n\}$. A *line matrix* is an i^{th} row matrix or a j^{th} column matrix.

Let $A = [a_{i,j}]$ be any matrix in $\mathbb{M}_{m,n}(\mathbb{B}_1)$. Then A can be written uniquely as $\sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}E_{i,j}$, which is called the *canonical form* of A. If $a_{i,j} = 1$ for some i and j, then we say that the cell $E_{i,j}$ *is in* the matrix A. Since $a_{i,j} \in \{0, 1\}$, the canonical form of A shows that A is a sum of cells.

For $A = [a_{i,j}]$, $B = [b_{i,j}] \in \mathbb{M}_{m,n}(\mathbb{B}_1)$, we say that B *dominates* A (written $B \geq A$ or $A \leq B$) if $b_{i,j} = 0$ implies $a_{i,j} = 0$ for all i and j. This provides a reflexive and transitive relation on $\mathbb{M}_{m,n}(\mathbb{B}_1)$.

Definition 2.1.2. Cells $E_1, E_2, \ldots, E_k$ are called *collinear* if $\sum_{i=1}^{k} E_i \leq L$ for some line matrix L.

Definition 2.1.3. A matrix $A \in \mathbb{M}_n(\mathbb{B}_1)$ is said to be *invertible* if there exists a matrix $B \in \mathbb{M}_n(\mathbb{B}_1)$ such that $AB = BA = I_n$.

In 1952, Luce([17]) showed that a matrix $A \in \mathbb{M}_n(\mathbb{B}_1)$ possesses a two-sided inverse if and only if A is an orthogonal matrix in the sense that $AA^T = I_n$, and that, in this case, A^T is a two-sided inverse of A. In 1963, Rutherford([23]) showed that if a matrix $A \in \mathbb{M}_n(\mathbb{B}_1)$ possesses a one-sided inverse, then the inverse is also a two-sided inverse. Furthermore such an inverse, if it exists, is unique and is A^T. Also, it is well known that the $n \times n$ permutation matrices are the only $n \times n$ invertible Boolean matrices.

The notion of generalized inverse of an arbitrary matrix apparently originated in the work of Moore([19]), and the generalized inverses have applications in network and switching theory and information theory([12]).

Definition 2.1.4. Let A be a matrix in $\mathbb{M}_{m,n}(\mathbb{B}_1)$. Consider a matrix $X \in \mathbb{M}_{n,m}(\mathbb{B}_1)$ in the equation

$$AXA = A. \tag{2.1.1}$$

If (2.1.1) has a solution X, then X is called a *generalized inverse* of A. Furthermore A is called *regular* if there exists a solution of (2.1.1); Otherwise, A is called *singular*.

Clearly $J_{m,n}$ and $O_{m,n}$ are regular in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ because $J_{m,n}J_{n,m}J_{m,n} = J_{m,n}$ and $O_{m,n}O_{n,m}O_{m,n} = O_{m,n}$.

In general, a solution of (2.1.1), although it exists, is not necessarily unique. For example, consider a matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{M}_2(\mathbb{B}_1)$. Then we can easily show that $X = \begin{bmatrix} 1 & a \\ b & c \end{bmatrix} \in \mathbb{M}_2(\mathbb{B}_1)$ are generalized inverses of A for all $a, b, c \in \mathbb{B}_1$.

The equation (2.1.1) have been studied by several authors ([19, 21, 22]). Plemmons ([21]) published algorithms for computing generalized inverses of Boolean matrices under certain conditions. Also Rao and Rao([22]) had characterizations of regular matrices in $\mathbb{M}_{m,n}(\mathbb{B}_1)$.

Proposition 2.1.5. *Let A be a matrix in $\mathbb{M}_{m,n}(\mathbb{B}_1)$. If $U \in \mathbb{M}_m(\mathbb{B}_1)$ and $V \in \mathbb{M}_n(\mathbb{B}_1)$ are invertible, then the following are equivalent:*

(a) A is regular in $\mathbb{M}_{m,n}(\mathbb{B}_1)$;

(b) UAV is regular in $\mathbb{M}_{m,n}(\mathbb{B}_1)$;

(c) A^T is regular in $\mathbb{M}_{n,m}(\mathbb{B}_1)$;

(d) UA^TV is regular in $\mathbb{M}_n(\mathbb{B}_1)$ (if $m = n$).

Proof. It is obvious. ∎

Also we can easily show that

$$A \text{ is regular if and only if } \begin{bmatrix} A & O \\ O & B \end{bmatrix} \text{ is regular} \tag{2.1.2}$$

for all matrices $A \in \mathbb{M}_{m,n}(\mathbb{B}_1)$ and for all regular matrices $B \in \mathbb{M}_{p,q}(\mathbb{B}_1)$.

In particular, all idempotent matrices in $\mathbb{M}_n(\mathbb{B}_1)$ are regular.

Definition 2.1.6. ([9]) The *Boolean rank*, $b(A)$, of a nonzero $m \times n$ Boolean matrix A is defined as the least integer k for which there exist $m \times k$ and $k \times n$ Boolean matrices B and C with $A = BC$. The Boolean rank of a zero matrix is zero.

We can easily obtain that

$$0 \leq b(A) \leq \min\{m, n\} \quad \text{and} \quad b(AB) \leq \min\{b(A), b(B)\} \tag{2.1.3}$$

for all $A \in \mathbb{M}_{m,n}(\mathbb{B}_1)$ and for all $B \in \mathbb{M}_{n,q}(\mathbb{B}_1)$.

Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n\,]$ be a matrix in $\mathbb{M}_{m,n}(\mathbb{B}_1)$, where $\mathbf{a}_j$ is the j^{th} column of A for all $j = 1, \ldots, n$. Then the *column space* of A is the set $\left\{ \sum\limits_{j=1}^{n} \alpha_j \mathbf{a}_j \,\middle|\, \alpha_j \in \mathbb{B}_1 \right\}$, and denoted by $< A >$; the *row space* of A is $< A^T >$.

Definition 2.1.7. ([22]) Let A be a matrix in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ with $b(A) = k$. Then A is said to be *space decomposable* if there exist matrices $B \in \mathbb{M}_{m,k}(\mathbb{B}_1)$ and $C \in \mathbb{M}_{k,n}(\mathbb{B}_1)$ such that $A = BC$, $< A > = < B >$ and $< A^T > = < C^T >$.

Theorem 2.1.8. ([22]) *A is regular in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ if and only if A is space decomposable.*

Proposition 2.1.9. *If A is a matrix in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ with $b(A) \leq 2$, then A is regular.*

Proof. If $b(A) = 0$, then $A = O$ is clearly regular. If $b(A) = 1$, then there exist permutation matrices P and Q such that $PAQ = \begin{bmatrix} J & O \\ O & O \end{bmatrix}$, and hence PAQ is regular by (2.1.2). It follows from Proposition 2.1.5 that A is regular.

Suppose that $b(A) = 2$. Then there exist $m \times 2$ matrix $B = [\mathbf{b}_1 \quad \mathbf{b}_2\,]$ and $2 \times n$ matrix $C = [\mathbf{c}_1 \quad \mathbf{c}_2\,]^T$ such that $A = BC$, where $\mathbf{b}_1$ and $\mathbf{b}_2$ are distinct nonzero columns of B, and $\mathbf{c}_1$ and $\mathbf{c}_2$ are distinct nonzero columns of C^T. Then we can easily show that all columns of A are of the forms $\mathbf{0}, \mathbf{b}_1, \mathbf{b}_2$ and $\mathbf{b}_1 + \mathbf{b}_2$ so that $< A >=< B >$. Similarly, all columns of A^T are of the forms $\mathbf{0}, \mathbf{c}_1, \mathbf{c}_2$ and $\mathbf{c}_1 + \mathbf{c}_2$ so that $< A^T >=< C^T >$. Therefore A is space decomposable and hence A is regular by Theorem 2.1.8. ■

The *weight* of a matrix A in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ is the number of nonzero entries of A and will be denoted by $\#(A)$. The number of elements in a set $\mathbb{S}$ is also denoted by $\#(\mathbb{S})$.

Corollary 2.1.10. *Let A be a matrix in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ with $\#(A) \leq 4$. Then A is regular.*

Proof. By Proposition 2.1.9, we lose no generality in assuming that $b(A) = 3$ or 4. Consider a matrix $B = \begin{bmatrix} A & O \\ O & O \end{bmatrix}$ in $\mathbb{M}_{m+1,n+1}(\mathbb{B}_1)$. Since $\#(A) \leq 4$ and $b(A) = 3$ or 4, we can easily show that there exist permutation matrices $P \in \mathbb{M}_{m+1}(\mathbb{B}_1)$ and $Q \in \mathbb{M}_{n+1}(\mathbb{B}_1)$ such that $PBQ = \begin{bmatrix} C & O \\ O & O \end{bmatrix}$ for some idempotent matrix C in $\mathbb{M}_4(\mathbb{B}_1)$ with $\#(C) = 3$ or 4. By (2.1.2) and Proposition 2.1.5, we have that B is regular and hence A is regular by (2.1.2). ■

Example 2.1.11. Consider a matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Then we can easily show that $b(A) = 3$.

Now we show that A is not space decomposable. If A is space decomposable, then there exist 3×3 matrices B and C such that $A = BC$, $< A >=< B >$ and $< A^T >=< C^T >$. It follows from (2.1.3) that $b(B) = b(C) = 3$, and hence both B and C have neither a zero row nor a zero column. Also, there exists a permutation matrix P such that $A = DE$, where $D = [d_{i,j}] = BP$, $E = [e_{i,j}] = P^T C$ and $D \geq I_3$. Then we have

$$< A >=< B >=< BP >=< D > \tag{2.1.4}$$

and

$$< A^T >=< C^T >=< C^T P >=< E^T > . \tag{2.1.5}$$

Furthermore we have that

$$E \text{ has neither a zero row nor a zero column} \tag{2.1.6}$$

because $b(E) = b(P^T C) = b(C) = 3$. From $A = DE$ with $a_{1,3} = a_{2,1} = a_{3,1} = a_{3,2} = 0$, we have $e_{1,3} = e_{2,1} = e_{3,1} = e_{3,2} = 0$. It follows from (2.1.6) that $e_{1,1} = e_{3,3} = 1$. Thus, $E = \begin{bmatrix} 1 & e_{1,2} & 0 \\ 0 & e_{2,2} & e_{2,3} \\ 0 & 0 & 1 \end{bmatrix}$. If $e_{1,2} = 0$ or $e_{2,3} = 0$, then we have $e_{2,2} = 1$ by (2.1.6). Then we have $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in < E^T >$, while $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \notin < A^T >$, a contradiction to (2.1.5). Thus we may assume that $e_{1,2} = e_{2,3} = 1$ so that $E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & e_{2,2} & 1 \\ 0 & 0 & 1 \end{bmatrix}$. If $e_{2,2} = 0$, then $b(E) = 2$, a contradiction. Hence $e_{2,2} = 1$. It follows from $A = DE$ that $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & d_{2,3} \\ 0 & 0 & 1 \end{bmatrix}$. In this case, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in < D >$, while $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \notin < A >$, a contradiction to (2.1.4). Therefore A is not space decomposable.

In the following, we give some properties of Boolean regular matrices.

If A and B are matrices in $\mathbb{M}_{m,n}(\mathbb{B}_1)$, we define $A \setminus B$ to be the matrix $C = [c_{i,j}]$ in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ such that $c_{i,j} = 1$ if and only if $a_{i,j} = 1$ and $b_{i,j} = 0$.

Define an upper triangular matrix Λ_n in $\mathbb{M}_n(\mathbb{B}_1)$ by

$$\Lambda_n = [\lambda_{i,j}] \equiv \left(\sum_{i \le j}^{n} E_{i,j} \right) \setminus E_{1,n} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 \\ & 1 & \cdots & 1 & 1 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}.$$

Then the following Lemma shows that Λ_n is not regular for $n \ge 3$.

Lemma 2.1.12. *Λ_n is regular in $\mathbb{M}_n(\mathbb{B}_1)$ if and only if $n \le 2$.*

Proof. If $n \le 2$, then Λ_n is regular by Corollary 2.1.10.

Conversely, assume that Λ_n is regular for some $n \ge 3$. Then there exists a nonzero matrix $B = [b_{i,j}]$ in $\mathbb{M}_n(\mathbb{B}_1)$ such that $\Lambda_n = \Lambda_n B \Lambda_n$. From $0 = \lambda_{1,n} = \sum_{i=1}^{n-1} \sum_{j=2}^{n} b_{i,j}$, we obtain that all entries of the second column of B are zero except for the entry $b_{n,2}$. From $0 = \lambda_{2,1} = \sum_{i=2}^{n} b_{i,1}$, we have that all entries of the first column of B are zero except for

$b_{1,1}$. Also, from $0 = \lambda_{3,2} = \sum_{i=3}^{n}\sum_{j=1}^{2} b_{i,j}$, we obtain that $b_{n,2} = 0$. If we combine these three results, we conclude that all entries of the first two columns are zero except for $b_{1,1}$. But we have $1 = \lambda_{2,2} = \sum_{i=2}^{n}\sum_{j=1}^{2} b_{i,j} = 0$, a contradiction. Hence Λ_n is singular for all $n \geq 3$. ■

In particular, $\Lambda_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is singular. By Proposition 2.1.5, we have that the lower triangular matrix Λ_n^T is singular for $n \geq 3$, while $\Lambda_n + \Lambda_n^T$ is regular by Proposition 2.1.9 because $b(\Lambda_n + \Lambda_n^T) = 2$. Let

$$\Phi_{m,n} = \begin{bmatrix} \Lambda_3 & O \\ O & O \end{bmatrix} \tag{2.1.7}$$

for all $\min\{m, n\} \geq 3$. Then $\Phi_{m,n}$ is singular by (2.1.2).

Corollary 2.1.13. *Let E and F be distinct cells in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ with $\min\{m,n\} \geq 3$. Then there exists a matrix A in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ such that $\#(A) = 3$ and $A + E + F$ is singular in $\mathbb{M}_{m,n}(\mathbb{B}_1)$.*

Proof. Since E and F are distinct cells, there exist permutation matrices P and Q such that

$$P(E + F)Q = E_{1,1} + E_{1,2}, \quad E_{1,2} + E_{2,2} \quad \text{or} \quad E_{1,1} + E_{2,2}.$$

Consider a matrix $A \in \mathbb{M}_{m,n}(\mathbb{B}_1)$ such that

$$PAQ = E_{2,2} + E_{2,3} + E_{3,3}, \quad E_{1,1} + E_{2,3} + E_{3,3} \quad \text{or} \quad E_{1,2} + E_{2,3} + E_{3,3}$$

according as $P(E + F)Q = E_{1,1} + E_{1,2}$, $E_{1,2} + E_{2,2}$ or $E_{1,1} + E_{2,2}$. Then we have that $P(A+E+F)Q = \Phi_{m,n}$ is singular in $\mathbb{M}_{m,n}(\mathbb{B}_1)$. Hence $A+E+F$ is singular in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ by Proposition 2.1.5. ■

Corollary 2.1.14. *Let A be a matrix in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ with $\#(A) = 3$. If $b(A) = 2$ or 3, then there exist cells E and F such that $A + E + F$ is singular.*

Proof. Consider the singular matrix $\Phi_{m,n}$ in (2.1.7). If $b(A) = 2$ or 3, then we can easily show that there exist permutation matrices U and V such that $UAV \leq \Phi_{m,n}$. Let E' and F' be cells satisfying $UAV + E' + F' = \Phi_{m,n}$. Then we obtain that

$$A + U^T E' V^T + U^T F' V^T = U^T \Phi_{m,n} V^T$$

is singular by Proposition 2.1.5. If we let $E = U^T E' V^T$ and $F = U^T F' V^T$, then the result follows. ■

Theorem 2.1.15. *For $m \geq 3$ and $n \geq 3$, let A be a matrix in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ with $\#(A) = k$ and $b(A) = k$, where $0 \leq k \leq \min\{m, n\}$. Then $J \setminus A$ is regular if and only if $k \leq 2$.*

Proof. If $k \leq 2$, then there exist permutation matrices P and Q such that $P(J \setminus A)Q = J \setminus (aE_{1,1} + bE_{2,2})$, where $a, b \in \{0, 1\}$, and hence

$$J \setminus E_{11} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{bmatrix},$$

$$J \setminus E_{22} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix},$$

$$J \setminus E_{11} + E_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \end{bmatrix}.$$

Thus $b(J \setminus A) = b(P(J \setminus A)Q) \leq 2$. Therefore we have $J \setminus A$ is regular by Proposition 2.1.9.

Conversely, assume that $J \setminus A$ is regular for some $k \geq 3$. It follows from $\#(A) = k$ and $b(A) = k$ that there exist permutation matrices U and V such that

$$U(J_{m,n} \setminus A)V = J \setminus \sum_{t=1}^{k} E_{t,t}.$$

Let $J \backslash \left(\sum_{t=1}^{k} E_{t,t} \right) = X = [x_{i,j}]$. By Proposition 2.1.5, X is regular, and hence there exists a nonzero matrix $B = [b_{i,j}] \in \mathbb{M}_{n,m}(\mathbb{B}_1)$ such that $X = XBX$. Then the $(t,t)^{th}$ entry of XBX becomes

$$\sum_{i \in I} \sum_{j \in J} b_{i,j} \tag{2.1.8}$$

for all $t = 1, \ldots, k$, where $I = \{1, \ldots, n\} \setminus \{t\}$ and $J = \{1, \ldots, m\} \setminus \{t\}$. From $x_{1,1} = 0$ and (2.1.8), we have that

$$b_{i,j} = 0 \quad \text{for all} \;\; i = 2, \ldots, n; \; j = 2, \ldots, m. \tag{2.1.9}$$

Consider the first row and the first column of B. It follows from $x_{2,2} = 0$ and (2.1.8) that

$$b_{i,1} = 0 = b_{1,j} \quad \text{for all} \;\; i = 1, 3, 4, \ldots, n; \; j = 1, 3, 4, \ldots, m. \tag{2.1.10}$$

Also, from $x_{3,3} = 0$, we obtain $b_{1,2} = b_{2,1} = 0$, and hence $B = O$ by (2.1.9) and (2.1.10). This contradiction shows that $k \leq 2$. ■

Proposition 2.1.16. *Let A be a matrix in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ with $\#(A) = 5$. If A has a row or a column that has at least 3 nonzero entries, then A is regular.*

Proof. Suppose that A has a row or a column that has at least 3 nonzero entries. Then we can easily show that $b(A) \leq 3$. By Proposition 2.1.9, we may assume that $b(A) = 3$. Then A has either a row or a column that has just 3 nonzero entries. Suppose that a row of A has just 3 nonzero entries. Since $b(A) = 3$, there exist permutation matrices U and V such that

$$UAV = E_{1,1} + E_{1,2} + E_{1,3} + E_{2,i} + E_{3,j}$$

for some $i, j \in \{1, \ldots, n\}$ with $i < j$. If $j \geq 4$, then UAV is regular by Corollary 2.1.10 and (2.1.2), and hence A is regular by Proposition 2.1.5. If $1 \leq i < j \leq 3$, then there exist permutation matrices U' and V' such that

$$U'UAVV' = \begin{bmatrix} B & O \\ O & O \end{bmatrix},$$

where $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We can easily show that B is idempotent in $\mathbb{M}_3(\mathbb{B}_1)$, and hence B is regular. It follows from (2.1.2) and Proposition 2.1.5 that A is regular.

If a column of A has just 3 nonzero entries, a parallel argument shows that A is regular. ■

2.2 Linear operators that preserve Boolean regular matrices

In this section we have characterizations of the linear operators that strongly preserve regular matrices over the binary Boolean algebra.

Definition 2.2.1. An operator T on $\mathbb{M}_{m,n}(\mathbb{B}_1)$ is said to be

(1) *linear* if $T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$ for all $\alpha, \beta \in \mathbb{B}_1$ and for all $A, B \in \mathbb{M}_{m,n}(\mathbb{B}_1)$.

(2) *preserve regularity* if $T(A)$ is regular whenever A is regular in $\mathbb{M}_{m,n}(\mathbb{B}_1)$.

Example 2.2.2. Let A be any regular matrix in $\mathbb{M}_{m,n}(\mathbb{B}_1)$. Define an operator T on $\mathbb{M}_{m,n}(\mathbb{B}_1)$ by

$$T(X) = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j} \right) A$$

for all $X = [x_{i,j}] \in \mathbb{M}_{m,n}(\mathbb{B}_1)$.

Then we can easily show that T is a linear operator that preserves regularity because $T(X)$ is either O or A for all $X \in \mathbb{M}_{m,n}(\mathbb{B}_1)$. But T does not preserve any singular matrix in $\mathbb{M}_{m,n}(\mathbb{B}_1)$.

Thus, we are interested in a linear operator T on $\mathbb{M}_{m,n}(\mathbb{B}_1)$ such that $T(X)$ is regular if and only if X is regular over $\mathbb{M}_{m,n}(\mathbb{B}_1)$.

Definition 2.2.3. A linear operator T on $\mathbb{M}_{m,n}(\mathbb{B}_1)$ is said to be *strongly preserve regularity* if $T(A)$ is regular if and only if A is regular in $\mathbb{M}_{m,n}(\mathbb{B}_1)$.

Theorem 2.2.4. *Let T be a linear operator on $\mathbb{M}_{m,n}(\mathbb{B}_1)$, where $\min\{m, n\} \leq 2$. Then T strongly preserves all regular matrices.*

Proof. If $\min\{m,n\} \leq 2$, then all matrices in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ are regular by (2.1.3) and Proposition 2.1.9. Hence $T(A)$ is always regular for all A in $\mathbb{M}_{m,n}(\mathbb{B}_1)$. Thus the result follows. ■

Definition 2.2.5. A linear operator T on $\mathbb{M}_{m,n}(\mathbb{B}_1)$ is said to be *singular* if $T(X) = O$ for some nonzero matrix X in $\mathbb{M}_{m,n}(\mathbb{B}_1)$; Otherwise, T is called *nonsingular*.

Lemma 2.2.6. *If T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{B}_1)$ that strongly preserves regularity for $m \geq 3$ and $n \geq 3$, then T is nonsingular.*

Proof. If $T(X) = O$ for some nonzero matrix X in $\mathbb{M}_{m,n}(\mathbb{B}_1)$, then we have $T(E) = O$ for all cells $E \leq X$. Let F be a cell different from E. By Corollary 2.1.13, there exists a matrix A with $\#(A) = 3$ such that $A + E + F$ is singular, while $A + F$ is regular by Corollary 2.1.10. Nevertheless, $T(A + E + F) = T(A + F)$, a contradiction to the fact that T strongly preserves regularity. Hence $T(X) \neq O$ for all nonzero matrix X in $\mathbb{M}_{m,n}(\mathbb{B}_1)$. Therefore T is nonsingular. ■

For any $i \in \{1, 2, \ldots, mn\}$, let S_i denote a sum of arbitrary distinct i cells in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ with $\#(S_i) = i$. Hereafter, we let $\min\{m,n\} = \alpha$ and $\max\{m,n\} = \beta$.

Proposition 2.2.7. *Let T be a linear operator on $\mathbb{M}_{m,n}(\mathbb{B}_1)$ that strongly preserves regularity, where $\min\{m,n\} = \alpha \geq 3$. Then we have*

$$\#(T(S_i)) \leq 2\alpha + i$$

for all $S_i \in \mathbb{M}_{m,n}(\mathbb{B}_1)$, where $i \in \{1, 2, \ldots, \alpha(\beta - 2)\}$.

Proof. We lose no generality in assuming that $\alpha = m$ and $\beta = n$. Thus we will show that $\#(T(S_i)) \leq 2m + i$ for all $S_i \in \mathbb{M}_{m,n}(\mathbb{B}_1)$, where $i \in \{1, 2, \ldots, m(n-2)\}$.

If $i = m(n-2)$ then clearly $\#(T(S_i)) \leq mn = 2m+i$. For arbitrary $i \in \{1, 2, \ldots, m(n-2)-1\}$, suppose that $\#(T(S_i)) \geq 2m+i+1$ for some $S_i \in \mathbb{M}_{m,n}(\mathbb{B}_1)$. Then $J \backslash T(S_i)$ dominates at most $mn-(2m+i+1)$ cells. Thus we have $\#(T(J) \backslash T(S_i)) \leq mn-(2m+i+1)$. Now for each cell G with $G \leq T(J) \setminus T(S_i)$, let H be a cell such that $G \leq T(H)$, and let X be the sum of all such cells H. Then we have

$$\#(X) \leq \#(T(J) \setminus T(S_i)) \leq mn - (2m + i + 1).$$

Now we claim that $T(J) = T(S_i) + T(X)$. It suffices to show $T(J) \leq T(S_i) + T(X)$. Let G be any cell such that $G \leq T(J)$. If $G \leq T(S_i)$, then we are done. If $G \not\leq T(S_i)$, then there exists a cell H with $H \leq X$ such that $G \leq T(H)$ by the construction of X. Thus, $G \leq T(H) \leq T(X)$. Therefore we have $T(J) \leq T(S_i) + T(X)$, and hence $T(J) = T(S_i) + T(X) = T(S_i + X)$.

Since $\#(X + S_i) \leq mn - (2m + 1)$, there exist distinct cells F_1, F_2, F_3 such that they are not dominated by $X + S_i$ and $b\Big(\sum_{j=1}^{3} F_j\Big) = 3$. It follows from $T(J) = T(X + S_i)$ and $X + S_i \leq J \setminus \sum_{j=1}^{3} F_j$ that

$$T(J) = T(X + S_i) \leq T\Big(J \setminus \sum_{j=1}^{3} F_j\Big) \leq T(J),$$

and hence $T(J) = T\Big(J \setminus \sum_{j=1}^{3} F_j\Big)$, a contradiction to the fact that T strongly preserves regularity because J is regular, while $J \setminus \sum_{j=1}^{3} F_j$ is not regular by Theorem 2.1.15. Therefore we have $\#(T(S_i)) \leq 2m + i$ for all S_i. We conclude that $\#(T(S_i)) \leq 2m + i$ for all $i = 1, 2, \ldots, m(n-2)$. ■

The next Lemma will be important in order to show that if E is any cell in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ with $\min\{m, n\} \geq 3$, then $T(E)$ is also a cell for any linear operator on $\mathbb{M}_{m,n}(\mathbb{B}_1)$ that strongly preserves regularity.

Lemma 2.2.8. *Let* $\min\{m,n\} = \alpha \geq 3$ *and* T *be a linear operator on* $\mathbb{M}_{m,n}(\mathbb{B}_1)$ *that strongly preserves regularity. Then for any* $h \in \{0, 1, 2, \ldots, 2\alpha\}$*, we have*

$$\#(T(S_i)) \leq 2\alpha + i - h$$

for all $S_i \in \mathbb{M}_{m,n}(\mathbb{B}_1)$*, where* $i \in \{1, 2, \ldots, 2\alpha - h + 1\}$*.*

Proof. Without loss of generality, we assume that $\alpha = m$. Thus we will show that if $h \in \{0, 1, 2, \ldots, 2m\}$, then we have $\#(T(S_i)) \leq 2m + i - h$ for all $S_i \in \mathbb{M}_{m,n}(\mathbb{B}_1)$, where $i \in \{1, 2, \ldots, 2m - h + 1\}$.

The proof proceeds by induction on h. It follows from Proposition 2.2.7 that $\#(T(S_i)) \leq 2m + i$ for all $S_i \in \mathbb{M}_{m,n}(\mathbb{B}_1)$, where $i \in \{1, 2, \ldots, 2m + 1\}$. Thus if $h = 0$, the result is obvious. Next, we assume that for some $h \in \{0, 1, 2, \ldots, 2m - 1\}$, the argument is true. That is, we have

$$\#(T(S_i)) \leq 2m + i - h \tag{2.2.1}$$

for all $S_i \in \mathbb{M}_{m,n}(\mathbb{B}_1)$, where $i \in \{1, 2, \ldots, 2m-h+1\}$. Now we will show that $\#(T(S_i)) \leq 2m + i - h - 1$ for all $S_i \in \mathbb{M}_{m,n}(\mathbb{B}_1)$, where $i \in \{1, 2, \ldots, 2m - h\}$. For arbitrary $i \in \{1, 2, \ldots, 2m - h\}$, suppose that $\#(T(S_i)) \geq 2m + i - h$ for some $S_i \in \mathbb{M}_{m,n}(\mathbb{B}_1)$. By (2.2.1), we have

$$\#(T(S_i)) = 2m + i - h \quad \text{and} \quad \#(T(S_i + F)) = 2m + i - h \text{ or } (2m + i - h) + 1$$

for all cells F with $F \not\leq S_i$. If $\#(T(S_i + F_1)) = 2m + i - h$ for some cell F_1 with $F_1 \not\leq S_i$, then we have $T(S_i + F_1) = T(S_i)$. Let F_2 and F_3 be distinct cells different from F_1 such that they are not dominated by S_i and $b\Big(\sum\limits_{j=1}^{3} F_j\Big) = 3$. Then we can select the matrix $Y \in \mathbb{M}_{m,n}(\mathbb{B}_1)$ such that $S_i + Y = J \setminus \sum\limits_{j=1}^{3} F_j$, and hence $S_i + Y + F_1 = J \setminus (F_2 + F_3)$. It follows from $T(S_i + F_1) = T(S_i)$ that $T(S_i + F_1) + T(Y) = T(S_i) + T(Y)$, equivalently

$$T\Big(J \setminus (F_2 + F_3)\Big) = T\Big(J \setminus \sum_{j=1}^{3} F_j\Big),$$

a contradiction because $J \setminus \sum\limits_{j=1}^{3} F_j$ is singular, while $J \setminus (F_2 + F_3)$ is regular by Theorem 2.1.15. Thus we may assume that $\#(T(S_i + F)) = (2m + i - h) + 1$ for all cells F with

$F \not\leq S_i$. This means that for any cell F with $F \not\leq S_i$, there exists only one cell C_F such that

$$C_F \not\leq T(S_i), \quad C_F \leq T(F) \quad \text{and} \quad T(S_i + F) = T(S_i) + C_F \tag{2.2.2}$$

because $\#(T(S_i)) = 2m + i - h$. Let $\mathbb{E}_{m,n}$ be the set of all cells in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ and let

$$\Omega = \{C_F \mid F \in \mathbb{E}_{m,n} \quad \text{and} \quad F \not\leq S_i\}.$$

Suppose that $C_H \neq C_F$ for all distinct cells F and H that are not dominated by S_i. Then we have $\#(\Omega) = mn - i$. Since $C_F \not\leq T(S_i)$ for any cell F with $F \not\leq S_i$, we have $\#(\Omega) \leq mn - (2m + i - h)$ because $\#(T(S_i)) = 2m + i - h$. This is impossible. Hence $C_H = C_F$ for some distinct cells F and H that are not dominated by S_i. It follows from (2.2.2) that

$$T(S_i + F + H) = T(S_i + F) + T(S_i + H) = T(S_i) + C_F = T(S_i + F). \tag{2.2.3}$$

Let H_1 and H_2 be distinct cells different from H such that they are not dominated by $S_i + F$ and $b(H + H_1 + H_2) = 3$. Let Y' be the matrix such that $S_i + F + Y' = J \setminus (H + H_1 + H_2)$. Then we have $S_i + F + H + Y' = J \setminus (H_1 + H_2)$. It follows from (2.2.3) that

$$T\big(J \setminus (H_1 + H_2)\big) = T\big(J \setminus (H + H_1 + H_2)\big),$$

a contradiction because $J \setminus (H_1 + H_2)$ is regular, while $J \setminus (H + H_1 + H_2)$ is singular by Theorem 2.1.15. Consequently, we have $\#(T(S_i)) \leq 2m + i - h$ for all $S_i \in \mathbb{M}_{m,n}(\mathbb{B}_1)$, where $i \in \{1, 2, \ldots, 2m - h\}$. Hence the result follows. ∎

Corollary 2.2.9. *Let T be a linear operator on $\mathbb{M}_{m,n}(\mathbb{B}_1)$ that strongly preserves regularity, where $\min\{m, n\} \geq 3$. Then $T(E)$ is a cell for all cells E in $\mathbb{M}_{m,n}(\mathbb{B}_1)$.*

Proof. Let $h = 2m$ in Lemma 2.2.8. Then we have $\#(T(S_1)) \leq 1$ for all $S_1 \in \mathbb{M}_{m,n}(\mathbb{B}_1)$. It follows from Lemma 2.2.6 that $\#(T(S_1)) = 1$ for all $S_1 \in \mathbb{M}_{m,n}(\mathbb{B}_1)$, equivalently $\#(T(E)) = 1$ for any cell E in $\mathbb{M}_{m,n}(\mathbb{B}_1)$. Therefore we have that $T(E)$ is a cell for any cell E in $\mathbb{M}_{m,n}(\mathbb{B}_1)$. ∎

As shown in Theorem 2.2.4, if T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{B}_1)$ with $\min\{m,n\} \le 2$, then T (strongly) preserves regularity because all matrices in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ are regular by Proposition 2.1.9.

If $\min\{m,n\} \ge 3$, there exists a linear operator on $\mathbb{M}_{m,n}(\mathbb{B}_1)$ such that T preserves regularity, while T does not strongly preserve regularity, see Example 2.2.2.

The next Lemmas are necessary to prove the main theorem of this section.

Lemma 2.2.10. *Let T be a linear operator on $\mathbb{M}_{m,n}(\mathbb{B}_1)$ that strongly preserve regularity for $\min\{m,n\} \ge 3$. Then T is bijective on the set of cells.*

Proof. By Corollary 2.2.9, we suffice to show that $T(E) \ne T(F)$ for all distinct cells E and F in $\mathbb{M}_{m,n}(\mathbb{B}_1)$. Suppose that $T(E) = T(F)$ for some distinct cells E and F. Then we have $T(E+F) = T(E)$. By Corollary 2.1.13, there exists a matrix A in $\mathbb{M}_{m,n}(\mathbb{B}_1)$ with $\#(A) = 3$ such that $A+E+F$ is singular. Since $T(E+F) = T(E)$, we have

$$T(A+E+F) = T(A+E),$$

a contradiction to the fact that T strongly preserves regularity because $A+E$ is regular by Corollary 2.1.10. Therefore T is bijective on the set of cells. ■

Let $A \in \mathbb{M}_{m,n}(\mathbb{B}_1)$ be a nonzero matrix dominated by a line matrix. Then we have $b(A) = 1$. If $\#(A) = s$, then we say that A is a *s-star matrix*. Therefore all s-star matrices are regular by Proposition 2.1.9.

Lemma 2.2.11. *Let T be a linear operator on $\mathbb{M}_{m,n}(\mathbb{B}_1)$ that strongly preserve regularity for $\min\{m,n\} \ge 3$. Then T preserves all 3-star matrices.*

Proof. Suppose that T does not preserve a 3-star matrix A in $\mathbb{M}_{m,n}(\mathbb{B}_1)$. Then we have that $b(T(A)) = 2$ or 3. By Corollary 2.1.14, there exist cells E and F such that $T(A)+E+F$ is singular. By Lemma 2.2.10, we can write $E = T(H_1)$ and $F = T(H_2)$ for some cells H_1 and H_2. Thus we have

$$T(A)+E+F = T(A+H_1+H_2).$$

But $A + H_1 + H_2$ is regular by Proposition 2.1.16. This contradicts to the fact that T strongly preserves regularity. Hence T preserves all 3-star matrices. ■

Corollary 2.2.12. *Let T be a linear operator on $\mathbb{M}_{m,n}(\mathbb{B}_1)$ that strongly preserve regularity for $\min\{m, n\} \geq 3$. Then T preserves all line matrices.*

Proof. Suppose that T does not preserve a line matrix A in $\mathbb{M}_{m,n}(\mathbb{B}_1)$. Then there exist two cells E and F dominated by A such that two cells $T(E)$ and $T(F)$ are not collinear. Let G be a cell such that $E+F+G$ is a 3-star matrix. By Lemma 2.2.11, $T(E+F+G)$ is a 3-star matrix, and hence $b(T(E + F + G)) = 1$. Thus, the three cells $T(E)$, $T(F)$ and $T(G)$ are collinear. This contradicts to the fact that the two cells $T(E)$ and $T(F)$ are not collinear. Therefore T preserves all line matrices. ■

We say that a linear operator T on $\mathbb{M}_{m,n}(\mathbb{B}_1)$ is a (U, V)-*operator* if there exist invertible matrices $U \in \mathbb{M}_m(\mathbb{B}_1)$ and $V \in \mathbb{M}_n(\mathbb{B}_1)$ such that either

$$T(X) = UXV \quad \text{for all} \quad X \in \mathbb{M}_{m,n}(\mathbb{B}_1), \quad \text{or}$$

$$m = n \quad \text{and} \quad T(X) = UX^TV \quad \text{for all} \quad X \in \mathbb{M}_{m,n}(\mathbb{B}_1).$$

We remind that the $n \times n$ permutation matrices are the only $n \times n$ invertible Boolean matrices.

Theorem 2.2.13. *Let T be a linear operator on $\mathbb{M}_{m,n}(\mathbb{B}_1)$ with $\min\{m, n\} \geq 3$. Then T strongly preserves regularity if and only if T is a (U, V)-operator.*

Proof. The sufficiency follows from Proposition 2.1.5. To prove the necessity, assume that T strongly preserves regularity. Then T is bijective on the set of cells by Lemma 2.2.10 and T preserves all line matrices by Corollary 2.2.12. Since no combination of s row matrices and t column matrices can dominate $J_{m,n}$ where $s + t = \min\{m, n\}$ unless $s = 0$ or $t = 0$, we have that either

(1) the image of each row matrix is a row matrix and the image of each column matrix is a column matrix, or

(2) the image of each row matrix is a column matrix and the image of each column matrix is a row matrix.

If (1) holds, then there exist permutations σ and τ of $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$, respectively such that $T(R_i) = R_{\sigma(i)}$ and $T(C_j) = C_{\tau(j)}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Let $U \in \mathbb{M}_m(\mathbb{B}_1)$ and $V \in \mathbb{M}_n(\mathbb{B}_1)$ be permutation (i.e., invertible) matrices corresponding to σ and τ, respectively. Then we have

$$T(E_{i,j}) = E_{\sigma(i),\tau(j)} = UE_{i,j}V$$

for all cells $E_{i,j}$ in $\mathbb{M}_{m,n}(\mathbb{B}_1)$. Let $X = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{i,j}E_{i,j}$ be any matrix in $\mathbb{M}_{m,n}(\mathbb{B}_1)$. By the action of T on the cells, we have that $T(X) = UXV$. If (2) holds, then $m = n$ and a parallel argument shows that there exist invertible matrices U and V in $\mathbb{M}_n(\mathbb{B}_1)$ such that $T(X) = UX^TV$ for all X in $\mathbb{M}_n(\mathbb{B}_1)$. Therefore T is a (U, V)-operator. ■

Thus, as shown in Theorems 2.2.4 and 2.2.13, we have characterizations of the linear operators that strongly preserve Boolean regular matrices.

3 Extreme sets of matrix pairs over general Boolean algebra and their preservers

3.1 Preliminaries and Basic results

Definition 3.1.1. ([11]) A *semiring* $\mathbb{S}$ consists of a set S and two binary operations, addition$(+)$ and multiplication$(\cdot)$, such that:

- S is an Abelian monoid under addition (the identity is denoted by 0);
- S is a monoid under multiplication (the identity is denoted by 1);
- multiplication is distributive over addition on both sides;
- $s0 = 0s = 0$ for all $s \in S$.

Definition 3.1.2. ([11]) A semiring $\mathbb{S}$ is called *antinegative* if the zero element is the only element with an additive inverse.

Definition 3.1.3. ([11]) A semiring $\mathbb{S}$ is called a *general Boolean algebra* if $\mathbb{S}$ is equivalent to a set of subsets of a given set M, the sum of two subsets is their union, and the product is their intersection. The zero element is the empty set and the identity element is the whole set M.

Let $S_k = \{a_1, a_2, \cdots, a_k\}$ be a set of k-elements, $\mathcal{P}(S_k)$ be the set of all subsets of S_k and $\mathbb{B}_k$ be a general Boolean algebra of subsets of $S_k = \{a_1, a_2, \cdots, a_k\}$, which is a subset of $\mathcal{P}(S_k)$. It is straightforward to see that a general Boolean algebra $\mathbb{B}_k$ is a commutative and antinegative semiring. Let $\mathbb{M}_{m,n}(\mathbb{B}_k)$ denote the set of $m \times n$ matrices with entries from the general Boolean algebra $\mathbb{B}_k$. If $m = n$, we use the notation $\mathbb{M}_n(\mathbb{B}_k)$ instead of $\mathbb{M}_{n,n}(\mathbb{B}_k)$.

Throughout the thesis, we assume that $m \leq n$. The matrix I_n is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones and $O_{m,n}$ is the $m \times n$ zero matrix. We omit the subscripts when the order is obvious from the context and we write I, J and O, respectively. The matrix $E_{i,j}$, which is called a *cell*, denotes the matrix with exactly one nonzero entry, that being a one in the $(i, j)^{th}$ entry. A *weighted cell* is any nonzero scalar multiple of a cell, that is, $\alpha E_{i,j}$ is a weighted cell for any $0 \neq \alpha \in \mathbb{B}_k$. Let R_i denote the matrix whose i^{th} row is all ones and is zero elsewhere, and C_j denote the matrix whose j^{th} column is all ones and is zero elsewhere. We denote by $\#(A)$ the number of nonzero entries in the matrix A. We denote by A[i,j|r,s] the 2×2 submatrix of A which lies in the intersection of the i^{th} and j^{th} rows with the r^{th} and s^{th} columns.

Definition 3.1.4. ([9], [13]) The matrix $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ is said to be of *Boolean rank r* if there exist matrices $B \in \mathbb{M}_{m,r}(\mathbb{B}_k)$ and $C \in \mathbb{M}_{r,n}(\mathbb{B}_k)$ such that $A = BC$ and r is the smallest positive integer that such a factorization exists. We denote $\mathrm{b}(A) = r$.

By definition, the unique matrix with Boolean rank equal to 0 is the zero matrix O.

If $\mathcal{F}$ is a field, then there is the usual rank function $\rho(A)$ for any matrix $A \in \mathbb{M}_{m,n}(\mathcal{F})$. These rank functions are not equal in general. However, the inequality $b(A) \geq \rho(A)$ always holds for any matrix $A \in \mathbb{M}_{m,n}(\mathcal{F}) \bigcap \mathbb{M}_{m,n}(\mathbb{S})$. Consider the matrix $M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \in \mathbb{M}_{4,4}(\mathbb{B}_k)$. Then M has Boolean rank 4 and has real rank 3 by Example 4.3 in [6].

The behavior of the function ρ with respect to matrix multiplication and addition is given by the following inequalities([2] and [18]):

- the rank-sum inequalities:

$$|\rho(A) - \rho(B)| \leq \rho(A + B) \leq \rho(A) + \rho(B),$$

- Sylvester's laws:

$$\rho(A) + \rho(B) - n \leq \rho(AB) \leq \min\{\rho(A), \rho(B)\},$$

- and the Frobenius inequality:

$$\rho(AB) + \rho(BC) \leq \rho(ABC) + \rho(B),$$

where A, B and C are conformal matrices with entries from a field.

The arithmetic properties of Boolean rank are restricted by the following list of inequalities([2]), since the Boolean algebra is antinegative:

(1) $\mathrm{b}(A+B) \le \mathrm{b}(A) + \mathrm{b}(B)$;

(2) $\mathrm{b}(AB) \le \min\{\mathrm{b}(A), \mathrm{b}(B)\}$;

(3) $\mathrm{b}(A+B) \ge \begin{cases} \mathrm{b}(A) & \text{if } B = O, \\ \mathrm{b}(B) & \text{if } A = O, \\ 1 & \text{if } A \ne O \text{ and } B \ne O; \end{cases}$

(4) $\mathrm{b}(AB) \ge \begin{cases} 0 & \text{if } \mathrm{b}(A) + \mathrm{b}(B) \le n, \\ 1 & \text{if } \mathrm{b}(A) + \mathrm{b}(B) > n. \end{cases}$

Below, we use the following notation in order to denote sets of matrices that arise as extremal cases in the inequalities listed above:

$\mathcal{R}_{SA}(\mathbb{B}_k) = \{\,(X,Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 |\ \mathrm{b}(X+Y) = \mathrm{b}(X) + \mathrm{b}(\mathrm{Y})\,\}$,

$\mathcal{R}_{S1}(\mathbb{B}_k) = \{(X,Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 |\ \mathrm{b}(X+Y) = 1\}$,

$\mathcal{R}_{SD}(\mathbb{B}_k) = \{(X,Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 |\ \mathrm{b}(X+Y) = |\mathrm{b}(X) - \mathrm{b}(\mathrm{Y})|\}$,

$\mathcal{R}_{MM}(\mathbb{B}_k) = \{(X,Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 |\ \mathrm{b}(XY) = \min\{\ \mathrm{b}(X), \mathrm{b}(\mathrm{Y})\}\}$,

$\mathcal{R}_{M0}(\mathbb{B}_k) = \{(X,Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 |\ \mathrm{b}(XY) = 0\}$,

$\mathcal{R}_{M1}(\mathbb{B}_k) = \{(X,Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 |\ \mathrm{b}(XY) = 1\}$,

$\mathcal{R}_{MA}(\mathbb{B}_k) = \{(X,Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 |\ \mathrm{b}(XY) = \mathrm{b}(X) + \mathrm{b}(\mathrm{Y}) - \mathrm{n}\}$,

$\mathcal{R}_{M3}(\mathbb{B}_k) = \{(X,Y,Z) \in \mathbb{M}_n(\mathbb{B}_k)^3 |\ \mathrm{b}(XYZ)+\mathrm{b}(Y) = \mathrm{b}(XY) + \mathrm{b}(YZ)\}$.

Definition 3.1.5. ([4]) We say that an operator T *preserves* a set $\mathcal{P}$ if $X \in \mathcal{P}$ implies that $T(X) \in \mathcal{P}$ or if $\mathcal{P}$ is the set of ordered pairs (triples) such that $(X,Y) \in \mathcal{P}$ (respectively, $(X,Y,Z) \in \mathcal{P}$) implies $((T(X), T(Y)) \in \mathcal{P}$ (respectively, $(T(X), T(Y), T(Z)) \in \mathcal{P}$).

Definition 3.1.6. An operator T *strongly preserves* the set $\mathcal{P}$ if $X \in \mathcal{P}$ if and only if $T(X) \in \mathcal{P}$ or if $\mathcal{P}$ is the set of ordered pairs (triples) such that $(X,Y) \in \mathcal{P}$ (respectively, $(X,Y,Z) \in \mathcal{P}$) if and only if $(T(X), T(Y)) \in \mathcal{P}$ (respectively, $(T(X), T(Y), T(Z)) \in \mathcal{P}$).

Definition 3.1.7. ([4]) For $X, Y \in \mathbb{M}_{m,n}(\mathbb{S})$, the matrix $X \circ Y$ denotes the *Hadamard or Schur product*, i.e., the $(i,j)^{th}$entry of $X \circ Y$ is $x_{i,j}y_{i,j}$.

Definition 3.1.8. ([4]) An operator T is called a (P, Q, B)-*operator* if there exist permutation matrices P and Q and a matrix $B \in \mathbb{M}_{m,n}(\mathbb{S})$ with no zero entries such that $T(X) = P(X \circ B)Q$ for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$ or if for $m = n$, $T(X) = P(X \circ B)^T Q$ for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$. A (P, Q, B)-operator is called a (P, Q)-*operator* if $B = J$, the matrix of all ones.

Definition 3.1.9. ([4]) Let $\mathbb{B}_k$ be a general Boolean algebra. An operator $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ is called *linear* if it satisfies $T(X + Y) = T(X) + T(Y)$ and $T(\alpha X) = \alpha T(X)$ for all $X, Y \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ and $\alpha \in \mathbb{B}_k$.

Definition 3.1.10. A *line* of a matrix A is a row or a column of the matrix A.

Definition 3.1.11. ([4]) We say that the matrix A *dominates* the matrix B if and only if $b_{i,j} \neq 0$ implies that $a_{i,j} \neq 0$, and we write $A \geq B$ or $B \leq A$.

Lemma 3.1.12. *Let P and Q be permutation matrices of m-square and n-square respectively. If $T : \mathbb{M}_{m,n}(\mathbb{S}) \to \mathbb{M}_{m,n}(\mathbb{S})$ is defined by $T(X) = PX$ or $T(X) = XQ$ for any $X \in \mathbb{M}_{m,n}(\mathbb{B}_k)$. Then T preserves Boolean rank. That is, $b(T(X)) = b(X)$.*

Proof. Let $A, B \in \mathbb{M}_{m,n}(\mathbb{S})$ and P be an $m \times m$ permutation matrix. Since, in any semiring $\mathbb{S}$,

$$\mathrm{b}(AB) \leq \min\{\mathrm{b}(A),\mathrm{b}(B)\},\ \mathrm{b}(PX) \leq \min\{\mathrm{b}(P),\ \mathrm{b}(X)\} \leq \mathrm{b}(X).$$

And

$$\mathrm{b}(X) = \mathrm{b}(IX) = \mathrm{b}((P^T P)X) = \mathrm{b}(P^T(PX)) \leq \mathrm{b}(PX).$$

Hence $\mathrm{b}(PX) = \mathrm{b}(X)$. Similarly $\mathrm{b}(XQ) = \mathrm{b}(X)$ for all $n \times n$ permutation matrix Q. ■

Lemma 3.1.13. *If* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_{2,2}(\mathbb{B}_k)$ *has Boolean rank 1, then* $ad = bc$.

Proof. Suppose that $\mathrm{b}(A) = 1$. Then there exist vectors $\mathbf{x} = [\mathbf{x_1}, \mathbf{x_2}]^{\mathbf{T}}$ and $\mathbf{y} = [\mathbf{y_1}, \mathbf{y_2}]$ such that $A = \mathbf{xy}$. Thus

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} y_1 & y_2 \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 \\ x_2y_1 & x_2y_2 \end{bmatrix}.$$

Hence $ad = x_1x_2y_1y_2 = bc$. ■

Lemma 3.1.14. *If* $ad \neq bc$*, then* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_{2,2}(\mathbb{B}_k)$ *has Boolean rank 2.*

Proof. Suppose that $ad \neq bc$ and $\mathrm{b}(A) \neq 2$. Then $\mathrm{b}(A) = 1$ and there exist vectors $\mathbf{x} = [\mathbf{x_1}, \mathbf{x_2}]^{\mathbf{T}}$ and $\mathbf{y} = [\mathbf{y_1}, \mathbf{y_2}]$ such that $A = \mathbf{xy}$. Thus

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} y_1 & y_2 \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 \\ x_2y_1 & x_2y_2 \end{bmatrix}.$$

Hence $ad = x_1x_2y_1y_2 = bc$, a contradiction to $ad \neq bc$. ■

The inverse of Lemma 3.1.13 is not true. Consider the following example:

Example 3.1.15. Let $\mathbb{B}_4 = \mathcal{P}(\{a, b, c, d\})$ and $A = \begin{bmatrix} \{a\} & \{b\} \\ \{c\} & \{d\} \end{bmatrix}$ be a 2×2 matrix over $\mathbb{B}_4$. Then $\{a\} \cdot \{d\} = 0 = \{b\} \cdot \{c\}$, but $\mathrm{b}(A) = 2 \neq 1$.

Lemma 3.1.16. *Let* $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$*, where* $m, n \geq 2$. $b(A) = 1$ *if and only if* $b(A') = 1$ *for any* 2×2 *submatrix* A' *of* A.

Proof. $\Rightarrow$) Suppose that $\mathrm{b}(A) = 1$, then there exist vectors $\mathbf{a} = [a_1, a_2, \ldots, a_m]^T$ and $\mathbf{b} = [\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_n}]$ such that $A = \mathbf{ab}$, i.e., $a_{i,j} = a_ib_j$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Thus for any 2×2 submatrix

$$A' = A[i, j|r, s] = \begin{bmatrix} a_ib_r & a_ib_s \\ a_jb_r & a_jb_s \end{bmatrix} = \begin{bmatrix} a_i \\ b_j \end{bmatrix} \begin{bmatrix} b_r & b_s \end{bmatrix},$$

i.e., $\mathrm{b}(A') = 1$.

$\Leftarrow$) Suppose that $\mathrm{b}(A) = r > 1$. Then there exist matrices $B \in \mathbb{M}_{m,r}(\mathbb{B}_k)$ and $C \in \mathbb{M}_{r,n}(\mathbb{B}_k)$ such that $A = BC$. Thus there exist matrices $B' \in \mathbb{M}_{m,2}(\mathbb{B}_k)$ and $C' \in \mathbb{M}_{2,n}(\mathbb{B}_k)$ with Boolean rank 2 such that $A' = B'C'$. Therefore there exist matrices $B'' \subset B'$ and $C'' \subset C'$ in $\mathbb{M}_{2,2}(\mathbb{B}_k)$ such that $A'' = B''C'' \in \mathbb{M}_{2,2}(\mathbb{B}_k)$ with $\mathrm{b}(A'') = 2$, a contradiction. ■

Theorem 3.1.17. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear operator. Then the following conditions are equivalent:*

(a) T is bijective;

(b) T is surjective;

(c) T is injective;

(d) there exists a permutation σ on $\{(i,j)|i = 1,2,\ldots,m; j = 1,2,\ldots,n\}$ such that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all $1 \le i \le m$ and $1 \le j \le n$.

Proof. (a), (b) and (c) are equivalent since $\mathbb{M}_{m,n}(\mathbb{B}_k)$ is a finite set.

(d)$\Rightarrow$(b) For any $D \in \mathbb{M}_n(\mathbb{B}_k)$, we may write

$$D = \sum_{i=1}^{m}\sum_{j=1}^{n} d_{i,j}E_{i,j}.$$

Since σ is a permutation, there exist $\sigma^{-1}(i,j)$ and

$$D' = \sum_{i=1}^{m}\sum_{j=1}^{n} d_{\sigma^{-1}(i,j)}E_{\sigma^{-1}(i,j)}$$

such that

$$\begin{aligned} T(D') &= T(\sum_{i=1}^{m}\sum_{j=1}^{n} d_{\sigma^{-1}(i,j)}E_{\sigma^{-1}(i,j)}) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} d_{\sigma\sigma^{-1}(i,j)}E_{\sigma\sigma^{-1}(i,j)} \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} d_{i,j}E_{i,j} = D. \end{aligned}$$

(a)$\Rightarrow$(d) We assume that T is bijective. Suppose that $T(E_{i,j}) \neq E_{\sigma(i,j)}$ where σ be a permutation on $\{(i,j)|i = 1,2,\ldots,m; j = 1,2,\ldots,n\}$. Then there exist some pairs

(i,j) and (r,s) such that $T(E_{i,j}) = \alpha E_{r,s}(\alpha \neq 1)$ or some pairs $(i,j),(r,s)$ and (u,v) $((r,s) \neq (u,v))$ such that $T(E_{i,j}) = \alpha E_{r,s} + \beta E_{u,v} + Z(\alpha \neq 0, \beta \neq 0, Z \in \mathbb{M}_{m,n}(\mathbb{B}_k))$, where the $(r,s)^{th}$ and $(u,v)^{th}$ entries of Z are zeros.

Case 1) Suppose that there exist some pairs (i,j) and (r,s) such that $T(E_{i,j}) = \alpha E_{r,s}(\alpha \neq 1)$. Since T is bijective, there exist $X_{r,s} \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ such that $T(X_{r,s}) = E_{r,s}$. Then $\alpha T(X_{r,s}) = \alpha E_{r,s} = T(E_{i,j})$, and $T(\alpha X_{r,s}) = T(E_{i,j})$. Hence $\alpha X_{r,s} = E_{i,j}$, which contradicts the fact that $\alpha \neq 1$.

Case 2) Suppose that there exist some pairs $(i,j),(r,s)$ and (u,v) such that $T(E_{i,j}) = \alpha E_{r,s} + \beta E_{u,v} + Z(\alpha \neq 0, \beta \neq 0, Z \in \mathbb{M}_{m,n}(\mathbb{B}_k))$, where the $(r,s)^{th}$ and $(u,v)^{th}$ entries of Z are zeros. Since T is bijective, there exist $X_{r,s}$, $X_{u,v}$ and $Z' \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ such that $T(X_{r,s}) = \alpha E_{r,s}$, $T(X_{u,v}) = \beta E_{u,v}$, and $T(Z') = Z$. Thus

$$T(E_{i,j}) = \alpha E_{r,s} + \beta E_{u,v} + Z = T(X_{r,s}) + T(X_{u,v}) + T(Z') = T(X_{r,s} + X_{u,v} + Z').$$

So $E_{i,j} = X_{r,s} + X_{u,v} + Z'$, a contradiction. ∎

Remark 3.1.18. One can easily verify that if $m = 1$ or $n = 1$, then all operators under consideration are (P,Q,B)-operators and if $m = n = 1$, then all operators under consideration are (P,P^T,B)-operators.

Henceforth we will always assume that $m,n \geq 2$.

Lemma 3.1.19. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear operator which maps a line to a line and T be defined by the rule $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$, where σ is a permutation on the set $\{(i,j)|i = 1,2,\ldots,m; j = 1,2,\ldots,n\}$ and $b_{i,j} \in \mathbb{B}_k$ are nonzero elements for $i = 1,2,\ldots,m; j = 1,2,\ldots,n$. Then T be a (P,Q,B)-operator.*

Proof. Since no combination of p rows and q columns can dominate J for any nonzero p and q with $p+q = m$, we have that either the image of each row is a row and the image of each column is a column, or $m = n$ and the image of each row is a column and image of each column is a row. Thus there are permutation matrices P and Q such that

$$T(R_i) \leq PR_iQ,\ T(C_j) \leq PC_jQ$$

or, if m = n,

$$T(R_i) \le P(R_i)^T Q,\ T(C_j) \le P(C_j)^T Q.$$

Since each nonzero entry of a cell lies in the intersection of a row and a column and T maps nonzero cells to nonzero (weighted) cells, it follows that

$$T(E_{i,j}) = Pb_{i,j}E_{i,j}Q = P(E_{i,j} \circ B)Q,$$

or, if m = n,

$$T(E_{i,j}) = P(b_{i,j}E_{i,j})^T Q = P(E_{i,j} \circ B)^T Q$$

where $B = (b_{i,j})$ is defined by the action of T on the cells. ■

Lemma 3.1.20. *If $T(X) = X \circ B$ for all $X \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ and $b(B) = 1$ then there exist diagonal matrices D and E such that $T(X) = DXE$ for all $X \in \mathbb{M}_{m,n}(\mathbb{B}_k)$.*

Proof. Since $\mathrm{b}(B) = 1$, there exist vectors $\mathbf{d} = [\mathbf{d_i}, \mathbf{d_2}, \ldots, \mathbf{d_m}]^{\mathbf{T}} \in \mathbb{M}_{\mathbf{m},\mathbf{1}}$ and $\mathbf{e} = [\mathbf{e_i}, \mathbf{e_2}, \ldots, \mathbf{e_n}] \in \mathbb{M}_{\mathbf{1},\mathbf{n}}$ such that $\mathbf{B} = \mathbf{de}$ or $\mathbf{b_{i,j}} = \mathbf{d_i e_j}$. Let $D = diag\{d_1, d_2, \ldots, d_m\}$ and $E = diag\{e_1, e_2, \ldots, e_n\}$. Now the $(i,j)^{th}$ entry of $T(X)$ is $b_{i,j}x_{i,j}$ and the $(i,j)^{th}$ entry of DXE is $d_i x_{i,j} e_j = b_{i,j}x_{i,j}$. Hence $T(X) = DXE$. ■

Example 3.1.21. Consider the linear operator $T : \mathbb{M}_{3,3}(\mathbb{B}_3) \to \mathbb{M}_{3,3}(\mathbb{B}_3)$ defined by $T(X) = X \circ B$ for all $X \in \mathbb{M}_{3,3}(\mathbb{B}_3)$ with $\mathbb{B}_3 = \mathcal{P}(\{a,b,c\})$. Then $\mathrm{b}(B) = 1$ but T does not preserves the Boolean rank.

Consider $X = \begin{bmatrix} \{a,b\} & \{a,b,c\} & \{a,b\} \\ \{a,c\} & \{a,b\} & \{a,c\} \\ \{a\} & \{b,c\} & \{a,b,c\} \end{bmatrix}$ and $B = \begin{bmatrix} \{a\} & \{b\} & \{a\} \\ \{a\} & \{b\} & \{a\} \\ \{a\} & \{b\} & \{a\} \end{bmatrix}$.

Then $\mathrm{b}(X) = 3$, but

$$T(X) = X \circ B = \begin{bmatrix} \{a\} & \{b\} & \{a\} \\ \{a\} & \{b\} & \{a\} \\ \{a\} & \{b\} & \{a\} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \{a\} & \{b\} & \{a\} \end{bmatrix}.$$

That is, $\mathrm{b}(T(X)) = \mathrm{b}(X \circ B) = 1 \neq 3 = \mathrm{b}(X)$. Thus $\mathrm{b}(B) = 1$ but T does not preserves the Boolean rank.

3.2 Linear preservers of $\mathcal{R}_{SA}(\mathbb{B}_k)$

Recall that

$$\mathcal{R}_{SA}(\mathbb{B}_k) = \{ (X, Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 |\ \mathrm{b}(X+Y) = \mathrm{b}(X) + \mathrm{b(Y)} \}.$$

Example 3.2.1. We show that $\mathcal{R}_{SA}(\mathbb{B}_2)$ is not an empty set.

Let $\mathbb{B}_2 = \mathcal{P}(\{a, b\}) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. Consider two matrices X and Y over $\mathbb{B}_2$:

$$X = \begin{bmatrix} \{a\} & \{a,b\} \\ \phi & \phi \end{bmatrix} = \begin{bmatrix} \{a,b\} \\ \phi \end{bmatrix} \begin{bmatrix} \{a\} & \{a,b\} \end{bmatrix}$$

and

$$Y = \begin{bmatrix} \phi & \phi \\ \{a,b\} & \{b\} \end{bmatrix} = \begin{bmatrix} \phi \\ \{a,b\} \end{bmatrix} \begin{bmatrix} \{a,b\} & \{b\} \end{bmatrix}.$$

Then $\mathrm{b}(X) = \mathrm{b}(Y) = 1$ and

$$X + Y = \begin{bmatrix} \{a\} & \{a,b\} \\ \{a,b\} & \{b\} \end{bmatrix}$$

has Boolean rank 2 by Lemma 3.1.14. Thus $(X, Y) \in \mathcal{R}_{SA}(\mathbb{B}_2)$. That is $\mathcal{R}_{SA}(\mathbb{B}_2) \neq \phi$.

We begin with some general observations on linear operators of special types that preserve $\mathcal{R}_{SA}(\mathbb{B}_k)$.

Lemma 3.2.2. *Let σ be a permutation of the set $\{(i,j)|i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$, and $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear operator defined by $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for some nonzero scalars $b_{i,j}$, $1 \leq i \leq m$ and $1 \leq j \leq n$. If T preserves $\mathcal{R}_{SA}(\mathbb{B}_k)$, then T is a (P, Q, B)-operator.*

Proof. We examine the action of T on rows and columns of a matrix. Suppose that the image of two cells are in the same line, but the cells are not, say, E and F are cells such that $\mathrm{b}(E + F) = 2$ and $\mathrm{b}(T(E + F)) = 1$. Then $(E, F) \in \mathcal{R}_{SA}(\mathbb{B}_k)$ but $(T(E), T(F)) \notin \mathcal{R}_{SA}(\mathbb{B}_k)$, a contradiction since T preserves $\mathcal{R}_{SA}(\mathbb{B}_k)$. Thus T maps any line to a line. By Lemma 3.1.19, we obtain the result. ■

Lemma 3.2.3. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$be a linear operator. If for some $B = (b_{i,j})$, where $b_{i,j}$ are nonzero scalars for all $1 \leq i \leq m$ and $1 \leq j \leq n$, $T(X) = X \circ B$ preserves $\mathcal{R}_{SA}(\mathbb{B}_k)$, then b$(B)$ = 1. Moreover, $T(X) = DXE$ for diagonal matrices D and E of appropriate sizes.*

Proof. If $\mathrm{b}(B) \geq 2$, then by Lemma 3.1.16, there is a 2×2 submatrix $B[i,j|r,s]$ such that $\mathrm{b}(B[i,j|r,s]) = 2$. Let $Y = E_{i,r} + E_{j,r} + E_{i,s} + E_{j,s}$. Thus $T(Y) = b_{i,r}E_{i,r} + b_{j,r}E_{j,r} + b_{i,s}E_{i,s} + b_{j,s}E_{j,s} = Z$ has Boolean rank 2 from $\mathrm{b}(B[i,j|r,s]) = 2$. Then for $q \neq r, s$, we have $\mathrm{b}(E_{i,q} + Y) = 2 = \mathrm{b}(E_{i,q}) + \mathrm{b}(Y)$, so that $(E_{i,q}, Y) \in \mathcal{R}_{SA}(\mathbb{B}_k)$, while $\mathrm{b}(T(E_{i,q} + Y))$ $= \mathrm{b}(b_{i,q}E_{i,q} + Z) = 2 \neq \mathrm{b}(b_{i,q}E_{i,q}) + \mathrm{b}(Z) = 1 + 2 = 3$, a contradiction since T preserves $\mathcal{R}_{SA}(\mathbb{B}_k)$. Thus $\mathrm{b}(B) = 1$. Moreover, by Lemma 3.1.20, there exist diagonal matrices D and E such that $T(X) = DXE$. ■

Theorem 3.2.4. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a surjective linear operator. The operator T preserves $\mathcal{R}_{SA}(\mathbb{B}_k)$ if and only if T is a (P, Q)-operator.*

Proof. $\Rightarrow$) If T is surjective, then by Theorem 3.1.17, we have that T is defined by a permutation σ on the set $\{(i,j)|i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$ such that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. By Lemma 3.2.2, we have that T is a (P, Q, J)-operator. Thus T is a (P, Q)-operator.

$\Leftarrow$) Assume that T is a (P, Q)-operator. For any $(X, Y) \in \mathcal{R}_{SA}(\mathbb{B}_k)$, we have $\mathrm{b}(X+Y)$ $= \mathrm{b}(X) + \mathrm{b}(Y)$. Thus

$$\begin{aligned}\mathrm{b}(T(X) + T(Y)) &= \mathrm{b}(T(X + Y)) = \mathrm{b}(P(X + Y)Q) = \mathrm{b}(X + Y)\\ &= \mathrm{b}(X) + \mathrm{b}(Y) = \mathrm{b}(PXQ) + \mathrm{b}(PYQ) = \mathrm{b}(T(X)) + \mathrm{b}(T(Y)).\end{aligned}$$

Hence the operator T preserves $\mathcal{R}_{SA}(\mathbb{B}_k)$. ■

Lemma 3.2.5. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear operator. Then there is a power of T which is idempotent.*

Proof. Since $\mathbb{B}_k$ is finite, there are only finitely many linear operators from $\mathbb{M}_{m,n}(\mathbb{B}_k)$ into $\mathbb{M}_{m,n}(\mathbb{B}_k)$. Thus the sequence $\{T, T^2, T^3, \ldots, T^m, \ldots\}$ is finite for sufficiently large n. That is, there exist integers $N \geq 1$ and $d \geq 1$ such that for $m, n \geq N$ with $m \equiv n \pmod d$, $T^m = T^n$. Let $p = Nd$. Then $2p \equiv p(\bmod\ d)$. Hence $(T^p)^2 = T^{2p} = T^p$. That is, T^p is idempotent. ■

Theorem 3.2.6. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear operator. If T strongly preserves $\mathcal{R}_{SA}(\mathbb{B}_k)$, then T is a (P, Q, B)-operator, where $B \in \mathbb{M}_{m,n}(\mathbb{B}_k)$.*

Proof. By Lemma 3.2.5, there is a power of T which is idempotent. Say $L = T^p$ with $L^2 = L$. If $X \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ and $(X, X) \in \mathcal{R}_{SA}(\mathbb{B}_k)$, then $\mathrm{b}(X) = \mathrm{b}(X + X) = \mathrm{b}(X) + \mathrm{b}(X)$. Thus $\mathrm{b}(X) = 0$, $X = O_{m,n}$. Similarly, if $(T(X), T(X)) \in \mathcal{R}_{SA}(\mathbb{B}_k)$, then $T(X) = O_{m,n}$. Thus $(X, X) \in \mathcal{R}_{SA}(\mathbb{B}_k)$ if and only if $(L(X), L(X)) \in \mathcal{R}_{SA}(\mathbb{B}_k)$ since T strongly preserves $\mathcal{R}_{SA}(\mathbb{B}_k)$. So, $\mathrm{b}(X) = 0$ if and only if $\mathrm{b}(L(X)) = 0$. That is, $X = O_{m,n}$ if and only if $L(X) = O_{m,n}$. Hence, if $A \neq O$, then we have $L(A) \neq O$ since T strongly preserves $\mathcal{R}_{SA}(\mathbb{B}_k)$. We examine the action of L on rows and columns. Assume that $L(R_i)$ is not dominated by R_i. Then there is some (r, s) such that $E_{r,s} \leq L(R_i)$ while $E_{r,s} \not\leq R_i$. Then it is easy to see that

$$(R_i, aE_{r,s}) \in \mathcal{R}_{SA}(\mathbb{B}_k). \tag{3.2.1}$$

Since $E_{r,s} \leq L(R_i)$, we can find a matrix $X = (x_{i,j}) \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ with $x_{r,s} = 0$ such that $L(R_i) = aE_{r,s} + X$ for nonzero a in $\mathbb{B}_k$. We have

$$\begin{aligned} L(R_i + aE_{r,s}) &= L(R_i) + L(aE_{r,s}) = L^2(R_i) + L(aE_{r,s}) \\ &= L(aE_{r,s} + X) + L(aE_{r,s}) = L(X) + L(aE_{r,s}) + L(aE_{r,s}) \\ &= L(X) + L(aE_{r,s}) = L(X + aE_{r,s}) = L(L(R_i)) = L^2(R_i) = L(R_i). \end{aligned}$$

That is,

$$\mathrm{b}(L(R_i) + L(aE_{r,s})) = \mathrm{b}(L(R_i + aE_{r,s})) = \mathrm{b}(L(R_i)).$$

But if $\mathrm{b}(L(R_i)) + \mathrm{b}(L(aE_{r,s})) = \mathrm{b}(L(R_i) + L(aE_{r,s})) = \mathrm{b}(L(R_i))$, then $\mathrm{b}(L(aE_{r,s})) = 0$. Then $L(aE_{r,s}) = 0$ and $aE_{r,s} = 0$; which is impossible. Thus $(L(R_i)), L(aE_{r,s}) \notin \mathcal{R}_{SA}(\mathbb{B}_k)$, contradiction from (3.2.1), since T and L strongly preserves $\mathcal{R}_{SA}(\mathbb{B}_k)$. Therefore we have established that $L(R_i) \leq R_i$ for all i. Similarly, $L(C_j) \leq C_j$ for all j. By considering that $E_{i,j}$ is dominated by both R_i and C_j, we have that $L(E_{i,j}) \leq R_i$ and $L(E_{i,j}) \leq C_j$, and hence $L(E_{i,j}) \leq E_{i,j}$. Since $\mathbb{B}_k$ is antinegative, T also maps a cell to a weighted cell and $T(J)$ has all nonzero entries. So, T induces a permutation σ on the set $\{(i,j)|i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$. That is, $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for some nonzero scalars $b_{i,j}$ in $\mathbb{B}_k$. By Lemma 3.2.2, T is a (P, Q, B)-operator. ■

3.3 Linear preservers of $\mathcal{R}_{S1}(\mathbb{B}_k)$

Recall that

$$\mathcal{R}_{S1}(\mathbb{B}_k) = \{(X,Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 | \ \mathrm{b}(X+Y) = 1\}.$$

Example 3.3.1. We show that $\mathcal{R}_{S1}(\mathbb{B}_2)$ is not an empty set.

Consider two matrices X and Y over $\mathbb{B}_2 = \mathcal{P}(\{a,b\})$:

$$X = \begin{bmatrix} \{a\} & \phi \\ \{a\} & \phi \end{bmatrix}$$

and

$$Y = \begin{bmatrix} \{a\} & \{a,b\} \\ \{a\} & \{a,b\} \end{bmatrix}.$$

Then $\mathrm{b}(X+Y) = 1$ and hence $\mathcal{R}_{S1}(\mathbb{B}_2) \neq \phi$.

Theorem 3.3.2. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a surjective linear operator. Then T preserves $\mathcal{R}_{S1}(\mathbb{B}_k)$ if and only if T is a (P,Q)-operator.*

Proof. If T is a surjective linear operator, by Theorem 3.1.17, we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. It is easy to see that the weighted cells $\alpha E_{i,j}$ and $\beta E_{r,s}$ are in the same line if and only if $\mathrm{b}(\alpha E_{i,j} + \beta E_{r,s}) = 1$ if and only if $(\alpha E_{i,j}, \beta E_{r,s}) \in \mathcal{R}_{S1}(\mathbb{B}_k)$. If T preserves $\mathcal{R}_{S1}(\mathbb{B}_k)$, then $(T(\alpha E_{i,j}), T(\beta E_{r,s})) \in \mathcal{R}_{S1}(\mathbb{B}_k)$ for $(\alpha E_{i,j}, \beta E_{r,s}) \in \mathcal{R}_{S1}(\mathbb{B}_k)$. And hence $\mathrm{b}(T(\alpha E_{i,j}) + T(\beta E_{r,s})) = 1$ which implies $T(\alpha E_{i,j})$ and $T(\beta E_{r,s})$ are weighted cells in the same line. Thus lines are mapped to lines by T, and we have that T is a (P,Q,B)-operator by Lemma 3.1.19. Here we have $B = J$ from $T(E_{i,j}) = E_{\sigma(i,j)}$. Thus T be a (P,Q)-operator.

Conversely let T be a (P,Q)-operator and consider any $(X,Y) \in \mathcal{R}_{S1}(\mathbb{B}_k)$. Then $\mathrm{b}(X+Y) = 1$. Thus

$$\mathrm{b}(T(X)+T(Y)) = \mathrm{b}(T(X+Y)) = \mathrm{b}(P(X+Y)Q) = \mathrm{b}(X+Y) = 1.$$

That is, $(T(X), T(Y)) \in \mathcal{R}_{S1}(\mathbb{B}_k)$. Hence T preserves $\mathcal{R}_{S1}(\mathbb{B}_k)$. ■

Theorem 3.3.3. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear operator preserving $\mathcal{R}_{S1}(\mathbb{B}_k)$. Then the following conditions are equivalent:*

(a) T is bijective;

(b) T is injective;

(c) T is surjective;

(d) T strongly preserves $\mathcal{R}_{S1}(\mathbb{B}_k)$;

(e) T is a (P, Q)-operator.

Proof. (a), (b) and (c) are equivalent by Theorem 3.1.17.

(c)⇒(e) If T is a surjective linear operator preserving $\mathcal{R}_{S1}(\mathbb{B}_k)$, then T is a (P, Q)-operator by Theorem 3.3.2.

(e)⇒(d) Assume that T is a (P, Q)-operator. Then $(X, Y) \in \mathcal{R}_{S1}(\mathbb{B}_k)$ if and only if $\mathrm{b}(X + Y) = 1$ if and only if $\mathrm{b}(P(X + Y)Q) = 1$ if and only if $\mathrm{b}(T(X + Y)) = 1$ if and only if $\mathrm{b}(T(X) + T(Y)) = 1$ if and only if $(T(X), T(Y)) \in \mathcal{R}_{S1}(\mathbb{B}_k)$. That is T strongly preserves $\mathcal{R}_{S1}(\mathbb{B}_k)$.

(d)⇒(c) Suppose T strongly preserves $\mathcal{R}_{S1}(\mathbb{B}_k)$. We claim that T is surjective. Assume that T is not surjective. Then, by Theorem 3.1.17, T is not injective and hence T is not injective on the set of all mn cells in $\mathbb{M}_{m,n}(\mathbb{B}_k)$. Therefore there exists two distinct cells $E_{i,j}, E_{h,l} \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ such that $T(E_{i,j}) = T(E_{h,l}) = E_{r,s}$. Then we have 3 cases as follows:

Case 1) Two cells in distinct lines are mapped to a cell. That is $T(E_{i,j}) = E_{r,s} = T(E_{h,l})$ with $i \neq h$, $j \neq l$. Let $X = E_{i,j}$, $Y = E_{h,l}$. Then $\mathrm{b}(X + Y) = 2$, but $\mathrm{b}(T(X) + T(Y)) = \mathrm{b}(E_{r,s}) = 1$; contradicts the fact that T strongly preserves $\mathcal{R}_{S1}(\mathbb{B}_k)$.

Case 2) Two cells in a row are mapped to a cell. That is $T(E_{i,j}) = E_{r,s} = T(E_{i,l})$ with $j \neq l$. Since T strongly preserves $\mathcal{R}_{S1}(\mathbb{B}_k)$, i^{th} row are mapped to r^{th} row or s^{th} column and j^{th} column are mapped to r^{th} row or s^{th} column. Say $T(E_{u,j}) = E_{v,s}$ with $i \neq u$. Let $X = E_{i,j} + E_{i,l}$ and $Y = E_{u,j}$. Then $\mathrm{b}(X + Y) = 2$, but $\mathrm{b}(T(X) + T(Y)) = \mathrm{b}(E_{r,s} + E_{v,s}) = 1$; contradicts the fact that T strongly preserves $\mathcal{R}_{S1}(\mathbb{B}_k)$.

Case 3) Two cells in a column are mapped to a cell. We have a similar contradiction as in the Case 2). Therefore these 3 cases implies that T is injective and hence T is surjective, by Theorem 3.1.17. ■

3.4 Linear preservers of $\mathcal{R}_{SD}(\mathbb{B}_k)$

Recall that

$$\mathcal{R}_{SD}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 | \ \mathrm{b}(X+Y) = |\mathrm{b}(X) - \mathrm{b}(\mathrm{Y})|\}.$$

Example 3.4.1. We show that $\mathcal{R}_{SD}(\mathbb{B}_k)$ is not an empty set.

Consider

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{M}_{2,2}(\mathbb{B}_k).$$

Then $\mathrm{b}(X+Y) = 1$, $\mathrm{b}(X) = 2$, $\mathrm{b}(Y) = 1$. Hence $\mathrm{b}(X+Y) = |\ \mathrm{b}(X) - \mathrm{b}(Y)\ |$. Thus $\mathcal{R}_{SD}(\mathbb{B}_k)$ contains $(X, Y) \in \mathbb{M}_{2,2}(\mathbb{B}_k)^2$ and hence $\mathcal{R}_{SD}(\mathbb{B}_k)$ is not an empty set.

Lemma 3.4.2. *Let σ be a permutation of the set $\{(i,j)|i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$, and $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be defined by $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ with nonzero $b_{i,j} \in \mathbb{B}_k$ for all $1 \le i \le m$ and $1 \le j \le n$, and min$\{m, n\} \ge 3$. If T preserves $\mathcal{R}_{SD}(\mathbb{B}_k)$, then T maps a line to a line.*

Proof. Since the sum of three weighted distinct cells has Boolean rank at most 3, it follows that $\mathrm{b}(T(E_{1,1} + E_{1,2} + E_{2,1})) \le 3$. Now, for $X = E_{1,1} + E_{1,2} + E_{2,1}$ and $Y = E_{2,2}$, we have that $(X, Y) \in \mathcal{R}_{SD}(\mathbb{B}_k)$, and the image of Y under T is a single weighted cell, and hence $\mathrm{b}(T(Y)) = 1$. Now, if $\mathrm{b}(T(X)) = 3$, then $T(X)$ is the sum of three weighted cells that lie in distinct lines. Thus $T(X+Y)$ must have Boolean rank 3 or 4, and hence $(T(X), T(Y)) \notin \mathcal{R}_{SD}(\mathbb{B}_k)$, a contradiction. If $\mathrm{b}(T(X)) = 1$, then $T(X+Y) \ne O$ and $(T(X), T(Y)) \notin \mathcal{R}_{SD}(\mathbb{B}_k)$, a contradiction. Consequently we have that $\mathrm{b}(T(X)) = 2$, and hence $\mathrm{b}(T(X+Y)) = 1$ from $(T(X), T(Y)) \in \mathcal{R}_{SD}(\mathbb{B}_k)$. However it is obvious that if a sum of four cells has the Boolean rank 1, then they lie either in a line or in the intersection of two rows and two columns. The matrix $T(X+Y)$ is a sum of four cells. These cells do not lie in a line since $\mathrm{b}(T(X)) = 2$. Thus $T(X+Y)$ must be the sum of four cells which lie in the intersection of two rows and two columns. Similarly, for any i, j, h and l, $T(E_{i,j} + E_{i,h} + E_{l,j} + E_{l,h})$ must lie in the intersection of two rows and two columns. It follows that any two rows must be mapped into two lines. By the bijectivity of T, if some pair of two rows is mapped into two rows (resp. columns), any pair of two rows is mapped into two rows(resp. columns). Similarly, if some pair of two columns is mapped into two rows(resp. columns), any pair of two columns is mapped into two rows (resp. columns). Now, the image of three rows is contained in three lines, two of which are the image of two rows, thus every row is mapped into a line. Thus T maps a line to a line. ∎

Theorem 3.4.3. *Let $m, n \geq 2$ and T be a surjective linear operator on $\mathbb{M}_{m,n}(\mathbb{B}_k)$. Then T preserves $\mathcal{R}_{SD}(\mathbb{B}_k)$ if and only if T is a (P, Q)-operator.*

Proof. $\Leftarrow$) Assume that T is surjective and a (P, Q)-operator. For any $(X, Y) \in \mathcal{R}_{SD}(\mathbb{B}_k)$, we have

$$\mathrm{b}(X + Y) = |\ \mathrm{b}\ (X) - \mathrm{b}(Y)|.$$

Thus

$$\begin{aligned} \mathrm{b}(T(X) + T(Y)) &= \mathrm{b}(T(X + Y)) = \mathrm{b}(P(X + Y)Q) = \mathrm{b}(X + Y) \\ &= |\ \mathrm{b}(X) - \mathrm{b}(Y)| = |\ \mathrm{b}(PXQ) - \mathrm{b}(PYQ)| = |\ \mathrm{b}(T(X)) - \mathrm{b}(T(Y))|. \end{aligned}$$

Hence $((T(X), T(Y)) \in \mathcal{R}_{SD}(\mathbb{B}_k)$. Therefore T preserves $\mathcal{R}_{SD}(\mathbb{B}_k)$.

$\Rightarrow$) Assume that T preserves $\mathcal{R}_{SD}(\mathbb{B}_k)$. Since T is a surjective linear operator, there exists permutation σ on $\{(i, j)|i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$ such that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$ by Theorem 3.1.17. Hence T maps any line to a line by Lemma 3.4.2. Therefore T is a (P, Q)-operator by Lemma 3.1.19 since all the entries of B are 1. ∎

3.5 Linear preservers of $\mathcal{R}_{MM}(\mathbb{B}_k)$

Recall that

$$\mathcal{R}_{MM}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 |\ \mathrm{b}(XY) = \min\{\mathrm{b}(X), \mathrm{b(Y)}\}\}.$$

Example 3.5.1. We show that $\mathcal{R}_{MM}(\mathbb{B}_k)$ is not an empty set.

Consider

$$X = Y = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in \mathbb{M}_{2,2}(\mathbb{B}_k).$$

Then $\mathrm{b}(XY) = 1$, $\mathrm{b}(X) = 1$ and $\mathrm{b}(Y) = 1$. Hence $\mathrm{b}(XY) = \min(\mathrm{b}(X),\ \mathrm{b}(Y))$. Thus $\mathcal{R}_{MM}$ $(\mathbb{B}_k)$ contains $(X, Y) \in \mathbb{M}_{2,2}(\mathbb{B}_k)^2$ and hence $\mathcal{R}_{MM}(\mathbb{B}_k)$ is not an empty set.

Theorem 3.5.2. *Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$be a linear operator. Then T is surjective and preserves $\mathcal{R}_{MM}(\mathbb{B}_k)$ if and only if there exists a permutation matrix P such that $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$.*

Proof. $\Leftarrow$) Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$be defined by $T(X) = PXP^T$ and $(X, Y) \in \mathcal{R}_{MM}(\mathbb{B}_k)$. Then $\mathrm{b}(XY) = \min\{\mathrm{b}(X),\ \mathrm{b}(Y)\}$ and hence

$$\mathrm{b}(T(X)T(Y)) = \mathrm{b}(PXP^T PYP^T) = \mathrm{b}(PXYP^T) = \mathrm{b}(XY) = min\{\mathrm{b}(X),\mathrm{b}(Y)\}.$$

Thus $(T(X), T(Y)) \in \mathcal{R}_{MM}(\mathbb{B}_k)$. That is T preserves $\mathcal{R}_{MM}(\mathbb{B}_k)$.

$\Rightarrow$) Assume that T is surjective and preserves $\mathcal{R}_{MM}(\mathbb{B}_k)$. By Theorem 3.1.17, we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for a permutation σ on $\{(i, j)|1 \le i, j \le n\}$. Consider $(E_{i,j}, E_{j,h}) \in \mathcal{R}_{MM}(\mathbb{B}_k)$ for all h. Then $\mathrm{b}(T(E_{i,j})T(E_{j,h})) = \min\{\mathrm{b}(T(E_{i,j}),\ \mathrm{b}(T(_{j,h}))\} = 1$, but $T(E_{i,j})T(E_{j,h}) = E_{\sigma(i,j)}E_{\sigma(j,h)}$. It follows that $E_{\sigma(j,h)}$ is in the same row as $E_{\sigma(j,1)}$ for any $h = 1, 2, \ldots, n$. That is, T maps rows to rows; similarly T maps columns to columns. By Lemma 3.1.19 with $b_{i,j} = 1$, it follows that $T(X) = PXQ$ for some permutation matrices P and Q. Let us show that $Q = P^T$. Indeed $T(E_{i,j}) = E_{\pi(i),\tau(j)}$, where π is the permutation corresponding to P and τ is the permutation corresponding to Q^T. But $(E_{1,i}, E_{i,1}) \in \mathcal{R}_{MM}(\mathbb{B}_k)$; thus $(E_{\pi(1),\tau(i)},\ E_{\pi(i),\tau(1)}) \in \mathcal{R}_{MM}(\mathbb{B}_k)$ and hence $\pi = \tau$. Therefore $Q = P^T$. ■

3.6 Linear preservers of $\mathcal{R}_{M0}(\mathbb{B}_k)$

Recall that

$$\mathcal{R}_{M0}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 |\ \mathrm{b}(XY) = 0\}.$$

Example 3.6.1. We show that $\mathcal{R}_{M0}(\mathbb{B}_k)$ is not an empty set.

Consider $X = E_{1,2}$ and $Y = E_{1,1}$. Then $(X, Y) \in \mathcal{R}_{M0}(\mathbb{B}_k)$. Thus $\mathcal{R}_{M0}(\mathbb{B}_k)$ is not an empty set.

Theorem 3.6.2. *Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$ be a nonsingular ($T(X) = O \Rightarrow X = O$) linear operator. Assume that $T(J) \geq P_J$, a permutation matrix. Then T preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$ if and only if there exists a permutation matrix P such that $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$.*

Proof. $\Leftarrow$) Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$be defined by $T(X) = PXP^T$ and $(X, Y) \in \mathcal{R}_{M0}(\mathbb{B}_k)$. Then $\mathrm{b}(XY) = 0$ and hence

$$\mathrm{b}(T(X)T(Y)) = \mathrm{b}(PXP^T PYP^T) = \mathrm{b}(PXYP^T) = \mathrm{b}(XY) = 0.$$

Thus $(T(X), T(Y)) \in \mathcal{R}_{M0}(\mathbb{B}_k)$. That is, T preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$.

$\Rightarrow$) Assume that T preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$. Since $T(J) \geq P_J$, a permutation matrix, there are n different cells whose images have nonzero entries in every column. Assume that these cells can be chosen such that their nonzero entries are in fewer than n columns, say $X = E_1 + E_2 + \ldots + E_n$ is the sum of n such cells and X has no nonzero entry in column h. Then $(X, R_h) \in \mathcal{R}_{M0}(\mathbb{B}_k)$ and hence $(T(X), T(R_h)) \in \mathcal{R}_{M0}(\mathbb{B}_k)$, since T preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$. But $T(X)$ has nonzero entry in every column, which implies $T(X)T(R_h) \neq O$, a contradiction. Thus, if T maps a column into two columns, then we have a contradiction from above. Furthermore, if T maps two columns into one column, there must be a column whose image is at least two column from $T(J) \geq P_J$ for some permutation matrix P_J. Thus in this case, we also have a contradiction as above. Consequently T maps a column into a column and all columns into all columns respectively. Hence T induces a permutation on the set of columns. Similarly T induces a permutation on the set of rows, i.e., $T(X) = P(X \circ B)Q$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$ and some permutation matrices P and Q. Let us show that $Q = P^T$. Indeed we have that $T(E_{i,j}) = b_{i,j}E_{\pi(i),\tau(j)}$. If $Q \neq P^T$, then $\pi \neq \tau$. Thus, for some i, we have $\pi(i) \neq \tau(i)$ and hence for some $j \neq i$, we have $\pi(j) = \tau(i)$. Here $(E_{i,i}, E_{j,i}) \in \mathcal{R}_{M0}(\mathbb{B}_k)$ but

$$T(E_{i,i})T(E_{j,i}) = b_{i,i}b_{j,i}E_{\pi(i),\tau(i)}E_{\pi(j),\tau(i)} = b_{i,i}b_{j,i}E_{\pi(i),\tau(i)} \neq O,$$

i.e., $(T(E_{i,i}), T(E_{j,i})) \notin \mathcal{R}_{M0}(\mathbb{B}_k)$; a contradiction. Thus $\pi = \tau$ and hence $T(X) = P(X \circ B)P^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$. Since T is nonsingular, all entries of B are nonzero and not zero divisors. But every elements α in $\mathbb{B}_k$ is a zero divisor if $\alpha \neq 1$. Thus $b_{i,j} = 1$. Hence $B = J$. Consequently $T(X) = PXP^T$. ■

Corollary 3.6.3. *Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$ be a surjective linear operator. Then T preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$ if and only if there exists a permutation matrix P such that $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$.*

Proof. If T be a surjective linear operator, then T is a bijective by Theorem 3.1.17. Thus T is a nonsingular. Hence, T preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$ if and only if $T(X) = PXP^T$ by Theorem 3.6.2. ■

Corollary 3.6.4. *Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$ be a linear operator. Then T strongly preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$ if and only if there exists a permutation matrix P such that $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$.*

Proof. $\Leftarrow$) It is easy to see that operator of the form $T(X) = PXP^T$ strongly preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$.

$\Rightarrow$) Suppose that T strongly preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$. We claim that (1) $T(J) \geq P_J$, some permutation matrix, i.e., $T(J)$ has a nonzero element in each row and each column and (2) T is a nonsingular operator. Then we apply Theorem 3.6.2.

Claim (1): $T(J) \geq P_J$. Assume, on the contrary, that $T(J)$ has a zero column (For the case of a zero row, the proof is quite similar). Up to a multiplication with permutation matrices, we may assume that there are nonzero elements in columns $1, 2, \ldots t$ of $T(J)$ and all elements in the column $(t+1), \ldots, n$ are zero. Then there exist column matrices $C_{j_1}, C_{j_2}, \ldots, C_{j_s}$ whose images dominate all nonzero entries in columns 1 through t. Let $l \neq j_h$ for all h, $1 \leq h \leq s$. Thus $(C_{j_1}, C_{j_2}, \ldots, C_{j_s})R_l = O$. Since T strongly preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$, it follows that $T(C_{j_1}, C_{j_2}, \ldots, C_{j_s})T(R_l) = O$. Then all the entries in rows 1 through t of $T(R_l)$ are zero, since in each of the first t columns of $T(C_{j_1}, C_{j_2}, \ldots, C_{j_s})$ there is a nonzero element. Therefore $T(E_{l,l})$ has nonzero entries only in rows $t+1, \ldots, n$ and only in columns $1, 2, \ldots t$. Thus $T(E_{l,l})^2 = O$, i.e., $(T(E_{l,l}), T(E_{l,l})) \in \mathcal{R}_{M0}(\mathbb{B}_k)$. This is a contradiction since T strongly preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$ and $(E_{l,l}, E_{l,l}) \notin \mathcal{R}_{M0}(\mathbb{B}_k)$. Thus $T(J)$ has neither a zero row nor a zero column, that is $T(J) \geq P_J$.

Claim (2): T is a nonsingular operator. Assume that there exists $O \neq X$ such that $T(X) = O$. Then $(T(X), T(I)) \in \mathcal{R}_{M0}(\mathbb{B}_k)$. But $(X, I) \notin \mathcal{R}_{M0}(\mathbb{B}_k)$. This contradicts the fact that T strongly preserves $\mathcal{R}_{M0}(\mathbb{B}_k)$. Thus T is a nonsingular.

Hence Theorem 3.6.2 is applicable, since claims (1) and (2) satisfy the condition in Theorem 3.6.2. Consequently we obtain $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$ and for some permutation matrix P. ■

3.7 Linear preservers of $\mathcal{R}_{M1}(\mathbb{B}_k)$

Recall that

$$\mathcal{R}_{M1}(\mathbb{B}_k) = \{(X,Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 |\ \mathrm{b}(XY) = 1\}.$$

Example 3.7.1. We show that $\mathcal{R}_{M1}(\mathbb{B}_k)$ is not an empty set.

Consider $X = E_{1,2} + E_{2,1}$ and $Y = E_{2,2}$. Then $XY = E_{1,2}$ has Boolean rank 1 and hence $(X,Y) \in \mathcal{R}_{M1}(\mathbb{B}_k)$. Thus $\mathcal{R}_{M1}(\mathbb{B}_k)$ is not an empty set.

Lemma 3.7.2. *Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$ be a linear operator defined by $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for some permutation σ of $\{(i,j)|1 \le i,j \le n\}$ and nonzero scalars $b_{i,j} \in \mathbb{B}_k$. Then T strongly preserves $\mathcal{R}_{M1}(\mathbb{B}_k)$ if and only if there exists a permutation matrix P such that $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$.*

Proof. $\Leftarrow$) Clearly linear operators of the form $T(X) = PXP^T$ strongly preserves $\mathcal{R}_{M1}(\mathbb{B}_k)$.

$\Rightarrow$) Assume that T strongly preserves $\mathcal{R}_{M1}(\mathbb{B}_k)$. Consider $(E_{i,i}, E_{i,h}) \in \mathcal{R}_{M1}(\mathbb{B}_k)$ for all $h = 1, \ldots, n$. If $T(E_{i,i}) = b_{i,i}E_{r,s}$ for some r and s, then $T(E_{i,h}) = b_{i,h}E_{s,\tau(h)}$, where τ is some permutation, since $(T(E_{i,i}), T(E_{i,h})) \in \mathcal{R}_{M1}(\mathbb{B}_k)$. That is, $T(R_i) \le R_s$. Thus T induces a permutation on the rows. Similarly T induces a permutation on the columns. Thus, for some permutations π and τ, $T(E_{i,j}) = b_{i,j}E_{\pi(i),\tau(j)}$. Now $\mathrm{b}(T(E_{i,i})T(E_{i,j}))$ must be 1 and hence $\pi(i) = \tau(i)$. Therefore $\pi = \tau$ and we have that $T(X) = P(X \circ B)P^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$, where P is the permutation corresponding to π. Now, if $B \neq J$, then $b_{p,q} \neq 1$ for some (p,q). But then, $(E_{i,i} + E_{i,q} + E_{p,i} + b_{p,q}E_{p,q}, I) \notin \mathcal{R}_{M1}(\mathbb{B}_k)$, while $(E_{i,i} + E_{i,q} + E_{p,i} + E_{p,q}, I) \in \mathcal{R}_{M1}(\mathbb{B}_k)$. However $T(E_{i,i} + E_{i,q} + E_{p,i} + b_{p,q}E_{p,q}) = T(E_{i,i} + E_{i,q} + E_{p,i} + E_{p,q})$, which contradicts the fact that T strongly preserves $\mathcal{R}_{M1}(\mathbb{B}_k)$. Thus $B = J$ and hence $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$. ■

Theorem 3.7.3. *Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$ be a surjective linear operator. Then T strongly preserves $\mathcal{R}_{M1}(\mathbb{B}_k)$ if and only if there exists a permutation matrix P such that $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$.*

Proof. $\Rightarrow$) Assume that T strongly preserves $\mathcal{R}_{M1}(\mathbb{B}_k)$. Since T is surjective, we have $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i and j with $1 \leq i, j \leq n$ by Theorem 3.1.17. By Lemma 3.7.2 with $b_{i,j} = 1$, we obtain the result.

$\Leftarrow$) If $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$ and some permutation matrix P, then

$$T(XY) = P(XY)P^T = PXP^T PYP^T = T(X)T(Y).$$

Thus

$$\mathrm{b}(T(X)T(Y)) = \mathrm{b}(T(XY)) = \mathrm{b}(PXYP^T) = \mathrm{b}(XY).$$

Hence $(X, Y) \in \mathcal{R}_{M1}(\mathbb{B}_k)$ if and only if $(T(X), T(Y)) \in \mathcal{R}_{M1}(\mathbb{B}_k)$. Therefore T strongly preserves $\mathcal{R}_{M1}(\mathbb{B}_k)$. ■

3.8 Linear preservers of $\mathcal{R}_{MA}(\mathbb{B}_k)$

Recall that

$$\mathcal{R}_{MA}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 |\ \mathrm{b}(XY) = \mathrm{b}(X) + \mathrm{b(Y) - n}\}.$$

Example 3.8.1. We show that $\mathcal{R}_{MA}(\mathbb{B}_k)$ is not an empty set.

Consider $X = I_n$ and $Y = E_{1,1}$. Then $\mathrm{b}(XY) = \mathrm{b}(E_{1,1}) = 1$ and hence $\mathrm{b}(XY) = \mathrm{b}(X) + \mathrm{b}(Y) - n$. That is $(X, Y) \in \mathcal{R}_{MA}(\mathbb{B}_k)$. Thus $\mathcal{R}_{MA}(\mathbb{B}_k)$ is not an empty set.

Theorem 3.8.2. *Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$, $n > 4$, be a surjective linear operator. Then T preserves $\mathcal{R}_{MA}(\mathbb{B}_k)$ if and only if there exists a permutation matrix P such that $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$.*

Proof. $\Leftarrow$) If $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$ and some permutation matrix P, then

$$T(XY) = P(XY)P^T = PXP^TPYP^T = T(X)T(Y)$$

for all $X,\ Y \in \mathbb{M}_n(\mathbb{B}_k)$. Thus for all $(X, Y) \in \mathcal{R}_{MA}(\mathbb{B}_k)$,

$$\begin{aligned} \mathrm{b}(T(X)T(Y)) &= \mathrm{b}(T(XY)) = \mathrm{b}(PXYP^T) = \mathrm{b}(XY) = \mathrm{b}(X) + \mathrm{b}(Y) - n \\ &= \mathrm{b}(PXP^T) + \mathrm{b}(PYP^T) - n = \mathrm{b}(T(X)) + \mathrm{b}(T(Y)) - n \end{aligned}.$$

That is, $(T(X), T(Y)) \in \mathcal{R}_{MA}(\mathbb{B}_k)$. Hence T preserves $\mathcal{R}_{MA}(\mathbb{B}_k)$.

$\Rightarrow$) Assume that T preserves $\mathcal{R}_{MA}(\mathbb{B}_k)$. Since T is surjective, by Theorem 3.1.17 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for some permutation σ. If $\mathrm{b}(A) = n$, then $(E_{i,j}, A) \in \mathcal{R}_{MA}(\mathbb{B}_k)$. Since $\mathrm{b}(T(E_{i,j})) = 1$ by Theorem 3.1.17 and T preserves $\mathcal{R}_{MA}(\mathbb{B}_k)$, it follows that $\mathrm{b}(T(A)) = n$. Therefore T maps the set of matrices with Boolean rank n to itself. If the preimage of a row is not dominated by any line, then there exist cells $E_{r,s}$ and $E_{p,q}$ such that $T(E_{r,s} + E_{p,q}) \le E_{i,h} + E_{i,l}$ with $r \neq p, s \neq q$. By extending $E_{r,s} + E_{p,q}$ to a permutation matrix by adding $n - 2$ cells, we find a matrix which is the image of a permutation matrix but is dominated by $n - 1$ lines; a contradiction since T maps the set of matrices with Boolean rank n to itself. Thus the preimage of every row is a row or column and, similarly, the preimage of every column is a row or a column. Hence T maps any line to a line. By Lemma 3.1.19, we have that T is a (P, Q, B)-operator with $B = J$. That is, T is a (P, Q)-operator. Since $(E_{1,1}, E_{2,1} + E_{3,2} + \ldots + E_{n,n-1}) \in \mathcal{R}_{MA}(\mathbb{B}_k)$ while $(E_{1,1}, E_{1,2} + E_{2,3} + \ldots + E_{n-1,n}) \notin \mathcal{R}_{MA}(\mathbb{B}_k)$, we have that the transpose operator does not preserve $\mathcal{R}_{MA}(\mathbb{B}_k)$, thus there exist permutation matrices P and Q such that $T(X) = PXQ$. Without loss of generality, we may assume that $P = I$. If $Q \neq I$, we assume that Q corresponds to the permutation π and $\pi(1) \neq 1$. Without loss of generality, $T(E_{1,1}) = E_{1,2}$. Then $(E_{1,1}, E_{2,2} + E_{3,3} + \ldots + E_{n,n}) \in \mathcal{R}_{MA}(\mathbb{B}_k)$, while $(T(E_{1,1}), T(E_{2,2} + E_{3,3} + \ldots + E_{n,n})) \notin \mathcal{R}_{MA}(\mathbb{B}_k)$ since $(E_{1,2})(E_{2,\pi(2)} + E_{3,\pi(3)} + \ldots + E_{n,\pi(n)}) = E_{1,2}E_{2,\pi(2)} \neq O$. This contradiction gives that $Q = P^T$ and hence $T(X) = PXP^T$. ■

3.9 Linear preservers of $\mathcal{R}_{M3}(\mathbb{B}_k)$

Recall that

$$\mathcal{R}_{M3}(\mathbb{B}_k) = \{(X, Y, Z) \in \mathbb{M}_n(\mathbb{B}_k)^3 | b(XYZ) + b(Y) = b(XY) + b(YZ)\}.$$

Example 3.9.1. We show that $\mathcal{R}_{M3}(\mathbb{B}_k)$ is not an empty set.

Consider $X = E_{1,1}$, $Y = E_{1,2}$ and $Z = E_{2,3}$. Then $\mathrm{b}(XYZ) = \mathrm{b}(E_{1,3}) = 1$, $\mathrm{b}(XY) = \mathrm{b}(E_{1,2}) = 1$ and $\mathrm{b}(YZ) = \mathrm{b}(E_{E1,3}) = 1$. Thus $(X, Y, Z) \in \mathcal{R}_{M3}(\mathbb{B}_k)$ and hence $\mathcal{R}_{M3}(\mathbb{B}_k)$ is not an empty set.

Lemma 3.9.2. *Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$, $n > 4$, be a surjective linear operator. If T preserves $\mathcal{R}_{M3}(\mathbb{B}_k)$, then there exists a permutation matrix P such that $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$.*

Proof. By Theorem 3.1.17, we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for a certain permutation σ on $\{(i, j)|1 \leq i, j \leq n\}$. It can be easily proved that $(E_{i,j}, E_{j,h}, E_{h,l}) \in \mathcal{R}_{M3}(\mathbb{B}_k)$ for all l and arbitrary fixed i, j and h. Thus

$$\begin{aligned} &\mathrm{b}(T(E_{i,j})T(E_{j,h})) + \mathrm{b}(T(E_{j,h})T(E_{h,l})) \\ &= \mathrm{b}(T(E_{i,j})T(E_{j,h})T(E_{h,l})) + \mathrm{b}(T(E_{j,h})). \end{aligned} \tag{3.9.1}$$

Now, by Theorem 3.1.17, we may assume that $T(E_{i,j}) = E_{p,q}$, $T(E_{j,h}) = E_{r,s}$, $T(E_{h,l}) = E_{u,v}$ for subscripts p, q, r, s, u, and v. Since $\mathrm{b}(E_{r,s}) = 1 \neq 0$, it follows from equality (3.9.1) that either $q = r$ or $s = u$ or both. If, for all $l = 1, \ldots, n$, both equalities hold, then for fixed i, j, and h, all matrices $T(E_{h,l})$, $l = 1, \ldots, n$, have their nonzero elements lying in one row. Thus T maps rows to rows. Similarly, it is easy to see that T maps columns to columns. Assume now that there exists an index l such that only one of the above equalities holds for the triple $(E_{i,j}, E_{j,h}, E_{h,l})$. Without loss of generality, assume that $s = u$ and $q \neq r$. Thus for arbitrary m, $1 \leq m \leq n$, one has that $(E_{i,j}, E_{j,h}, E_{h,m}) \in \mathcal{R}_{M3}(\mathbb{B}_k)$. Let $T(E_{h,m}) = E_{w,z}$ for certain w and z depending on h and m. In the above notation, $(E_{p,q}, E_{r,s}, E_{w,z}) \in \mathcal{R}_{M3}(\mathbb{B}_k)$ since T preserves $\mathcal{R}_{M3}(\mathbb{B}_k)$. Since $q \neq r$, it follows that $w = s$ for any w. Thus, in this case, we also obtain that rows are transformed to rows. By the same arguments with the first matrix, it is easy to see that columns are transformed to columns. In the other case ($s \neq u$ and $q = r$), one obtains similarly that rows are transformed to rows and columns to columns. By Lemma 3.1.19, it follows that there exist permutation matrices P and Q such that $T(X) = P(X \circ B)Q$ with $B = J$. (I.e., $T(X) = PXQ$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$.) In order to show that $Q = P^T$ it suffices to note that $(E_{i,j}, E_{j,j}, E_{j,i}) \in \mathcal{R}_{M3}(\mathbb{B}_k)$. Let π be the permutation corresponding to P and τ be the permutation corresponding to Q^T. Therefore

$$(T(E_{i,j}), T(E_{j,j}), T(E_{j,i})) = (PE_{i,j}Q, PE_{j,j}Q, PE_{j,i}Q)$$

$$= (E_{\pi(i),\tau(j)}), E_{\pi(j),\tau(j)}, E_{\pi(j),\tau(i)}) \in \mathcal{R}_{M3}(\mathbb{B}_k).$$

Thus $\pi = \tau$ and $Q = P^T$. ∎

Theorem 3.9.3. *Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$, $n > 4$, be a surjective linear operator. Then T preserves $\mathcal{R}_{M3}(\mathbb{B}_k)$ if and only if $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$, where P is a permutation matrix.*

Proof. $\Leftarrow$) If $(X, Y, Z) \in \mathcal{R}_{M3}(\mathbb{B}_k)$, then $\mathrm{b}(XYZ) + \mathrm{b}(Y) = \mathrm{b}(XY) + \mathrm{b}(YZ)$. Thus

$$b(T(X)T(Y)T(Z)) + b(T(Y)) = b(PXP^TPYP^TPZP^T) + b(PYP^T)$$

$$= b(PXYZP^T) + b(PYP^T) = b(XYZ) + b(Y).$$

Similarly

$$b(T(X)T(Y)) + b(T(Y)T(Z)) = b(PXP^TPYP^T) + b(PYP^TPZP^T)$$

$$= b(PXYP^T) + b(PYZP^T) = b(XY) + b(YZ).$$

Hence

$$\mathrm{b}(T(X)T(Y)T(Z)) + \mathrm{b}(T(Y)) = \mathrm{b}(T(X)T(Y)) + \mathrm{b}(T(Y)T(Z)).$$

That is, $(T(X), T(Y), T(Z)) \in \mathcal{R}_{M3}(\mathbb{B}_k)$. Therefore T preserves $\mathcal{R}_{M3}(\mathbb{B}_k)$.

$\Rightarrow$) Assume that T preserves $\mathcal{R}_{M3}(\mathbb{B}_k)$. Then, by Lemma 3.9.2, T has the form $T(X) = PXP^T$ for some permutation matrix P. ■

As a concluding remark, we have characterized the linear operators that preserve the extreme sets of matrix pairs over general Boolean algebra which come from certain Boolean rank inequalities.

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<국문초록>

부울 行列들의 定規性과
階數 不等式의 極値集合 保存者

本 論文에서는 1887年 以後 100년이 넘도록 研究되고 있는 線形保存者 問題의 一環으로 두 가지 主題를 研究하였다.

한 가지 主題는 二項 부울 代數 上의 行列의 定規性을 保存하는 線形演算者를 糾明하는 研究이다. 한 行列 M이 定規行列이라는 定義는 적당한 行列 X가 存在하여 $MXM=M$을 滿足하는 것이다. 本 研究에서는 定規行列의 性質들을 調査 分析하여 그 特性들을 밝혔다. 그리고 定規行列 X를 線形演算者로 보내어 다시 定規行列이 되게 할 境遇에 그 線形演算者의 形態는 적당한 可逆行列 U와 V가 存在하여 $T(X)=UXV$ 또는 $T(X)=UX^TV$ 形態로 糾明됨을 밝혔고, 이를 부울 行列의 特性을 活用하여 證明하였다.

다른 하나의 主題는 一般的인 부울 代數 上의 行列 雙들을 保存하는 線形演算者를 糾明하는 研究이다. 一般的인 부울 代數 上의 集合에서 두 行列의 合과 곱에 대하여 부울 階數의 값에 관한 不等式을 分析 調査하였다. 그 結果 그들 사이에 成立하는 階數 不等式들을 調査하여 그 不等式들이 等式이 되는 경우의 行列 짝들로 構成되는 8가지 極値 集合들을 構成하였다.

$$\mathcal{R}_{SA}(\mathbb{B}_k)=\{(X,Y)\in\mathbb{M}_{m,n}(\mathbb{B}_k)^2 \mid \mathrm{b}(X+Y)=\mathrm{b}(X)+\mathrm{b}(Y)\},$$

$$\mathcal{R}_{S1}(\mathbb{B}_k)=\{(X,Y)\in\mathbb{M}_{m,n}(\mathbb{B}_k)^2 \mid \mathrm{b}(X+Y)=1\},$$

$$\mathcal{R}_{SD}(\mathbb{B}_k)=\{(X,Y)\in\mathbb{M}_{m,n}(\mathbb{B}_k)^2 \mid \mathrm{b}(X+Y)=|\mathrm{b}(X)-\mathrm{b}(Y)|\},$$

$$\mathcal{R}_{MM}(\mathbb{B}_k)=\{(X,Y)\in\mathbb{M}_{n}(\mathbb{B}_k)^2 \mid \mathrm{b}(XY)=\min\{\mathrm{b}(X),\mathrm{b}(Y)\}\},$$

$\mathcal{R}_{M0}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 | \ \mathrm{b}(XY) = 0\}$,

$\mathcal{R}_{M1}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 | \ \mathrm{b}(XY) = 1\}$,

$\mathcal{R}_{MA}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 | \ \mathrm{b}(XY) = \mathrm{b}(X) + \mathrm{b}(Y) - n\}$,

$\mathcal{R}_{M3}(\mathbb{B}_k) = \{(X, Y, Z) \in \mathbb{M}_n(\mathbb{B}_k)^3 | \ \mathrm{b}(XYZ)+\mathrm{b}(Y) = \mathrm{b}(XY) + \mathrm{b}(YZ)\}$.

以上의 行列 짝들의 集合을 線形演算者로 보내어 그 集合의 性質들을 保存하는 線形演算者를 硏究하여 그 形態를 糾明하였다. 곧, 이러한 行列 짝들의 集合을 保存하는 線形演算者의 形態는 $T(X) = PXQ$, $T(X) = PXP^T$ 또는 $T(X) = PX^TQ$와 같은 形態로 나타남을 보이고, 이들을 證明하였다. 그리고 이 線形演算者와 同値가 되는 條件들을 찾고, 이 條件들이 同値가 됨을 證明하였다.

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