J. of Basic Sciences, Cheju Nat'l Univ.
18(1), 91~98, 2005

기초과학연구 제주대학교
18(1), 91~98, 2005

LINEAR MAPS PRESERVING ZERO-TERM RANK OF MATRICES OVER RINGS

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Abstract. Zero-term rank of a matrix is the minimum number of lines (row or columns) needed to cover all the zero entries of the given matrix. We characterize the linear operators that preserve zero-term rank of the $m \times n$ matrices over any ring whose characteristic is not 2. That is, a linear operator T preserves zero-term rank if and only if it has the form $T(A) = P(A \circ B)Q$, where P, Q are permutation matrices and $A \circ B$ is the Schur product with B whose entries are all nonzero and not zero-divisors.

1. Introduction and preliminaries

There are many papers on the study of linear operators on matrices that preserve certain matrix functions. Matrices over rings containing integers also have been the subject of research by many authors. But there are few papers on zero-term rank of the matrices over any ring. Recently Beasley,

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[1]1991 Mathematics Subject Classification : 15A03, 15A04, 15A36.
Key words and phrases : Linear operator, term-rank, zero-term rank, (P, Q, B)-operator.
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Song and Lee [2] obtained characterizations of zero-term rank preservers of matrices over antinegative semiring.

In this article, we obtain characterizations of the linear operators that preserve zero-term rank of matrices over any ring whose characteristic is not 2.

Let $M_{m,n}(\mathbb{R})$ denote the set of all $m \times n$ matrices with entries in a ring $\mathbb{R}$. Let $\mathbb{B} = \{0, 1\}$ be the Boolean algebra. For $A \in M_{m,n}(\mathbb{R})$, let $\overline{A}$ denote the $m \times n$ matrix with entries in $\mathbb{B}$ such that $\overline{a_{ij}} = 0$ if and only if $a_{ij} = 0$. Let E_{ij} be the matrix in $M_{m,n}(\mathbb{R})$ with exactly one nonzero entry, that being a 1 in the (i, j)entry. We call E_{ij} a *cell.* A weighted cell is any nonzero scalar multiple of a cell, that is, $\alpha E_{i,j}$ is a weighted cell for any $0 \neq \alpha \in \mathbb{R}$. Let J denote the $m \times n$ matrix all of whose entries are 1. A matrix A is said to *dominate* a matrix B if $a_{ij} = 0$ implies that $b_{ij} = 0$ and we write $A \geq B$.

The *zero-term rank* [3] of a matrix A, $z(A)$, is the minimum number of lines (row or columns) needed to cover all the zero entries of A. Of course, the *term rank* [1] of A, $t(A)$, is defined similarly for all the nonzero entries of A. Evidently the zero-term rank(or term rank) of a matrix is the zero-term rank(term-rank, respectively) of $\overline{A}$.

A linear operator $T : M_{m,n}(\mathbb{R}) \to M_{m,n}(\mathbb{R})$ *preserves zero-term rank* k if $z(T(A)) = k$ whenever $z(A) = k$. So a linear operator T *preserves zero-term rank* on $M_{m,n}(\mathbb{R})$ if it preserves zero-term rank k for every $k \leq \min\{m, n\}$.

Which linear operators over $M_{m,n}(\mathbb{R})$ preserve zero-term rank? The operations of (1) permuting rows, (2) permuting columns and (3) (if $m = n$) transposing the matrices in $M_{m,n}(\mathbb{R})$ are all linear, zero-term rank preserving operators on $M_{m,n}(\mathbb{R})$.

If we take a fixed $m \times n$ matrix B in $M_{m,n}(\mathbb{R})$, all of whose entries are nonzero and not zero-divisors, then its *Schur product* $A \circ B = [a_{ij}b_{ij}]$ with A

has the same zero-term rank as does A. The operator $A \longmapsto A \circ B$ is linear. Similarly $A \longmapsto A \circ B$ is linear zero-term rank preserving operator. That these operators and their compositions are the only zero-term rank preservers is one of the consequence of Theorem 2.3 below.

A linear operator $T : M_{m,n}(\mathbb{R}) \rightarrow M_{m,n}(\mathbb{R})$ is called a (P, Q, B)*-operator* if there exist permutation matrices P and Q, and a matrix B, all of whose entries are nonzero and not zero-divisors, such that $T(A) = P(A \circ B)Q$ for all $A \in M_{m,n}(\mathbb{R})$ or if $m = n$, $T(A) = P(A \circ B)^t Q$ for all $A \in M_{m,n}(\mathbb{R})$.

In [2], the linear operators which preserve zero-term rank of matrices over antinegative semiring were shown to be (P, Q, B)-operators.

We now state the result for later reference.

Theorem 1.1 [2]. Let $\mathbb{S}$ be an antinegative semiring. Suppose T is a linear operator on $M_{m,n}(\mathbb{S})$. Then the following statements are equivalent :

(i) T is a (P, Q, B)-operator ;

(ii) T preserves zero-term rank ;

(iii) T preserves zero-term rank 1 and $T(J) \geq J$.

In the followings, we assume that $\mathbb{R}$ is a ring whose characteristic is not 2 and T is a linear operator on $M_{m,n}(\mathbb{R})$ with $m, n > 1$.

2. Zero-term rank preservers on $M_{m,n}(\mathbb{R})$

In this section we investigate the preservers of the zero-term rank on the set of $m \times n$ matrices over $\mathbb{R}$.

We give a lemma upon which the main theorem will rely.

Lemma 2.1. If T preserves zero-term ranks 0 and 1, then T maps each cell to a weighted cell which induces a bijection on the set of indices $\{1, 2, \cdots, m\} \times \{1, 2, \cdots, n\}$.

Proof. Since T preserves zero-term rank 0, we have

$$T(J) = V = (v_{ij}) \tag{1}$$

for some $V \in M_{m,n}(\mathbb{R})$ with $v_{ij} \neq 0$ for all (i, j). If $T(E_{ij}) = 0$, then $T(J - E_{ij}) = T(J)$. But the zero-term rank of $T(J) = V$ is zero while the zero-term rank of $T(J - E_{ij})$ is 1 since T preserves zero-term rank 1. This contradiction implies that

$$T(E_{ij}) \neq 0 \tag{2}$$

for all (i, j). Since the zero-term rank of $T(J - E_{ij})$ is 1, there is some pair (r, s) such that the (r, s) entry of $T(J - E_{ij})$ is zero. Let

$$T(E_{ij}) = U = (u_{hk}). \tag{3}$$

Then $V = T(J) = T(J - E_{ij}) + T(E_{ij})$ and hence $v_{rs} = u_{rs}$. If some nonzero entries u_{hk} and v_{hk} are distinct, then the zero-term rank of $u_{hk}J - v_{hk}E_{ij}$ is zero while the (h, k) entry of $T(u_{hk}J - v_{hk}E_{ij}) = u_{hk}T(J) - v_{hk}T(E_{ij})$ is $u_{hk}v_{hk} - v_{hk}u_{hk} = 0$. This is a contradiction to the fact that T preserves zero-term rank 0. Thus if u_{hk} is not zero, then

$$u_{hk} = v_{hk}. \tag{4}$$

If we put $T(J - E_{ij}) = W = (w_{ij})$, we must have $U + W = V$. Hence if u_{rs} is not zero, then we have $u_{rs} + w_{rs} = v_{rs}$ and hence $v_{rs} + w_{rs} = v_{rs}$ by (4). Thus $w_{rs} = 0$. Since the zero-term rank of $T(J - E_{ij})$ is 1, all the zero entries of W lie in a single row or column. Without loss of generality, we may assume that all zero entries of W and hence all nonzero entries of U lie in row r.

Suppose that $T(E_{ij}) = U = (u_{hk})$ and $T(E_{cd}) = G = (g_{ef})$ with $(i,j) \neq (c,d)$. If the (r,s) entries of both U and G are not zero, then $v_{rs} = u_{rs} = g_{rs}$ by (4) and hence $T(E_{ij} + E_{cd})$ has (r,s) entry $2v_{rs}$, which is not zero since characteristic of R is not 2. Hence $T(2J - E_{ij} - E_{cd})$ has zero in the (r,s) entry and then $z(T(2J - E_{ij} - E_{cd})) \geq 1$ while $z(2J - E_{ij} - E_{cd}) = 0$, which is a contradiction to the fact that T preserves zero-term rank 0. Thus $u_{r,s} \neq 0$ implies $g_{r,s} = 0$ and vice versa. Since $T(E_{ij}) \neq 0$ for all (i,j) by (2), $T(E_{ij})$ must be a weighted cell by pigeon hole principle. Since $T(J) = V$ has zero-term rank 0, the mapping T must induce a bijection on the set of indices $\{1,2,\cdots,m\} \times \{1,2,\cdots,n\}$. □

If $T : M_{m,n}(\mathbb{R}) \to M_{m,n}(\mathbb{R})$ is a linear operator, define $\overline{T} : M_{m,n}(\mathbb{B}) \to M_{m,n}(\mathbb{B})$ by $\overline{T}(\overline{A}) = \sum_{i=1}^{m} \sum_{j=1}^{n} \overline{T(a_{ij}E_{ij})}$ for any $A \in M_{m,n}(\mathbb{R})$.

Proposition 2.2. If T preserves zero-term ranks 0 and 1, then we have the following :

(1) T preserves term rank 1;

(2) T maps a row into a row (or a column if $m = n$) and T maps a column into a column (or a row if $m = n$) ;

(3) For $m = n$, if T maps a row into a row(or a column), then all rows must be mapped to some rows(columns, respectively) under T;

(4) T preserves term rank k for $k \geq 2$.

Proof. (1) Suppose that T does not preserve term rank 1. Then there exist some distinct cells E_{ij} and E_{il} on the same row(or column) such that

$$T(E_{ij} + E_{il}) = b_{ij}E_{rs} + b_{il}E_{pq}$$

with $p \neq r$ and $q \neq s$ by Lemma 2.1. So the zero-term rank of $J - E_{ij} - E_{il}$ is 1. But the zero-term rank of its image is 2 since

$$\begin{aligned} z(T(J - E_{ij} - E_{il})) &= z(\overline{T(J - E_{ij} - E_{il})}) \\ &= z(J - E_{rs} - E_{pq}) \\ &= 2 \end{aligned}$$

This shows that T does not preserve zero-term rank 1, which is a contradiction. Hence T preserves term rank 1.

(2) Suppose T does not map a row into a row(or a column if $m = n$). Then T does not preserve term rank 1. This contradicts (1).

(3) Let R_i and C_j denote the ith row and jth column respectively. If $T(R_1) \subseteq R_i$ but $T(R_2) \subseteq C_j$ then $R_1 + R_2$ has $2n$ cells but $R_i + C_j$ has $2n - 1$ cells. This contradicts the bijectivity of the corresponding map of T on the set of indices $\{1, 2, \cdots, m\} \times \{1, 2, \cdots, n\}$ by Lemma 2.1. Thus we have the result.

(4) Since T maps a row into a row(or a column if $m = n$) by (2). T does not increase term rank of a matrix. Suppose there exists a matrix X such that $t(X) = k \geq 2$ and $t(T(X)) < k$. Since the corresponding map of T on indices is bijective by Lemma 2.1, we can take a 2×2 submatrix A of X such that $A = a_1 E_{ij} + a_2 E_{kl}$ but $T(A)$ has term rank 1, where a_1, a_2 are nonzero entries and $i \neq k, j \neq l$. Then T maps two rows into one row(or one column if $m = n$), which is a contradiction. Therefore T preserves term rank k. □

Theorem 2.3. If $T : M_{m,n}(\mathbb{R}) \rightarrow M_{m,n}(\mathbb{R})$ preserves zero-term ranks 0 and 1, then T is a (P, Q, B)-operator.

Proof. From the Lemma 2.1, $\overline{T}$ is bijective on the set of cells in $M_{m,n}(\mathbb{B})$.

Thus for any $A \in M_{m,n}(\mathbb{R})$,

$$\overline{T(A)} = \overline{\sum_{i=1}^{m}\sum_{j=1}^{n} T(a_{ij}E_{ij})} = \sum_{i=1}^{m}\sum_{j=1}^{n} \overline{T(a_{ij}E_{ij})} = \overline{T}(\overline{A}).$$

Thus, since T preserves zero-term rank 1, we have that $\overline{T}$ does also. By Theorem 1.1, $\overline{T}$ is a (P, Q, B)-operator on $M_{m,n}(\mathbb{B})$, where $B = J$. Thus the mapping $\overline{A} \mapsto \overline{P^tT(A)Q^t}$ is the identity linear operator on $M_{m,n}(\mathbb{B})$. That is, $P^tT(E_{ij})Q^t = b_{ij}E_{ij}$ for some pair (i, j)(or perhaps $P^tT(E_{ij})Q^t = b_{ij}E_{ji}$ in case $m = n$). Then, $T(A) = P(A \circ B)Q$ for all $A \in M_{m,n}(\mathbb{R})$ or $m = n$ and $T(A) = P(A \circ B)^tQ$ for all $A \in M_{m,n}(\mathbb{R})$. □

Theorem 2.4. For a linear operator T on $M_{m,n}(\mathbb{R})$, the following are equivalent :

(i) T preserves zero-term ranks 0 and 1;

(ii) T is a (P, Q, B)-operator;

(iii) T preserves zero-term rank.

Proof. Theorem 2.3 shows that i) implies ii). Obviously ii) implies iii) and iii) implies i). □

Thus we had characterizations of the linear operators that preserve zero-term rank of matrices over any ring whose characteristic is not 2.

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