J. of Basic Sciences, Cheju Nat'l Univ.
15(1), 91~104, 2002

기초과학연구 제주대학교
15(1), 91~104, 2002

Analytic types of the surface singularities defined by some weighted homogeneous polynomials [1]

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abstract

We classify analytically surface singularities defined by some weighted homogeneous polynomials which are topologically equivalent to the type $z_0^n + z_1^k + z_2^l = 0$.

1 Introduction

The aim in this paper is to provide the analytic classification of isolated surface singularities defined by some weighted homogeneous polynomials, which are topologically equivalent to the type $z_0^n + z_1^k + z_2^l = 0$.

Let ${}_{n+1}\mathcal{O}$ or $\mathbb{C}\{z_1, \ldots, z_n\}$ be the ring of convergent power series at the origin in $\mathbb{C}^{n+1}$. It is a natural question to ask for necessary and sufficient conditions to analytically equivalent between two germs of complex hypersurface with isolated singular points. It is known by Theorem 2.5([Ma-Ya]) that two germs of complex analytic hypersurace singularities defined by f and g with isolated singular points at the origin in $\mathbb{C}^{n+1}$ are analytically equivalent if and only if their moduli algebra ${}_{n+1}\mathcal{O}/(f, \Delta f)$ and ${}_{n+1}\mathcal{O}/(g, \Delta g)$

[1] 1995. Mathematics Subject Classification, Primary 32S15, 14E15.

keywords: weighted homogeneous polynomial, surface singularities.

are isomorphic as a $\mathbb{C}$-algebra where $(f, \Delta f) = (f, \partial f/\partial z_0, \ldots, \partial f/\partial z_n)$ is an ideal in ${}_{n+1}\mathcal{O}$ generated by $f, \partial f/\partial z_0, \ldots, \partial f/\partial z_n$ and so on. In spite of the above theorem, it is still difficult to find a concrete criterion for analytic equivalence between two surfaces with isolated singular points at the origin.

By Theorem 2.6([Xu-Ya]), if f and g are surface singularities at the origin defined by weighted homogeneous polynomials with the same weights, then f and g are topologically equivalent. It is well known by Theorem 2.7([Xu-Ya]) that surface singularities defined by weighted homogeneous polynomials can be topologically classified by seven classes.

But, for the analytic case, two weighted homogeneous polynomials with isolated singularities at the origin may not be analytically equivalent even though they have the same weights. So, we will find the necessary and sufficient conditions for a type of surface singularities, which are topologically equivalent to the type $z_0^n + z_1^k + z_2^l = 0$, to be analytically equivalent.

2 Definitions and Known Preliminaries

Let ${}_{n+1}\mathcal{O}$ be the ring of germs of holomorphic functions at the origin in $\mathbb{C}^{n+1}$ and $f(z_0, \ldots, z_n)$ and $g(z_0, \ldots, z_n)$ are in ${}_{n+1}\mathcal{O}$ which have isolated singular points at the origin in $\mathbb{C}^{n+1}$

Definition 2.1 f and g are said to have the same analytic type of singularity at the origin if there is a germ at the origin of biholomorphisms $\psi : (U_1, O) \to (U_2, O)$ such that $\psi(V) = W$ and $\psi(O) = O$ where U_1 and U_2 are open subsets in $\mathbb{C}^{n+1}$, that is, $f \circ \psi = ug$ where u is a unit in ${}_{n+1}\mathcal{O}$. Then we write $f \approx g$. If not, we write $f \not\approx g$.

Definition 2.2 Two germs of holomorphic function $f, g : (\mathbb{C}^{n+1}, O) \to (\mathbb{C}, O)$ are called right equivalent if there exists a biholomorphic map $\varphi : (\mathbb{C}^{n+1}, O) \to (\mathbb{C}^{n+1}, O)$ such that $f = g \circ \varphi$.

Definition 2.3 $f(z_0, \ldots, z_n)$ is called a weighted homogeneous polynomial with weights $(\omega_0, \ldots, \omega_n)$, where $(\omega_0, \ldots, \omega_n)$ are fixed positive rational numbers, if it can be expressed as a linear combination of monomials $z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n}$ for which $i_0/\omega_0 + \cdots + i_n/\omega_n = 1$.

Theorem 2.4 (Sc) *If (V, O) and (W, O) be germs of hypersurface singularities isolated singular points at the origin in $\mathbb{C}^{n+1}$ defined by weighted homogeneous polynomials f and g respectively, then (V, O) and (W, O) are analytically equivalent if and only if f and g are right equivalent. That is, there exists a biholomorphism $\varphi : (\mathbb{C}^{n+1}, O) \to (\mathbb{C}^{n+1}, O)$ such that $f \circ \varphi = g$.*

Theorem 2.5 (Ma-Ya) *Suppose that $V = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid f(z_0, \ldots , z_n) = 0\}$ and $W = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid g(z_0, \ldots, z_n) = 0\}$ have the isolated singular point at the origin. Then the following conditions are equivalent.*

(i) $f \approx g$

(ii) *$A(f)$ is isomorphic to $A(g)$ as a $\mathbb{C}$-algebra where $A(f) = {}_{n+1}\mathcal{O}/(f, \Delta(f))$, $A(g) = {}_{n+1}\mathcal{O}/(g, \Delta(g))$ w $(f, \Delta(f))$ is the ideal in ${}_{n+1}\mathcal{O}$ generated by $f, \partial f/\partial z_0, \ldots, \partial f/\partial z_n$.*

(iii) *$B(f)$ is isomorphic to $B(g)$ as a $\mathbb{C}$-algebra where $B(f) = {}_{n+1}\mathcal{O}/(f, \mathbf{m}\Delta(f))$, $B(g) = {}_{n+1}\mathcal{O}/(g, \mathbf{m}\Delta(g))$ where $(f, \mathbf{m}\Delta(f))$ is the ideal in ${}_{n+1}\mathcal{O}$ generated by f a $z_i \partial f/\partial z_j$ for all $i, j = 0, 1, \ldots, n$.*

Theorem 2.6 (Xu-Ya) *Suppose that $f(z_0, z_1, z_2)$ and $g(z_0, z_1, z_2)$ are weig -hted homogeneous polynomials with the same weights $(\omega_0, \omega_1, \omega_2)$. If f and g have isolated singularities at the origin in $\mathbb{C}^3$, then f is topologically equivalent to g.*

Theorem 2.7 (Xu-Ya) *Let $(V, 0)$ and $(W, 0)$ be two isolated quasihomogeneous surface singularities having the same topological type. Then $(V, 0)$ is connected to $(W, 0)$ by a family of constant topological type.*

3 Main Results

We will find the concrete criterion for analytically equivalent of two surface singularities defined by some weighted homogeneous polynomials with isolated singular points at the origin in $\mathbb{C}^3$, which are topologically equivalent to the type $z_0^n + z_1^k + z_2^l = 0$, as follows:

$$
\begin{aligned}
T_0 &= z_0^n + z_1^k + z_2^l \\
T_1 &= z_0^n + z_1^k + z_2^l + \sum_{\substack{\alpha_1,\beta_1 \\ \alpha_1 \le n-2}} A^1_{\alpha_1\beta_1} z_0^{\alpha_1} z_1^{\beta_1} + \sum_{\substack{\gamma_1,\delta_1 \\ \gamma_1 \le n-2}} B^1_{\gamma_1\delta_1} z_0^{\gamma_1} z_2^{\delta_1} \\
T_2 &= z_0^n + z_1^k + z_2^l + \sum_{\substack{\alpha_2,\beta_2 \\ \alpha_2 \le n-2}} A^2_{\alpha_2\beta_2} z_0^{\alpha_2} z_1^{\beta_2} + \sum_{\gamma_2,\delta_2} B_{\gamma_2\delta_2} z_1^{\gamma_2} z_2^{\delta_2} \\
T_3 &= z_0^n + z_1^k + z_2^l + \sum_{\substack{\alpha_3,\beta_3 \\ \alpha_3 \le n-2}} A^3_{\alpha_3\beta_3} z_0^{\alpha_3} z_1^{\beta_3} + \sum_{\substack{\gamma_3,\delta_3,\varepsilon_3 \\ \gamma_3 \le n-2}} B_{\gamma_3\delta_3\varepsilon_3} z_0^{\gamma_3} z_1^{\delta_3} z_2^{\varepsilon_3} \\
T_4 &= z_0^n + z_1^k + z_2^l + \sum_{\substack{\alpha_4,\beta_4 \\ \alpha_4 \le n-2}} A^4_{\alpha_4\beta_4} z_0^{\alpha_4} z_2^{\beta_4} + \sum_{\gamma_4,\delta_4} B^4_{\gamma_4\delta_4} z_1^{\gamma_4} z_2^{\delta_4} \\
T_5 &= z_0^n + z_1^k + z_2^l + \sum_{\substack{\alpha_5,\beta_5 \\ \alpha_5 \le n-2}} A^5_{\alpha_5\beta_5} z_0^{\alpha_5} z_2^{\beta_5} + \sum_{\substack{\gamma_5,\delta_5,\varepsilon_5 \\ \gamma_5 \le n-2}} B_{\gamma_5\delta_5\varepsilon_5} z_0^{\gamma_5} z_1^{\delta_5} z_2^{\varepsilon_5} \\
T_6 &= z_0^n + z_1^k + z_2^l + \sum_{\alpha_6,\beta_6} A^6_{\alpha_6\beta_6} z_1^{\alpha_6} z_2^{\beta_6} + \sum_{\substack{\gamma_6,\delta_6,\varepsilon_6 \\ \gamma_6 \le n-2}} B^6_{\gamma_6\delta_6\varepsilon_6} z_0^{\gamma_6} z_1^{\delta_6} z_2^{\varepsilon_6}
\end{aligned}
$$

In the representations above, all coefficients are nonzero complex numbers.

Definition 3.8 It is said that a weighted homogeneous polynomial f belongs to the type T_i if f can be written in the form T_i for $i = 0, \ldots, 6$. In this case, $f \in T_i$. Otherwise, $f \notin T_i$.

Note that the surface singularities defined by the above six different weighted homogeneous polynomials are topologically equivalent to the surface singularity defined by $z_0^n + z_1^k + z_2^l = 0$. It is a consequence of Theorem 2.6([Xu-Ya]). But, for the analytic case, we will prove the different result.

Remark: If f and g have the type T_1 in a sense of Definition 3.8, then f and g can be written as follows:

$$(1)\qquad \begin{aligned} f &= z_0^n + z_1^k + z_2^l + \sum_{\substack{\alpha,\beta \\ \alpha \le n-2}} A_{\alpha\beta} z_0^\alpha z_1^\beta + \sum_{\substack{\gamma,\delta \\ \gamma \le n-2}} B_{\gamma\delta} z_0^\gamma z_2^\delta, \\ g &= z_0^n + z_1^k + z_2^l + \sum_{\substack{\varepsilon,\eta \\ \varepsilon \le n-2}} C_{\varepsilon\eta} z_0^\varepsilon z_1^\eta + \sum_{\substack{\mu,\nu \\ \mu \le n-2}} D_{\mu\nu} z_0^\mu z_2^\nu \end{aligned}$$

for some nonzero complex numbers $A_{\alpha\beta}, B_{\gamma\delta}, C_{\varepsilon\eta}$ and $D_{\mu,\nu}$. Set $I_{01}(f) = \{(\alpha,\beta) : A_{\alpha\beta} \neq 0 \text{ in } f\}$ and $I_{02}(f) = \{(\gamma,\delta) : B_{\gamma\delta} \neq 0 \text{ in } f\}$. For the convenience, we will use the notation $I_{01}(f) = \{(\alpha_i,\beta_i) : i = 1,\ldots,s\}, I_{02}(f) = \{(\gamma_j,\delta_j) : j = 1,\ldots,s'\}$ where $s = \#I_{01}(f), s' = \#I_{02}(f)$, respectively. Note that $s \ge 1, s' \ge 1$.

On the other hand, if $f \approx g$, then there exists a biholomorphism $\varphi : (\mathbb{C}^3, O) \to (\mathbb{C}^3, O)$ at the origin such that $f \circ \varphi = g$. If we set $\varphi(z_0,z_1,z_2) = (H, L, M)$ where

$$(2)\qquad \begin{aligned} H &= a_1 z_0 + b_1 z_1 + c_1 z_2 + H_2 + \cdots, \quad H_s = \textstyle\sum_{p+q+r=s} A_{p,q,r} z_0^p z_1^q z_2^r, \\ L &= a_2 z_0 + b_2 z_1 + c_2 z_2 + L_2 + \cdots, \quad L_s = \textstyle\sum_{p+q+r=s} B_{p,q,r} z_0^p z_1^q z_2^r, \\ M &= a_3 z_0 + b_3 z_1 + c_3 z_2 + M_2 + \cdots, \quad M_s = \textstyle\sum_{p+q+r=s} C_{p,q,r} z_0^p z_1^q z_2^r, \end{aligned}$$

then,

$$(3)\qquad H^n + L^k + M^l + \sum_{(\alpha,\beta)\in I_{01}(f)} A_{\alpha\beta} H^\alpha L^\beta + \sum_{(\gamma,\delta)\in I_{02}(f)} B_{\gamma\delta} H^\gamma L^\delta = g.$$

Definition 3.9 Let $f \in T_1$ in a sense of Definition 3.8. We will define $\min(f)$ by

$$\min(f) = \min\left\{\min\{\alpha+\beta : (\alpha,\beta) \in I_{01}(f)\}, \min\{\gamma+\delta : (\gamma,\delta) \in I_{02}(f)\}\right\}.$$

Since $I_{01}(f) \neq \emptyset$ and f is a weighted homogeneous polynomial, we have $\min(f) < k$.

Lemma 3.10 *Suppose that f and g have the type T_i for $i = 1, \ldots, 6$ in a sense of Definition 3.8 and $f \approx g$. Let $\varphi : (\mathbb{C}^3, O) \to (\mathbb{C}^3, O)$ be a biholomorphism at the origin such that $f \circ \varphi = g$ as in the above Remark.*

Then $a_1^n = 1, b_1 = c_1 = 0$ and

$$\begin{aligned} H &= a_1 z_0 + H_{\min\{k, \min(f)\}-n+1} + \cdots \\ L &= z_0 P + z_1 Q + L_e + \cdots \end{aligned}$$

for some polynomials P and Q where $e = \min\{l - k + 1, \mu - k + 1\}$ with $\mu = \{\alpha_i(\min(f) - n + 1) + \beta_i : (\alpha_i, \beta_i) \in I_{01}(f)\}$. In particular $ke > l$

Proof. By comparison with degrees between $f \circ \varphi$ and g in (3), we have $a_1^n = 1, b_1 = c_1 = 0$ and

$$H = a_1 z_0 + H_{\min\{k, \min(f)\}-n+1} + \cdots$$

For the proof of the second fact, we must show that if $\tau < \min\{l - k + 1, \mu - k + 1\} = e$, then $z_2^\tau \notin L_\tau$ for all τ with $2 \leq \tau < e$. Recall that

$$L_\tau = \sum_{p+q+r=\tau} B_{pqr} z_0^p z_1^q z_2^r.$$

In L^k, every monomial with degree $k - 1 + 2 = k + 1$ belongs to $(a_2 z_0 + b_2 z_1)^{k-1} L_2$ only. If $z_1^{k-1} z_2^2$ belongs to one of $H^n, M^l, H^{\alpha_i} L^{\beta_i}$ or $H^{\gamma_j} M^{\delta_j}$ for some i, j, then $k - 1 + 2 \geq \min\{l, \mu\}$. Thus if $k - 1 + 2 < \min\{l, \mu\}$, i.e., $2 < \min\{l - k + 1, \mu - k + 1\}$, then $z_1^{k-1} z_2^2$ does not belong to $H^n, M^l, H^{\alpha_i} L^{\beta_i}$ and $H^{\gamma_j} M^{\delta_j}$ for all i, j. Note that $\eta \leq k - 2$ for all η in g. Thus the coefficient $k b_2^{k-1} B_{0,0,2}$ of the monomial $z_1^{k-1} z_2^2$ is zero by comparison with g. Since $b_2 \neq 0$, we have $B_{0,0,2} = 0$. i.e., z_2^2 does not belong to L_2. Similarly, in the expansion L^k, the monomial $z_1^{k-1} z_2^3$ has coefficient $k b_2^{k-1} B_{0,0,3}$. If $k - 1 + 3 < \min\{l, \mu\}$, i.e., $3 < \min\{l - k + 1, \mu - k + 1\}$, then $z_1^{k-1} z_2^3$ does not belong to $H^n, M^l, H^{\alpha_i} L^{\beta_i}$ and $H^{\gamma_j} M^{\delta_j}$ for all i and j. Since $b_2 \neq 0$, we have $B_{0,0,3} = 0$ by comparison with g. i.e., z_2^3 does not belong to L_3. Continuing this process, we have the desired results.

Note that the inequalities:

$$k[\ \alpha_i(\min(f)-n+1)+\beta_i-k+1]$$

$$=k\alpha_i(\min(f)-n+1)-k(k-\beta_i)+k$$

$$>n(k-\beta_i)(\min(f)-n+)-k(k-\beta_i)+k \quad (\text{by } k\alpha_i>n(k-\beta_i))$$

$$=(k-\beta_i)(n(\min(f)-n+1)-k)+k>l,$$

$$k(\ l-k+1)>l$$

for all $(\alpha_i,\beta_i)\in I_{01}(f)$. Thus $ke>l$. This proves the Lemma.

Theorem 3.11 (Main result) *Suppose that f and g have the type T_1 as in the equation (1). Let n,k and l be positive integers with $2\le n<k<l$. Suppose that $n(\min(f)-n+1)>l$. Then, we get the following:*

(i) *$f\approx g$ if and only if $I_{01}(f)=I_{01}(g), I_{02}(f)=I_{02}(g)$ and there exist complex numbers a,b and c with $a^n=b^k=c^l=1$ such that $A_{\alpha\beta}a^\alpha b^\beta=C_{\alpha\beta}$ and $B_{\gamma\delta}a^\gamma c^\delta=D_{\gamma\delta}$ for all $(\alpha,\beta)\in I_{01}(f)$, $(\gamma,\delta)\in I_{02}(f)$.*

(ii) *If $f\in T_1$ and $h\in T_i$ for $i\neq 1$, then $f\not\approx h$.*

Proof. The converse of (i) is trivial. Thus we will prove the other case of (i) and (ii).

Suppose that f and g have the type T_1 in a sense of Definition 3.8 and $f\approx g$. Choose a biholomorphism $\varphi:(\mathbb{C}^3,O)\to(\mathbb{C}^3,O)$ at the origin such that $f\circ\varphi=g$ as in the above Lemma. For the proof of this Theorem, we must show the following facts in the expansion (3) of $f\circ\varphi$:

1. The monomial z_1^k has nonzero coefficient b_2^k.

2. The monomial z_2^l has nonzero coefficient c_3^l.

3. For each $(\alpha,\beta)\in I_{01}(f),(\gamma,\delta)\in I_{02}(f)$, the monomial $z_0^\alpha z_1^\beta$ and $z_0^\gamma z_2^\delta$ belongs to only the expansion of $H^\alpha L^\beta$ and $H^\gamma M^\delta$, respectively.

4. $\min(f) = \min(g)$.

In the Lemma 3.10, H can be written as follows:

$$H = a_1 z_0 + H_{\min(f)-n+1} + \cdots$$

By the inequalities

$$\begin{aligned} n(\min(f) - n + 1) &> k, \\ \alpha_i(\min(f) - n + 1) + \beta_i &> k, \\ \gamma_j(\min(f) - n + 1) + \delta_j &> k \end{aligned} \tag{4}$$

for all $(\alpha_i, \beta_i) \in I_{01}(f), (\gamma_j, \delta_j) \in I_{02}(f)$, we have $b_2^k = 1$ and $c_2 = 0$. Since φ is a biholomorphism at the origin, we have

$$|J_\varphi(O)| = \det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \neq 0. \tag{5}$$

Since $a_1^n = 1, b_1 = c_1 = 0$ in the Lemma 3.10, we have $|J_\varphi(O)| = a_1 b_2 c_3 \neq 0$. So $c_3 \neq 0$. In particular, by the inequality

$$\min\{n(\min(f) - n + 1), ke, \alpha_i(\min(f) - n + 1) + \beta_i e, \gamma_j e + \delta_j\} > l$$

for all $(\alpha_i, \beta_i) \in I_{01}(f)$ and $(\gamma_j, \delta_j) \in I_{12}(f)$, we have $c_3^l = 1$. Thus we have proved 1 and 2 as desired. Now, we will prove that 3 holds. First, we will claim that the monomial $z_0^{\alpha_i} z_1^{\beta_i}$ does not belongs to $H^{\alpha_j} L^{\beta_j}$ if $(\alpha_i, \beta_i) \neq (\alpha_j, \beta_j)$ in $I_{01}(f)$ and H^n, L^k, M^l. Note that if $\alpha_i + \beta_i = \alpha_j + \beta_j$, then $(\alpha_i, \beta_i) = (\alpha_j, \beta_j)$ for all $(\alpha_i, \beta_i) \in I_{01}(f)$ and $(\alpha_j, \beta_j) \in I_{01}(f)$. Thus it is nothing to prove if $\alpha_i + \beta_i \leq \alpha_j + \beta_j$. Suppose that $\alpha_i + \beta_i > \alpha_j + \beta_j$. From now on, we will use the notation $a_{ij} = a_i - a_j$, frequently. If $\beta_{ji} \geq 0$, then

$\frac{\alpha_{ij}}{n} = \frac{\beta_{ji}}{k} < \frac{\beta_{ji}}{n}$. That is, $\alpha_j + \beta_j > \alpha_i + \beta_i$. This is a contradiction. Thus $\beta_{ji} < 0$, so that $\alpha_j > \alpha_i$ if $\alpha_i + \beta_i > \alpha_j + \beta_j$. Therefore, the monomial $z_0^{\alpha_i} z_1^{\beta_i}$ belongs to $(a_1 z_0)^{\sigma}(H_{\min(f)-n+1} + \cdots)^{\alpha_j - \sigma} L^{\beta_j}$ for some σ where $0 \le \sigma \le \alpha_i$ if the monomial $z_0^{\alpha_i} z_1^{\beta_i}$ belongs to $H^{\alpha_j} L^{\beta_j}$. According to the inequalities

$$
(6)\qquad \begin{aligned}
\sigma + (\alpha_j - \sigma)(\min(f) - n + 1) + \beta_j &\ge \alpha_i + \alpha_{ji}(\min(f) - n + 1) + \beta_j \\
&> \alpha_i + \frac{k}{n}\alpha_{ji} + \beta_j \\
&= \alpha_i + \beta_i,
\end{aligned}
$$

every monomial in the expansion of $(a_1 z_0)^{\sigma}(H_{\min(f)-n+1} + \cdots)^{\alpha_j - \sigma} L^{\beta_j}$ has a greater degree than $\alpha_i + \beta_i$ if $0 \le \sigma \le \alpha_i$. Consequently, $z_0^{\alpha_i} z_1^{\beta_i} \notin H^{\alpha_j} L^{\beta_j}$ if $(\alpha_i, \beta_i) \ne (\alpha_j, \beta_j)$. Next, claim that $z_0^{\alpha_i} z_1^{\beta_i} \notin H^{\gamma_j} M^{\delta_j}$ for all $(\gamma_j, \delta_j) \in I_{02}(f)$. Note that if $\alpha_i + \beta_i = \gamma_j + \delta_j$, then $\alpha_i < \gamma_j$, since $\frac{\alpha_i - \gamma_j}{n} = \frac{\delta_j}{l} - \frac{\beta_i}{k} < \frac{\delta_j - \beta_i}{k}$. Thus, if $\alpha_i + \beta_i \le \gamma_j + \delta_j$, then it is nothing to prove. Suppose that $\alpha_i + \beta_i > \gamma_j + \delta_j$. If $\alpha_i \ge \gamma_j$, then $\delta_j = \frac{l}{k}\beta_i + \frac{l}{n}(\alpha_i - \gamma_j) > \beta_i + \alpha_i - \gamma_j$. It is a contradiction. Thus $\alpha_i < \gamma_j$. If the monomial $z_0^{\alpha_i} z_1^{\beta_i}$ belongs to the expansion of $H^{\gamma_j} L^{\delta_j}$, then it belongs to $(a_1 z_0)^{\sigma}(H_{\min(f)-n+1} + \cdots)^{\gamma_j - \sigma} M^{\delta_j}$ for some σ where $0 \le \sigma \le \alpha_i$. Note that the inequalities:

$$
(7)\qquad \begin{aligned}
\sigma + (\gamma_j - \sigma)(\min(f) - n + 1) + \delta_j &\ge \alpha_i + \frac{l}{n}(\gamma_j - \alpha_i) + \delta_j \\
&= \alpha_i - \delta_j + \frac{l}{k}\beta_i + \delta_j \\
&> \alpha_i + \beta_i
\end{aligned}
$$

for all σ where $0 \le \sigma \le \alpha_i$. This say that every monomial in the expansion of $(a_1 z_0)^{\sigma}(H_{\min(f)-n+1} + \cdots)^{\gamma_j - \sigma} M^{\delta_j}$ has a greater degree than $\alpha_i + \beta_i$ if $0 \le \sigma \le \alpha_i$. This is a contradiction. Thus $z_0^{\alpha_i} z_1^{\beta_i}$ does not belong to the expansion of $H^{\gamma_j} M^{\delta_j}$ even if $\alpha_i + \beta_i > \gamma_j + \delta_j$. Clearly, $z_0^{\alpha_i} z_1^{\beta_i}$ does not belong to the expansion of H^n, L^k and M^l. Thus the monomial $z_0^{\alpha_i} z_1^{\beta_i}$ has nonzero coefficient $a_1^{\alpha_i} b_2^{\beta_i} A_{\alpha_i \beta_i}$ in the expansion of $f \circ \varphi$ for all $(\alpha_i, \beta_i) \in I_{01}(f)$. Similarly, we will show that the monomial $z_0^{\gamma_j} z_2^{\delta_j}$ has nonzero coefficient

$a_1^{\gamma_j} c_3^{\delta_j} B_{\gamma_j \delta_j}$ in the expansion of $f \circ \varphi$. For each $(\gamma_i, \delta_i), (\gamma_j, \delta_j)$ in $I_{02}(f)$, if $\gamma_i + \delta_i \leq \gamma_j + \delta_j$, then $z_0^{\gamma_i} z_2^{\delta_i} \notin H^{\gamma_j} M^{\delta_j}$ is clear. Suppose that $\gamma_i + \delta_i > \gamma_j + \delta_j$. If $\gamma_i \geq \gamma_j$, then $\frac{\gamma_{ij}}{n} = \frac{\delta_{ji}}{l} > \frac{\gamma_{ij}}{l}$. Thus $\gamma_j + \delta_j > \gamma_i + \delta_i$ if $\gamma_i \geq \gamma_j$. It is a contradiction. That is, $\gamma_i < \gamma_j$ if $\gamma_i + \delta_i > \gamma_j + \delta_j$. If the monomial $z_0^{\gamma_i} z_2^{\delta_i}$ belongs to $H^{\gamma_j} M^{\delta_j}$, then the monomial $z_0^{\gamma_i} z_2^{\delta_i}$ belongs to the expansion of $(a_1 z_0)^\sigma (H_{\min(f)-n+1} + \cdots)^{\gamma_j - \sigma} M^{\delta_j}$ for some σ where $0 \leq \sigma \leq \gamma_i$. Note that the inequalities:

$$
\begin{aligned}
\sigma + (\gamma_j - \sigma)(\min(f) - n + 1) \quad & \geq \gamma_i + \gamma_{ji}(\min(f) - n + 1) + \delta_j \\
& > \gamma_i + \frac{l}{n}\gamma_{ji} + \delta_j \\
& = \gamma_i + \delta_{ij} + \delta_j = \gamma_i + \delta_i.
\end{aligned} \tag{8}
$$

This shows that every monomial in the expansion of $(a_1 z_0)^\sigma (H_{\min(f)-n+1} + \cdots)^{\gamma_j - \sigma} M^{\delta_j}$ has a greater degree than $\gamma_i + \delta_i$ if $0 \leq \sigma \leq \gamma_i$. It is a contradiction. Thus $z_0^{\gamma_i} z_2^{\delta_i} \notin H^{\gamma_j} M^{\delta_j}$ even if $\gamma_i + \delta_i > \gamma_j + \delta_j$. Next, claim that for each $(\gamma_j, \delta_j) \in I_{02}(f)$, $z_0^{\gamma_j} z_2^{\delta_j}$ does not belong to the expansions of $H^{\alpha_i} L^{\beta_i}$ for all $(\alpha_i, \beta_i) \in I_{01}(f)$. If $\gamma_j + \delta_j \leq \alpha_i + \beta_i$, then it is nothing to prove. Suppose that $\gamma_j + \delta_j > \alpha_i + \beta_i$. Note that $\alpha_i + \beta_i > \gamma_j$, since $\frac{\gamma_j}{n} + \frac{\delta_j}{l} = \frac{\alpha_i}{n} + \frac{\beta_i}{k}$ and $\frac{\gamma_j}{n} < \frac{\alpha_i}{n} + \frac{\beta_i}{k} < \frac{\alpha_i + \beta_i}{n}$. If the monomial $z_0^{\gamma_j} z_2^{\delta_j}$ belongs to $H^{\alpha_i} L^{\beta_i} = (a_1 z_0 + H_{\min(f)-n+1} + \cdots)^{\alpha_i} (z_0 P + z_1 Q + L_e + \cdots)^{\beta_i}$, then the monomial $z_0^{\gamma_j} z_2^{\delta_j}$ belongs to the expansion of $(a_1 z_0)^\sigma (H_{\min(f)-n+1} + \cdots)^{\alpha_i - \sigma} (z_0 P)^\eta (L_e + \cdots)^{\beta_i - \eta}$ for some σ and η where

$$
\begin{aligned}
& 0 \leq \sigma \leq \alpha_i, \\
& 0 \leq \eta \leq \beta_i, \\
& 0 \leq \sigma + \eta \leq \gamma_j.
\end{aligned} \tag{9}
$$

We will set

$$
\begin{aligned}
h(\sigma, \eta) \quad & = \sigma + (\alpha_j - \sigma)(\min(f) - n + 1) + \eta + (\beta_i - \eta)e \\
& = \alpha_i(\min(f) - n + 1) + \beta_i e - \sigma(\min(f) - n) + (1 - e)\eta.
\end{aligned}
$$

If $\gamma_j > \beta_i$, then $h(\sigma,\eta)$ has the minimum at $(\gamma_j - \beta_i, \beta_i)$ or at $(\alpha_i, \gamma_j - \alpha_i)$ if $\gamma_j > \alpha_i$, and $h(\sigma,\eta)$ has the minimum at $(\gamma_j - \beta_i, \beta_i)$ or $(\gamma_j, 0)$ if $\gamma_j \leq \alpha_i$ under the conditions (9). On the other hand, if $\gamma_j \leq \beta_i$, then $h(\sigma,\eta)$ has the minimum at $(0,\gamma_j)$ or $(\alpha_i, \gamma_j - \alpha_i)$ if $\gamma_j > \alpha_i$, and $h(\sigma,\eta)$ has the minimum at $(0,\gamma_j)$ or $(\gamma_j, 0)$ if $\gamma_j \leq \alpha_i$ under the conditions (9). Consider the inequalities:

(10)

$$\begin{aligned}
h(\gamma_j - \beta_i, \beta_i) &= \alpha_i(\min(f) - n + 1) + \beta_i e - (\gamma_j - \beta_i)(\min(f) - n) + (1-e)\beta_i \\
&= (\alpha_i + \beta_i - \gamma_j)(\min(f) - n + 1) + \gamma_j \\
&\geq \gamma_j + \frac{l}{n}(\alpha_i + \beta_i - \gamma_j) \\
&= \gamma_j + \delta_j + \left(\frac{l}{n} - \frac{l}{k}\right)\beta_i > \gamma_j + \delta_j \\
h(\alpha_i, \gamma_j - \alpha_i) &= \gamma_j + (\beta_i - \gamma_j + \alpha_i)e \\
&\geq \gamma_j + \frac{l}{k}(\beta_i - \gamma_j + \alpha_i) \\
&= \gamma_j + \delta_j + \left(\frac{l}{n} - \frac{l}{k}\right)(\gamma_j - \alpha_i) > \gamma_j + \delta_j \text{ (in the case } \gamma_j > \alpha_i),
\end{aligned}$$

since $\frac{l}{k}\beta_i = \delta_j + \frac{l}{n}(\gamma_j - \alpha_i)$ and

$$\begin{aligned}
h(0,\gamma_j) &= \alpha_i(\min(f) - n + 1) + \gamma_j + (\beta_i - \gamma_j)e \\
&> \frac{l}{n}\alpha_i + \frac{l}{k}(\beta_i - \gamma_j) + \gamma_j \\
&= \frac{l}{n}\gamma_j - \frac{l}{k}\gamma_j + \gamma_j + \delta_j > \gamma_j + \delta_j,
\end{aligned}$$

(11)

$$\begin{aligned}
h(\gamma_j, 0) &= \alpha_i(\min(f) - n + 1) + \beta_i e - \gamma_j(\min(f) - n) \\
&= (\alpha_i - \gamma_j)(\min(f) - n + 1) + \beta_i e + \gamma_j \\
&> \frac{l}{n}(\alpha_j - \gamma_j) + \frac{l}{k}\beta_i + \gamma_j \\
&= \gamma_j + \delta_j \quad \text{(in the case } \gamma_j \leq \alpha_i).
\end{aligned}$$

Thus $h(\sigma,\eta) > \gamma_j+\delta_j$ at any case. That is, every monomial in the expansion of $(a_1z_0)^\sigma(H_{\min(f)-n+1}+\cdots)^{\alpha_i-\sigma}(z_0P)^\eta(L_e+\cdots)^{\beta_i-\eta}$ has a greater degree than $\gamma_j+\delta_j$ under the conditions (9). It is a contradiction if the monomial $z_0^{\gamma_j}z_2^{\delta_j}$ belongs to $H^{\alpha_i}L^{\beta_i}$. Thus $z_0^{\gamma_j}z_2^{\delta_j}$ does not belong to $H^{\alpha_i}L^{\beta_i}$ for all $(\alpha_i,\beta_i)\in I_{01}(f)$. By the inequalities

$$\begin{aligned}\gamma_j+(n-\gamma_j)(\min(f)-n+1) &> \gamma_j+\delta_j,\\ \gamma_j+(k-\gamma_j)(\min(f)-n+1) &> \gamma_j+\delta_j,\end{aligned}\tag{12}$$

the monomial $z_0^{\gamma_j}z_2^{\delta_j}$ does not belong to the expansions of H^n and L^k. Therefore, for each $(\gamma_j,\delta_j)\in I_{02}(f)$, the monomial $z_0^{\gamma_j}z_2^{\delta_j}$ has nonzero coefficient $a_1^{\gamma_j}c_3^{\delta_j}B_{\gamma_j\delta_j}$ in the expansion of $f\circ\varphi$. Thus we have proved 3 as desired in this case.

If there exists $(\varepsilon',\eta')\in I_{01}(g)=\{(\varepsilon,\eta): C_{\varepsilon\eta}\neq 0 \text{ in } g\}$ with $\varepsilon'+\eta'<\min(f)$, then the monomial $z_0^{\varepsilon'}z_1^{\eta'}$ must appear in the expansion of $f\circ\varphi$. It is a contradiction. Similarly, the case of $I_{02}(g)=\{(\mu,\nu): D_{\mu\nu}\neq 0 \text{ in } g\}$ is the same. That is, $s=s'$ and $(\alpha_i,\beta_i)=(\gamma_j,\delta_j)$ up to order and $\min(f)=\min(g)$. This finishes proving (i).

If h has the type T_i with $i\neq 1$, then $f\not\approx h$ is obvious by (i). This proves (ii).

This is the complete proof of Theorem 3.11.

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가중동차 다항식으로 정의된 어떤 곡면 특이점의 해석적 분류

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$z_0^n + z_1^k + z_2^l = 0$와 위상적으로 동형인 어떤 가중동차 다항식으로 정의된 곡면 특이점을 해석적으로 분류한다.