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NEARNESS FRAMES RELATED TO THE REGULAR NEARNESS SPACES

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ABSTRACT. In this pater, we study the relationship between regular nearness spaces and nearness frames determined by regular nearness spaces. We show that there is a compact regular frame in the frame $(\Omega(X), \mu)$ determined by a weakly normal nearness spaces. Moreover, a regular nearness space (X, ξ) is complete if $(\Omega(X), \mu)$ is complete in the sense of ([2]).

0. INTRODUCTION

The concept of nearness frames is introduced by Banaschewski and Pultr ([2]) to generalize uniform frames.

A *frame* is a complete lattice L in which binary meet distributes over arbitrary joins:

$$a \wedge \bigvee S = \bigvee\{a \wedge t | t \in S\} \quad (a \in L, S \subseteq L)$$

and a *frame homomorphism* is a map $h : L \to M$ between frames which preserves finitary meets, including the unit e, and arbitrary joins, including the zero 0.

For a topological space X, the open set lattice $\Omega(X)$ of X under the inclusion is a frame.

Herrlich has introduced a concept of nearness spaces to cover various topological structures ([4]) and characterized strict extension of topological spaces by completions of nearness spaces ([3,4,5]).

We recall some additional definitions. In any frame L, a *cover* of L is any subset whose join is e, and Cov(L) will be the set of all covers of L. Also, for any subsets A and B of L, $A \leq B$ means that, for each $a \in A$, there exists $b \geq a$ in B; this relation will be of particular importance on Cov(L).

Futher, we have:

For any $A \subseteq L$ and $x \in L, Ax = \bigvee\{a \in A | a \wedge x \neq 0\}$.

For any $A, B \subset L, AB = \{Ax | x \in B\}$.

For any $\mathcal{N} \subseteq \mathrm{Cov}(L)$, define a relation $\triangleleft_{\mathcal{N}}$, or simply $\triangleleft$ on L by $x \triangleleft y$ iff $Ax \leq y$ for some $A \in \mathcal{N}$.

A set $\mathcal{N} \subseteq \mathrm{Cov}(L)$ is called *admissible* if $a = \bigvee\{x \in L | x \triangleleft a\}$ for each $a \in L$.

A *nearness* on L is an admissible filter $\mathcal{N}$ in $(\mathrm{Cov}(L), \leq)$. A *nearness frame* L is a frame together with a specified nearness, denoted by $\mathcal{N}L$.

For any nearness space (X, ξ), we have the open set frame $\Omega(X)$ and the associated uniform covering structure $\mu_\xi = \{\mathcal{A} : \{X - A : A \in \mathcal{A}\} \notin \xi\}$ ([4]). Noticing that for $A, B \in \Omega(X), A <_\xi B$ iff $A \triangleleft_\mu B$, where μ is the family of open uniform covers in (X, ξ).

The purpose of this paper is to study the relationship between regular nearness spaces and nearness frames determined by regular nearness spaces. We define a strong inclusion relation on $(\Omega(X), \mu)$ which is a nearness frame determined by a weakly normal nearness space. Moreover, we have that there is a compact regular frame in the frame $(\Omega(X), \mu)$ determined by a weakly normal nearness space. We show that a strict extension between regular spaces induces a dense surjection ([2]), and using this, we show that a regular nearness space (X, ξ) is complete if $(\Omega(X), \mu)$ is complete in the sense of ([2]). We do not know yet the converse is true.

For the terminology, we refer to ([8]) for frames, ([4]) for nearness spaces and ([2]) for nearness frames.

1. Nearness frames determined by regular nearness spaces

We recall that for subsets A, B of a nearness space (X, ξ), $A <_\xi B$ iff the cover $\{X - A, B\}$ is a uniform cover in (X, ξ) and that an element x of X belongs to $intA$ iff $\{x\} <_\xi A$ ([4]). In the following, for a nearness space $(X, \xi), \Omega(X)$ denotes the underlying open set lattice on the space and μ the family of open uniform covers in (X, ξ). We note that μ generates the associated uniform covering structure on (X,ξ), because for any uniform cover $\mathcal{A}$ in (X, ξ), $\{intA : A \in \mathcal{A}\}$ is also a uniform cover in (X,ξ) by the condition of axioms for (X,ξ).

For the following, recall that, in any frame $L, a \prec b$ means there exist c such that $a \wedge c = 0$ and $b \vee c = e$. We note that $u \prec v$ in $\Omega(X)$ means $\overline{u} \subseteq v$, for a topological space $(X, \Omega(X))$ and L is called *regular* whenever $a = \bigvee\{x \in L | x \prec a\}$ for all $a \in L$. Note that $a \prec b$ is also expressed by the condition that $b \vee a^* = e$ where $a^* = \bigvee\{x \in L | x \wedge a = 0\}$ is the *pseudocomplement* of a, L is said to be a *compact* frame, if for any subset S of L with $\bigvee S = e$, there is a finite subset C of S such that $\bigvee C = e$.

Proposition 1.1. *A frame has a nearness iff it is regular.*

Proof. ($\Rightarrow$) For any cover $A, Ax \leq y$ implies $x \prec y$ since $e = \bigvee A = (Ax) \vee z, z = \bigvee\{t \in A | t \wedge x = 0\}$ and hence $x \wedge z = 0$ and $y \vee z = e$. Thus, regularity follows from the admissibility condition for a nearness.

($\Leftarrow$) If $x \prec y$ then, for the cover $A = \{y, x^*\}, Ax = y$. Hence, regularity implies that the filter in Cov(L) generated by all finite covers is a nearness. □

Proposition 1.2. *Let (X,ξ) be a nearness space and let $x \in X$ and $A \subset X$. Then the followings are equivalent :*

(i) If $\{x\} <_\xi A$ then there is a subset B of X with $\{x\} <_\xi B <_\xi A$.

(ii) The family μ of open uniform covers in (X,ξ) is a nearness on the frame $\Omega(X)$.

Proof. Clearly μ forms a filter in the family Cov$\Omega(X)$ of covers in the frame $\Omega(X)$. For A, B in $\Omega(X), A <_\xi B$ iff $A \lhd B$ because $A <_\xi B$ implies $\{X - A, B\}A \subseteq B$, and if $A \lhd B$, then there is a cover $\mathcal{A}$ in μ with $\mathcal{A}A \subseteq B$, which implies that $\mathcal{A}$ refines $\{X - A, B\}$, i.e., $A <_\xi B$. Thus μ is admissible iff the condition (i) holds, for $\{x\} <_\xi A$ iff $\{x\} <_\xi intA$. □

Definition 1.3. A nearness space (X,ξ) is called *regular* iff for any $\mathcal{A} \in \mu, \widetilde{\mathcal{A}} =$

$\{B \subset X : B <_\xi A$ for some $A \in \mathcal{A}\} \in \mu$, or equivalently, $\widehat{\mathcal{A}} = \{B \subset X : A <_\xi B$ for some $A \in \mathcal{A}\} \in \xi$ implies $\mathcal{A} \in \xi$. Moreover, (X, ξ) is called ***weakly normal*** iff for any $\mathcal{A} \in \mu$, then there exists a function $f : \mathcal{A} \to PX$ such that $f(\mathcal{A}) <_\xi \mathcal{A}$ and $\{f(A) | A \in \mathcal{A}\} \in \mu$.

Every weakly normal nearness space is obviousely regular.

Proposition 1.4. *If (X, ξ) is a regular nearness space, then it satisfies the condition (i) of Proposition 1.2*

Proof. Since $x \in A, \{\{x\}, X - A\} \notin \xi$. Put $\mathcal{A} = \{\{x\}, X - A\}$. Then $\widehat{\mathcal{A}} \in \xi$, for (X, ξ) is regular. Thus $\cap\{cl_\xi B | B \in \widehat{\mathcal{A}}\} = \emptyset$. So there is $B \in \widehat{\mathcal{A}}$ with $x \notin cl_\xi B$. It holds that $\{\{x\}, B\} \notin \xi$, and hence $\{x\} <_\xi X - B$. Since $C \in \widehat{\mathcal{A}}$ and $x \notin cl_\xi B, X - A <_\xi B$ and so that $X - B <_\xi X - (X - A) = A$. Therefore $\{x\} <_\xi X - B <_\xi A$.

Corollary 1.5. *Suppose (X, ξ) is a regular nearness space. Then the open set frame $\Omega(X)$ is a regular frame and it has the nearness μ.*

Proof. It is clear from Proposition 1.1 and 1.2.

Remark 1.6. *The relation $<_\xi$ on a nearness space (X, ξ) holds the followings : for any subsets A, B, A_1, B_1 of X,*

(1) $A_1 \subseteq A <_\xi B \subseteq B_1$ *implies* $A_1 <_\xi B_1$.

(2) $A <_\xi B$ *implies* $cl_\xi A \subset B$.

(3) $A <_\xi B$ *iff* $X - B <_\xi X - A$.

Definition 1.7. A binary relation $◂$ on a frame L is said to be a ***strong inclusion***, if it satisfies:

1) if $x \leq a ◂ b \leq y$ then $x ◂ y$.

2) $◂$ is a sublattic of $L \times L$.

3) $a ◂ b$ implies $a \prec b$.

4) $a ◂ b$ implies $b^* ◂ a^*$.

5) for any $a \in L, a = \bigvee\{x \in L | x \blacktriangleleft a\}$.

6) $\blacktriangleleft$ interpoles. i.e., if $a \blacktriangleleft b$ then $a \blacktriangleleft c \blacktriangleleft b$ for some c.

Proposition 1.8. *Let $(\Omega(X), \mu)$ be a nearness frame determined by regular nearness spaces (X, ξ). Define a relation $\blacktriangleleft$ on $\Omega(X)$ by*

$$A \blacktriangleleft B \text{ iff } \triangleleft_\mu B \text{ iff } A <_\xi B \text{ for any } A, B \text{ in } \Omega(X).$$

Then $\blacktriangleleft$ satisfies 1), 2), 3), 4) and 5) in Definition 1.7.

Proof. 1), 3) and 4) follow from (1), (2), (3) in Remark 1.6, respectively.

2) follows from the fact that $\Omega(X)$ is closed under finite intersections and arbitrary unions.

5) follows from the admissibility condition for μ. □

Proposition 1.9. *Let $(\Omega(X), \mu)$ be a nearness frame determined by weakly normal nearness spaces (X, ξ). Then $\blacktriangleleft$ interpoles.*

Proof. Assume that $A \blacktriangleleft B$ in Ω. Then $\{X - A, B\}$ is a uniform cover in (X, ξ). Since (X, ξ) is weakly normal, there is an open uniform cover $\{C, D\}$ such that $C <_\xi X - A$ and $D <_\xi B$. Thus $A <_\xi X - C \subseteq D <_\xi B$, which implies $A \blacktriangleleft D \blacktriangleleft B$. □

Since a weakly normal nearness space is regular, we have the following.

Corollary 1.10. *In a weakly normal nearness space, there is a strong inclusion on $(\Omega(X), \mu)$.*

In([1]), construct a compact regular frame on a frame L as follows: First of all, define a strong inclusion $\blacktriangleleft$ on L and then consider the family SR of strongly regular ideals realtive to $\blacktriangleleft$, that is, for any I in SR is a subset of L, which is a lower set and closed under finite joins and there is $b \in L$ with $a \blacktriangleleft b$ for all $a \in I$.

Using Corollary 1.10, we have also a compact regular frame on the frame $\Omega(X)$.

Theorem 1.11. *Let (X,ξ) be a weakly normal nearness space and $(\Omega(X),\mu)$ be the nearness frame determined by (X,ξ). Then there is a compact regular frame on the frame $(\Omega(X),\mu)$.*

2. Strict extensions and completeness

We investigate the relationship between strict extensions and surjections. The following is due to Bentley and Herrlich [3, 5].

Definition 2.1. Let $f:(X,\xi)\to(Y,\xi')$ be a nearness preserving map.

1) For any $A\subseteq X, opA = Y - cl_{\xi'}f(X-A)$ and for $\mathcal{A}\subseteq P(X), op\mathcal{A}=\{opA : A\in\mathcal{A}\}$.

2) The map f is said to be *strict* if μ' is generated by $\{op\mathcal{A}:\mathcal{A}\in\mu\}$, where μ,μ' denote the nearness on $\Omega(X)$ and $\Omega(Y)$, respectively.

3) The map f is said to be a *strict extension* if it is a strict dense embedding. i.e., (X,ξ) is a dense subspace of (Y,ξ').

Definition 2.2. Let $(L.\mathcal{N}L),(M,\mathcal{N}M)$ be nearness frames. A frame homomorphism $h: L\to M$ is said tc be a *surjection* if h is onto and for any $C\in\mathcal{N}M, h^*(C)$ is a cover of L and $\{h^*(C):C\in\mathcal{N}M\}$ generates $\mathcal{N}L$, where $h^*:M\to L$ denotes the right adjoint of h.

Theorem 2.3. *Let (X,ξ) and (Y,ξ') be regular nearness spaces. If $f:(X,\xi)\to(Y,\xi')$ is a strict extension then $\Omega(f):(\Omega(Y),\mu')\to(\Omega(X),\mu)$ is a dense surjection.*

Proof. Let h denote $\Omega(f)$. Since f is a dense embedding, h is onto dense. Take any $\mathcal{A}\in\mu$, then there is $\mathcal{B}\in\mu'$ such that $\mathcal{B}_X=\{B\cap X : B\in\mathcal{B}\}$ refines $\mathcal{A}$, for f is initial. For any $B\in\mathcal{B}$, there is $A\in\mathcal{A}$ with $h(B)=B\cap X\subseteq A$, i.e., $B\subseteq h^*(A)$. Thus $\mathcal{B}$ refines $h^*(\mathcal{A})$; hence $h^*(\mathcal{A})\in\mu'$ and $h^*(\mathcal{A})$ is a cover of $\Omega(Y)$. Suppose $\mathcal{B}\in\mu'$, then there is $\mathcal{A}\in\mu$ such that $op\mathcal{A}$ refines $\mathcal{B}$, for f is strict. For any $A\in\mathcal{A}$, take any $x \in h^*(A)$, then there is U in $\Omega(Y)$ such that $x \in U$ and $h(U) = U \cap X \subseteq A$. Since U is an open neighborhood of x in Y, $x \in Y - cl_{\xi'}(X - A) = opA$. Thus $h^*(A) \subseteq opA$. Hence $h^*(\mathcal{A})$ refines $\mathcal{B}$. Thus $\{h^*(\mathcal{A}) : \mathcal{A} \in \mu\}$ generates μ'. □

In [4], the completion (X^*, ξ^*) of a nearness space (X, ξ) was constructed, where $X^* = X \cup \{\mathcal{A} : \mathcal{A}$ is a ξ–cluster without adherence points$\}$ and the inclusion map $c : (X, \xi) \to (X^*, \xi^*)$ is a strict extension. Note that a nearness space (X, ξ) is regular iff (X, ξ^*) is regular.

We recall that a nearness frame L is *complete* if every dense surjection $h : M \to L$ is an isomorphism (see [2]).

Theorem 2.4. *Let (X, ξ) be a regular nearness space. If $(\Omega(X), \mu)$ is a complete nearness frame, then (X, ξ) is also a complete nearness space.*

Proof. Let $c : (X, \xi) \to (X^*, \xi^*)$ be a completion. Then c is a strict extension, and hence $\Omega(c) : (\Omega(X^*), \mu^*) \to (\Omega(X), \mu))$ is a dense surjection by Theorem 2.3 and $\Omega(c)$ is an isomorphism, for $(\Omega(X), \mu)$ is complete. Noticing that a map $f : (X, \xi) \to (Y, \xi')$ is a nearness preserving map iff the map $\Omega(f) : (\Omega(Y), \mu') \to (\Omega(X), \mu)$ defined by $\Omega(f)(U) = f^{-1}(U)$ is uniform, where (X, ξ) and (Y, ξ') are regular nearness spaces. Thus $c : (X, \xi) \to (X^*, \xi^*)$ is also an isomorphism. Therefore (X, ξ) is a complete nearness space.

Remark 2.5. *It doesn't know whether the converse of Theorem 2.4 is true or not.*

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