J. of Basic Sciences, Cheju Nat. Univ.
9(2), 9~16, 1996

기초과학연구 제주대학교
9(2), 9~16, 1996

ON CONTRACTIBLE NONNEGATIVE CONVERTIBLE MATRICES

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ABSTRACT. An $n \times n$ matrix A is called convertible if there is an $n \times n$ (1,-1)-matrix H such that per$A = \det(A \circ H)$ where $A \circ H$ denotes the Hadamard product of A and H. A convertible (0,1)- matrix is called extremal if replacing any zero entry with a 1 breaks the convertibility. In this paper, some properties of contractible nonnegative convertible matrices are investigated and finds an extremal matrix. We present new proofs for the extremality of U_n and $V_{n,k}$.

1. INTRODUCTION

Let $M_n(\mathbf{R})$ denote the vector space of all $n \times n$ real matrices. For $A \in M_n(\mathbf{R})$, the *permanent* of A is defined by

$$\text{per}A = \sum_{\sigma} a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

where σ runs over all permutations of the set $\{1, 2, \cdots, n\}$.

In 1961, Marcus and Minc [6] proved that there is no linear transformation $T : M_n(\mathbf{R}) \longrightarrow M_n(\mathbf{R})$ such that per$A = \det T(A)$ for all $A \in M_n(\mathbf{R})$. However, there are matrices A such that per$A = \det B$ for some matrix B obtained from A by affixing $\pm$ signs to entries of A, i.e., such that per$A = \det(A \circ H)$ for some (1,-1)-matrix H of the same size as A where $A \circ H$ denotes

1980 *Mathematics Subject Classification* (1985 *Revision*). 05B20, 15A15.
Key words and phrases. convertible, permanent, contractible.
This paper was supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1995 .

the Hadamard product of A and H. Such a matrix A is called *convertible* and such a matrix H is called a *converter* of A.

For matrices A, B of the same size, A is said to be *permutation equivalent* to B if there exist permutation matrices P, Q such that $PAQ = B$. If both A, B are real, we denote by $A \leq B$ that every entry of A is less than or equal to the corresponding entry of B. Let $T_n = [t_{ij}]$ be the $n \times n$ (0,1)-matrix defined by $t_{ij} = 0$ if and only if $j > i + 1$.

Gibson proved that every $n \times n$ real matrix A such that $A \leq T_n$ is convertible [3] and also that the number of 1's of an $n \times n$ convertible (0,1)-matrix B is less than or equal to $(n^2 + 3n - 2)/2$ with equality if and only if B is permutation equivalent to T_n [2].

Let A be an *extremal* matrix if replacing any one of the zero entries of it by a 1 yields a nonconvertible matrix. For example, the matrix T_n is an extremal matrix. In [4] and [5], Hwang, Kim and song investigated some properties of nonnegative convertible matrices and find some classes of extremal matrices different from T_n.

Let $A = [a_{ij}]$ be an $m \times n$ real matrix with row vectors $\alpha_1, \cdots, \alpha_m$. We say A is *contractible* on column (resp. row) k if column (resp. row) k contains exactly two nonzero entries. Suppose A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A by replacing row i with $a_{jk}\alpha_i + a_{ik}\alpha_j$ and deleting row j and column k is called the *contraction* of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = (A^T_{ij:k})^T$ is called the *contraction* of A on row k relative to columns i and j. We say that A can be *contracted* to a matrix B if either $B = A$ or there exist matrices $A_0, A_1, \cdots, A_t$ $(t \geq 1)$ such that $A_0 = A$, $A_t = B$ and A_r is a contraction of A_{r-1} for $r = 1, 2, \cdots, t$.

In this paper, we investigate some properties of contractible convertible matrices and propose the other proofs for the extremality of U_n and $V_{n,k}$. And, we find an extremal matrix.

2. RESULTS

For positive integers k and n with $k \leq n$, let $Q_{k,n}$ denote the set of all strictly increasing k-sequences from $\{1, 2, \cdots, n\}$. For an $n \times n$ matrix A and for $\alpha, \beta \in Q_{k,n}$, let $A[\alpha|\beta]$ denote the matrix lying in rows α and columns β and let $A(\alpha|\beta)$ denote the matrix complementary to $A[\alpha|\beta]$ in A.

The following lemma shows a property of contraction for a real matrix A.

Lemma 1. [1]. *Let A be a nonnegative real matrix of order $n > 1$ and let B be a contraction of A. Then*

$$\mathrm{per}A = \mathrm{per}B.$$

Let A be a contractible matrix. If A is a convertible matrix, then, there is a converter H such that $\mathrm{per}A = \det(A \circ H)$. Since the permanent is permutation invariant, there are permutation matrices P, Q such that $\mathrm{per}A = \det(A \circ H) = \mathrm{per}PAQ = \det(P(A \circ H)Q)$ and $\det PQ = 1$. Thus, if A is a contractible matrix, then, without loss of generality, we can write A as follows;

$$A = \begin{bmatrix} x & \alpha \\ y & \beta \\ 0 & C \end{bmatrix}$$

where $x \neq 0 \neq y$, $\alpha = (\alpha_1, \cdots, \alpha_{n-1})$, $\beta = (\beta_1, \cdots, \beta_{n-1})$ and C is a $(n-2) \times (n-1)$ matrix.

For an $n \times n$ matrix $A = [a_{ij}]$, the $n \times n$ (0,1)-matrix $\mathrm{supp}A = [a_{ij}^*]$ defined by

$$a_{ij}^* = \begin{cases} 1, & \text{if } a_{ij} \neq 0 \\ 0, & \text{if } a_{ij} = 0 \end{cases}$$

is called the *support* of A.

Theorem 2. [4]. *A nonnegative square matrix is convertible if and only if its support is.*

Through this paper, let $p_j = \mathrm{per}A(1,2|1,j)$ and let $d_j = \det B(1,2|1,j)$ for $j = 2, 3, \cdots, n$.

Lemma 3. *Let A be a nonnegative contractible matrix. If A is a convertible matrix, then there exists converter $H = [h_{ij}]$ of A such that either $h_{11} = h_{21}$ and $h_{1j} \neq h_{2j}$ or $h_{11} \neq h_{21}$ and $h_{1j} = h_{2j}$ for $j = 2, 3, \cdots, n$.*

Proof. Since A is a nonnegative matrix, without loss of generality, we may assume that

$$A = \begin{bmatrix} 1 & \alpha \\ 1 & \beta \\ 0 & C \end{bmatrix}$$

and let $H = [h_{ij}]$ be the converter of A. And let $B = A \circ H$. Since A is a convertible matrix,

$$\begin{aligned} \text{per}A &= \text{per}A(1|1) + \text{per}A(2|1) \\ &= \det B \\ &= h_{11} \det B(1|1) - h_{21} \det B(2|1). \end{aligned}$$

So,

$$(\text{per}A(1|1) - h_{11} \det B(1|1)) + (\text{per}A(2|1) + h_{21} \det B(2|1)) = 0.$$

Suppose that $h_{11} \neq h_{21}$. Without loss of generality, we may assume that $h_{11} = 1, h_{21} = -1$. Then

$$\begin{aligned} \text{per}A(1|1) &= \beta_1 p_2 + \beta_2 p_3 + \cdots + \beta_{n-1} p_n, \\ h_{11} \det B(1|1) &= h_{22}\beta_1 d_2 - h_{23}\beta_2 d_3 + \cdots + (-1)^n h_{2n}\beta_{n-1} d_n, \\ \text{per}A(2|1) &= \alpha_1 p_2 + \alpha_2 p_3 + \cdots + \alpha_{n-1} p_n, \\ h_{21} \det B(2|1) &= -h_{12}\alpha_1 d_2 + h_{13}\alpha_2 d_3 + \cdots + (-1)^{n+1} h_{1n}\alpha_{n-1} d_n. \end{aligned}$$

So,

$$\beta_1(p_2 - h_{22}d_2) + \beta_2(p_3 + h_{23}d_3) + \cdots + \beta_{n-1}(p_n + (-1)^{n+1}h_{2n}d_n) +$$
$$\alpha_1(p_2 - h_{12}d_2) + \alpha_2(p_3 + h_{13}d_3) + \cdots + \alpha_{n-1}(p_n + (-1)^{n+1}h_{1n}d_n) = 0$$

Since p_i and d_i, $i = 2, \cdots, n$, are same zero pattern, $p_i \geq d_i$. If $\alpha_i = 0$ for some i, then we can take $h_{1i} = h_{2i}$. If $\beta_j = 0$ for some j, then we can take $h_{2j} = h_{1j}$. Now, for any nonzero α_i, β_i,

$$p_i + (-1)^{i+1} h_{2i} d_i = 0$$
$$p_i + (-1)^{i+1} h_{1i} d_i = 0$$

If $d_i = 0$, then we can take $h_{2i} = h_{1i}$. If $d_i \neq 0$, then $h_{2i} = h_{1i}$. ■

Theorem 4. *Let A be a nonnegative contractible matrix and let A_1 be a contraction of A. Then A is convertible if and only if A_1 is convertible.*

Proof. Suppose that

$$A = \begin{bmatrix} 1 & \alpha \\ 1 & \beta \\ 0 & C \end{bmatrix}, \quad \alpha = (\alpha_1, \cdots, \alpha_{n-1}), \quad \beta = (\beta_1, \cdots, \beta_{n-1}),$$

is convertible matrix with converter

$$H = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & \cdots & h_{2n} \\ 1 & & & \\ \vdots & & H_2 & \\ 1 & & & \end{bmatrix}.$$

Let $B = A \circ H$ and let $\tilde{\alpha}_i = \alpha_i h_{1i+1}$, $\tilde{\beta}_i = \beta_i h_{2i+1}$. Then,

$$\begin{aligned} \operatorname{per} A &= \det B \\ &= h_{11} \det B(1|1) - h_{21} \det B(2|1) \\ &= h_{11}(\tilde{\beta}_1 d_2 - \tilde{\beta}_2 d_3 + \cdots + (-1)^n \tilde{\beta}_{n-1} d_n) \\ &\quad - h_{21}(\tilde{\alpha}_1 d_2 - \tilde{\alpha}_2 d_3 + \cdots + (-1)^n \tilde{\alpha}_{n-1} d_n) \\ &= (h_{11}\tilde{\beta}_1 - h_{21}\tilde{\alpha}_1)d_2 - (h_{11}\tilde{\beta}_2 - h_{21}\tilde{\alpha}_2)d_3 \\ &\quad + \cdots + (-1)^n (h_{11}\tilde{\beta}_{n-1} - h_{21}\tilde{\alpha}_{n-1})d_n. \end{aligned}$$

By above lemma 3, in any cases,

$$\begin{aligned} \operatorname{per} A &= h_{12}(\beta_1 + \alpha_1)d_2 - h_{13}(\beta_2 + \alpha_2)d_3 + \cdots + (-1)^n h_{1n}(\beta_{n-1} + \alpha_{n-1})d_n \\ &= \det(A_1 \circ H_1), \end{aligned}$$

where

$$H_1 = \begin{bmatrix} h_{12} & h_{13} & \cdots & h_{1n} \\ & & H_2 & \end{bmatrix}.$$

Since A_1 is a contraction of A, $\operatorname{per} A = \operatorname{per} A_1$. Thus, $\operatorname{per} A_1 = \det(A_1 \circ H_1)$.

Conversely, suppose that A_1 is a convertible matrix with converter

$$H_1 = \begin{bmatrix} h_{12} & h_{13} & \cdots & h_{1n} \\ & & H_2 & \end{bmatrix} \text{ for some } H_2.$$

Let $B_1 = A_1 \circ H_1$ and let $B = A \circ H$ where

$$H = \begin{bmatrix} 1 & h_{12} & h_{13} & \cdots & h_{1n} \\ -1 & h_{12} & h_{13} & \cdots & h_{1n} \\ 1 & & & & \\ \vdots & & H_2 & & \\ 1 & & & & \end{bmatrix}$$

Since A_1 is convertible,

$$\begin{aligned} \text{per}A_1 &= \det B_1 \\ &= h_{12}(\beta_1 + \alpha_1)\det B_1(1|1) + (-1)h_{13}(\beta_2 + \alpha_2)\det B_1(1|2) + \\ &\quad \cdots + (-1)^n h_{1n}(\beta_{n-1} + \alpha_{n-1})\det B_1(1|n-1) \\ &= h_{12}(\beta_1 + \alpha_1)d_2 + (-1)h_{13}(\beta_2 + \alpha_2)d_3 + \\ &\quad \cdots + (-1)^n h_{1n}(\beta_{n-1} + \alpha_{n-1})d_n \\ &= (h_{12}\beta_1 d_2 + (-1)h_{13}\beta_2 d_3 + \cdots + (-1)^n h_{1n}\beta_{n-1}d_n) \\ &\quad + (h_{12}\alpha_1 d_2 + (-1)h_{13}\alpha_2 d_3 + \cdots + (-1)^n h_{1n}\alpha_{n-1}d_n) \\ &= \det B(1|1) + \det B(2|1) \\ &= \det B \\ &= \det(A \circ H) = \text{per}A. \end{aligned}$$

Thus, A is convertible. ■

Let

$$U_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and let

$$U_n = \begin{bmatrix} U_{n-1} & \mathbf{b} \\ \mathbf{a} & 1 \end{bmatrix}$$

for $n \geq 3$ where

$$\mathbf{a} = (0, \cdots, 0, \frac{1-(-1)^n}{2}, 1), \quad \mathbf{b} = (0, \cdots, 0, \frac{1+(-1)^n}{2}, 1)^T$$

Let $T_{n-1} = [\mathbf{x}_1, \cdots, \mathbf{x}_{n-1}]$ and for $k = 1, 2, \cdots, n-1$ let

$$V_{n,k} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \mathbf{x}_k & \mathbf{x}_1 & \cdots & \mathbf{x}_{k-1} & \mathbf{x}_k & \mathbf{x}_{k+1} & \cdots & \mathbf{x}_{n-1} \end{bmatrix}.$$

Let J_n be a matrix such that all entries are 1 of order n. Well known fact, J_3 is a nonconvertible matrix.

In [4], Hwang and Kim proved that for each $n \geq 2$, U_n and for $n \geq 4$, $V_{n,k}$ are extremal. Now, we propose the other proofs for extremality of U_n and $V_{n,k}$.

Theorem 5. *For each $n \geq 2$, the matrix U_n is extremal.*

Proof. By contraction, if replacing any one of the zero entries of it except u_{1n} by a 1, then yields J_3. But J_3 is a nonconvertible matrix.

Now, we consider $U_n + E_{1n}$. We proceed by induction on n. Let $n \geq 4$ since the cases $n = 2, 3$ are trivial. Then, we have two cases. The first case, if n is odd, then the proof is same as [4]. The second case, if n is even, then we produce the odd case by contraction. Therefore, U_n is extremal by induction. ■

Theorem 6. *For $n \geq 4$ and for each $k = 1, 2, \cdots, n-1$, the matrix $V_{n,k}$ is extremal and convertible.*

Proof. Since $V_{n-1,k}$ is extremal, $V_{n,k} + E_{1n}$ is an extremal. By contraction, theorem 3, and theorem 4, the matrix $V_{n,k}$ is extremal and convertible. ■

For a matrix A, let $\pi(A)$ denote the number of positive entries of A. For example, $\pi(T_n) = \frac{n^2+3n-2}{2}$. Let μ_n be the minimum number of entries of matrix which is extremal. Then $\mu_n \leq 4n - 4$ because U_n is an extremal matrix with $\pi(U_n) = 4n - 4$.

The following theorem finds a extremal and convertible matrix using contraction of matrix with $\pi(A) = \frac{n^2+n+4}{2}$.

Theorem 7. *Let*

$$T_{1,n-1} = \begin{bmatrix} 1 & \alpha \\ \beta & T_{n-1} \end{bmatrix}$$

where $\alpha = (1, 1, 0, \cdots, 0)$ and $\beta = (1, 0, \cdots, 0)^T$. Then $T_{1,n-1}$ is extremal *and convertible matrix with*

$$\pi(T_{1,n-1}) = \frac{n^2+n+4}{2}$$

for $n \geq 3$.

Proof. Since the first column of $T_{1,n-1}$ have exactly two nonzero entries, the matrix $T_{1,n-1}$ can be contracted to a form of T_{n-1}. The support matrix of a form of T_{n-1} is T_{n-1}. And a form of T_{n-1} is convertible matrix because the matrix T_{n-1} is convertible. Thus, $T_{1,n-1}$ is convertible matrix having exactly

$$\begin{aligned}\pi(T_{1,n-1}) &= \frac{(n-1)^2+3(n-1)-2}{2}+4 \\ &= \frac{n^2+n+4}{2}\end{aligned}$$

positive entries.

Now, we consider the extremality of $T_{1,n-1}$. If replacing any one of the zero entries of the first column of $T_{1,n-1}$, then, by contraction, yields J_3. If replacing any one of the zero entries from second column to nth column of $T_{1,n-1}$, then, by contraction and extremality of T_{n-1}, $T_{1,n-1}$ is extremal. This proves the theorem. ■

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